Spectral commutativity of multioperators

M. Chō* and V. Müller **

Dedicated to Professor Tadasi Huruya on his 60th birthday

Abstract. We give an example of pairs $A = (A_1, A_2)$, $B = (B_1, B_2)$ of operators such that $AB = (A_1B_1, A_2, B_2)$ and $BA = (B_1A_1, B_2A_2)$ are commuting pairs but $\sigma_T(AB) \setminus \{(0,0)\} \neq \sigma_T(BA) \setminus \{(0,0)\}$. This gives a negative answer to a problem posed by S. Li. Further, we show that $\sigma_T(AB) = \sigma_T(BA)$ if A and B are criss-cross commuting n-tuples and A is normal. This gives a positive answer to a problem studied in [ChCH].

Denote by $\mathcal{B}(X)$ the set of all bounded linear operators on a Banach space X.

It is well-known for two operators $A, B \in \mathcal{B}(X)$ that the spectra of AB and BA are almost equal,

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}. \tag{1}$$

Let $A = (A_n, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be two n-tuples of operators on a Banach space X. We denote by AB the n-tuple

$$AB = (A_1B_1, A_2B_2, \dots, A_nB_n). \tag{2}$$

In [L1], S. Li posed the following problem:

Is it true that

$$\sigma_T(AB) \setminus \{(0, \dots, 0)\} = \sigma_T(BA) \setminus \{(0, \dots, 0)\}$$

for all n-tuples A, B such that the n-tuples AB and BA are commuting (so that the Taylor spectrum σ_T is defined)?

In [L1], a positive answer was given under the assumption that the n-tuples $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are criss-cross commuting, i.e.,

$$A_iB_jA_k = A_kB_jA_i, \qquad B_iA_jB_k = B_kA_jB_i \tag{3}$$

for all i, j, k. Criss-cross commuting tuples were further studied in [L2], [H], [ChCH].

Remark 1. Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_m)$ be two tuples of operators on a Banach space X. Another natural possibility how to define the product of A and B is to consider the nm-tuple consisting of all products

$$(A_1B_1, A_1B_2, \dots, A_1B_m, A_2B_1, \dots, A_2B_m, \dots, A_nB_m).$$

Keywords and phrases: criss-cross commuting, spectral commutativity.

2000 Mathematics Subject Classification: Primary 47A10.

* partially supported by the Grant-in-Aid No. 14540190.

** partially supported by the grant No. 201/00/0208 of GA ČR.

This mn-tuple is commuting if A and B are criss-cross commuting in the sense of (3). However, this mn-tuple can be expressed as $\tilde{A}\tilde{B}$ where

$$\tilde{A} = (A_1, \dots, A_1, A_2, \dots, A_2, \dots, A_n \dots, A_n)$$

and

$$\tilde{B} = (B_1, B_2, \dots, B_m, B_1, \dots, B_m, \dots, B_1, \dots, B_m).$$

Thus all problems concerning this more general type of product can be reduced to the case of $m = n$ and the product defined by (2).

The first result of this paper gives a negative answer to the above mentioned problem of S. Li.

Example 2. We give an example of pairs $A = (A_1, A_2)$ and $B = (B_1, B_2)$ of operators such that $AB = (A_1B_1, A_2B_2)$ and $BA = (B_1A_1, B_2A_2)$ are commuting pairs but $\sigma_T(AB) \setminus \{(0,0)\} \neq \sigma_T(BA) \setminus \{(0,0)\}$.

Let H be a separable Hilbert space with an orthonormal basis $\{e_i, f_i, g_i\}_{i\in\mathbb{Z}}$. Define operators $A_1, A_2, B_1, B_2 \in \mathcal{B}(H)$ by

$$\begin{array}{llll} A_1e_i = 0, & A_2e_i = 0, & B_1e_i = f_i, & B_2e_i = g_i, \\ A_1f_i = e_i, & A_2f_i = 0, & B_1f_i = 0, & B_2f_i = 0, \\ A_1g_i = 0, & A_2g_i = e_{i+1}, & B_1g_i = 0, & B_2g_i = 0. \end{array}$$

It is easy to check that A_1B_1 and A_2B_2 are commuting. Similarly, B_1A_1 and B_2A_2 are commuting. However, A and B are not criss-cross commuting since $A_2B_1A_1 \neq A_1B_1A_2$ and $B_1A_2B_2 \neq B_2A_2B_1$.

For $i \in \mathbb{Z}$ we have $B_2A_2f_i = 0$ and $(B_1A_1 - I)f_i = 0$, so the pair $(B_1A_1 - I, B_2A_2)$ is Taylor singular and $(1,0) \in \sigma_T(BA)$.

We show that $(A_1B_1 - I, A_2B_2)$ is Taylor regular. We have

$$\begin{aligned} (A_1B_1 - I)e_i &= 0, \\ (A_1B_1 - I)f_i &= -f_i, \\ (A_1B_1 - I)g_i &= -g_i \end{aligned}$$

and

$$\begin{aligned} A_2B_2e_i &= e_{i+1}, \\ A_2B_2f_i &= 0, \\ A_2B_2g_i &= 0. \end{aligned}$$

Thus $\mathrm{Ker}(A_1B_1 - I) \cap \mathrm{Ker}(A_2B_2) = \{0\}$ and $\mathrm{Ran}(A_1B_1 - I) + \mathrm{Ran}(A_2B_2) = H$.

It is sufficient to show that the Koszul complex of the pair $(A_1B_1 - I, A_2B_2)$ is exact in the middle. Let $x = \sum_{i\in\mathbb{Z}}(\alpha_ie_i + \beta_if_i + \gamma_ig_i)$ and $y = \sum_{i\in\mathbb{Z}}(\alpha'_ie_i + \beta'_if_i + \gamma'_ig_i)$ be vectors in H satisfying $A_2B_2x = (A_1B_1 - I)y$. Thus

$$A_2B_2x = \sum_{i\in\mathbb{Z}} \alpha_ie_{i+1} = (A_1B_1 - I)y = \sum_{i\in\mathbb{Z}}(-\beta'_if_i - \gamma'_ig_i).$$

So $\alpha_i = 0, \beta_i' = 0$ and $\gamma_i' = 0$ for all i. Set $u = \sum_{i\in\mathbb{Z}}(\alpha_{i+1}' e_i - \beta_i f_i - \gamma_i g_i)$. Then $A_2B_2u = \sum_{i\in\mathbb{Z}} \alpha_i' e_i = y$ and $(A_1B_1 - I)u = \sum_{i\in\mathbb{Z}}(\beta_i f_i + \gamma_i g_i) = x$. Hence $(A_1B_1 - I, A_2B_2)$ is Taylor regular and $(1,0) \notin \sigma_T(AB)$.

In the second half of this paper we consider criss-cross commuting normal tuples. Note that if A, B are operators on a Hilbert space and A is normal then the equality (1) is true in a stronger form: $\sigma(AB) = \sigma(BA)$. The analogous question for n-tuples of operators was investigated in [ChCH] and partial results were obtained. We show that $\sigma_T(AB) = \sigma_T(BA)$ whenever A and B are criss-cross commuting tuples and A is normal, i.e., A consists of mutually commuting normal operators. This gives a positive answer to a problem studied in [ChCH].

We start with a version of the Fuglede-Putnam theorem.

Theorem 3. Let H, K be Hilbert spaces, let $A = (A_1, A_2) \in \mathcal{B}(H)^2$ and $B = (B_1, B_2) \in \mathcal{B}(K)^2$ be commuting pairs of normal operators, let $S : H \to K$ be a bounded linear operator. Then the following statements are equivalent:
(i) $B_1SA_1 + B_2SA_2 = 0$;
(ii) $SH_A(F) \subset K_B(F^\perp)$ for each closed subset $F \subset \mathbb{C}^2$, where

$$F^\perp = \{(\mu_1, \mu_2) \in \mathbb{C}^2 : \lambda_1\mu_1 + \lambda_2\mu_2 = 0 \text{ for some } (\lambda_1, \lambda_2) \in F\}$$

and $H_A(\cdot)$, $K_B(\cdot)$ are the spectral subspaces of A and B, respectively.

Proof. Without loss of generality we can assume that A_1, A_2, B_1, B_2 are contractions. Denote by $E_A(\cdot)$ and $E_B(\cdot)$ the spectral projections corresponding to A and B, respectively.

(i)$\Rightarrow$(ii): Suppose on the contrary that there is a closed subset $F \subset \sigma_T(A)$ such that $SH_A(F) \not\subset K_B(F^\perp)$. Equivalently, $E_B\big(\sigma_T(B) \setminus F^\perp\big)SE_A(F) \neq 0$.

Since

$$\sigma_T(B) \setminus F^\perp = \bigcup_{n=1}^{\infty} \big\{(\mu_1, \mu_2) \in \sigma_T(B) : \inf_{(\lambda_1,\lambda_2)\in F} |\lambda_1\mu_1 + \lambda_2\mu_2| \geq n^{-1}\big\},$$

it is easy to see that there are $\varepsilon > 0$ and a closed subset $M \subset \sigma_T(B)$ such that $E_B(M)SE_A(F) \neq 0$ and $|\lambda_1\mu_1 + \lambda_2\mu_2| \geq \varepsilon$ for all $(\lambda_1, \lambda_2) \in F$ and $(\mu_1, \mu_2) \in M$.

Choose a positive number $\delta < \varepsilon/8$. Since F and M can be covered by a finite number of balls of radius δ, there are $(\lambda_1, \lambda_2) \in F$, $(\mu_1, \mu_2) \in M$ and Borel sets F', M' such that $E_B(M')SE_A(F') \neq 0$, $M' \subset M \cap \{(z, w) : |z - \mu_1| \leq \delta, |w - \mu_2| \leq \delta\}$ and $F' \subset F \cap \{(z, w) : |z - \lambda_1| \leq \delta, |w - \lambda_2| \leq \delta\}$. Set $S' = E_B(M')SE_A(F')$.

Choose $x \in H_A(F')$ of norm one such that $\|S'x\| > \|S'\|/2$. We have

$$\|B_1S'A_1x - \lambda_1\mu_1S'x\| \leq \|(B_1 - \mu_1)S'A_1x\| + \|\mu_1S'(A_1x - \lambda_1x)\| \leq 2\delta\|S'\|,$$

and similarly, $\|B_2S'A_2x - \lambda_2\mu_2S'x\| \leq 2\delta\|S'\|$. Since

$$B_1S'A_1 + B_2S'A_2 = E_B(M')(B_1SA_1 + B_2SA_2)E_A(F') = 0,$$

we have $\|(\lambda_1\mu_1 + \lambda_2\mu_2)S'x\| \leq 4\delta\|S'\|$. On the other hand,

$$\|(\lambda_1\mu_1 + \lambda_2\mu_2)S'x\| \geq \|S'x\| \cdot |\lambda_1\mu_1 + \lambda_2\mu_2| \geq \varepsilon\|S'x\| > 4\delta\|S'\|,$$

a contradiction.

(ii)$\Rightarrow$(i): Let $\varepsilon > 0$. Let $(C_i)_{i=1}^\infty$ be nonempty disjoint Borel sets with diameters $< \varepsilon$ such that $\bigcup_i C_i = \mathbb{C}$. For each i fix $\lambda_i \in C_i$. Thus $C_i \subset \{z \in \mathbb{C} : |z - \lambda_i| < \varepsilon\}$. Set $F_0 = \{(0, w) : w \in \mathbb{C}\}$ and, for $i \in \mathbb{N}$, $F_i = \{(z, cz) : z \neq 0, c \in C_i\}$. Then $(F_i)_{i=0}^\infty$ are disjoint sets, $\bigcup_{i=0}^\infty F_i = \mathbb{C}^2$, $F_0^\perp = \{(z, 0) : z \in \mathbb{C}\}$ and $F_i^\perp = \{(-cz, z) : c \in C_i, z \in \mathbb{C}\} \quad (i \geq 1)$.

We have $E_B(F_0^\perp)(B_1SA_1 + B_2SA_2)E_A(F_0) = 0$. Clearly for each $i \geq 1$ we have $\|(A_2 - \lambda_i A_1)|H_A(F_i)\| < \varepsilon$ and $\|(B_1 + \lambda_i B_2)|K_B(F_i^\perp)\| < \varepsilon$. For $x \in H_A(F_i)$ we have

$$\begin{aligned}\|B_1SA_1x + B_2SA_2x\| &= \|B_1SA_1x + \lambda_i B_2SA_1x - \lambda_i B_2SA_1x + B_2SA_2x\|\\ &\leq \|(B_1 + \lambda_i B_2)SA_1x\| + \|B_2S(A_2 - \lambda_i A_1)x\| < 2\varepsilon\|S\| \cdot \|x\|.\end{aligned}$$

Thus $\|(B_1SA_1 + B_2SA_2)|H_A(F_i)\| \leq 2\varepsilon\|S\|$ for all i.

For $x \in H_A(F_i)$ we have $SA_1x \in K_B(F_i^\perp)$ and

$$B_1SA_1x = B_1E_B\big(F_i^\perp \setminus \{(0,0)\}\big)SA_1x + B_1E_B\big(\{(0,0)\}\big)SA_1x \in K_B\big(F_i^\perp \setminus \{(0,0)\}\big).$$

Similarly $B_2SA_2x \in K_B\big(F_i^\perp \setminus \{(0,0)\}\big)$ and we have $(B_1SA_1 + B_2SA_2)H_A(F_j) \subset K_B(F_j^\perp \setminus \{(0,0)\})$. Since the sets $F_j \setminus \{(0,0)\}$ are mutually disjoint, the spaces $K_B(F_j \setminus \{(0,0)\})$ are orthogonal. Thus $\|B_1SA_1 + B_2SA_2\| \leq 2\varepsilon\|S\|$. Since ε was arbitrary, we have $B_1SA_1 + B_2SA_2 = 0$. $\square$

Remark 4. Let A_1, A_2, B_1, B_2, S satisfy the conditions of the previous theorem. Since the spectral subspaces of A and A^* coincide and satisfy $H_A(F) = H_{A^*}(\bar{F})$ where $\bar{F} = \{\bar{z} : z \in F\}$, and similar relations hold for B and B^*, Theorem 3 implies the following general form of the Fuglede-Putnam theorem, see [P], [W]: if $B_1SA_1 + B_2SA_2 = 0$ then $B_1^*SA_1^* + B_2^*SA_2^* = 0$.

Theorem 5. Let $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$ be criss-cross commuting tuples, let A be normal (i.e, $A_1, \dots, A_n$ are commuting normal operators). Then $\sigma_T(AB) = \sigma_T(BA)$.

Proof. If $0 \in \sigma_T(A)$ then both AB and BA are Taylor singular by [ChCH], Theorem 2.1. Thus we may assume that A is Taylor regular. For $j = 1, \dots, n$ write

$$M_j = \{(z_1, \dots, z_n) \in \sigma(A) : |z_j| > |z_i| \quad (i < j) \text{ and } |z_j| \geq |z_i| \quad (i > j)\}.$$

Let H_j be the corresponding spectral subspaces $H_j = H_A(M_j)$. Clearly $H = \bigoplus_{j=1}^n H_j$ and $A_iH_j \subset H_j \quad (i, j = 1, \dots, n)$. Set $c_j = \min\{|z_j| : (z_1, \dots, z_n) \in M_j\}$. Then $c_j > 0$ and $A_j|H_j$ is invertible for each $j = 1, \dots, n$.

Fix k, i, j, $1 \leq k, i, j \leq n$, $i \neq j$. We have $A_iB_kA_j - A_jB_kA_i = 0$. By Theorem 3 for the pairs $(A_i, A_j), (A_j, -A_i)$ we have

$$B_kH_A\big(\{(z_1, \dots, z_n) : |z_i| \leq |z_j|, \|z_j\| \geq c_j/2\}\big) \subset H_A\big(\{(z_1, \dots, z_n) : |z_i| \leq |z_j|\}\big)$$

and

$$\begin{aligned}B_kH_A\big(\{(z_1, \dots, z_n) : |z_i| < |z_j|\}\big) &= \bigcup_{r=1}^\infty B_kH_A\big(\{(z_1, \dots, z_n) : |z_i| + r^{-1} \leq |z_j|\}\big)\\ &\subset H_A\big(\{(z_1, \dots, z_n) : |z_i| < |z_j| \text{ or } z_i = z_j = 0\}\big).\end{aligned}$$

Hence the spaces $H_j \quad (j = 1, \dots, n)$ are invariant with respect to the operators B_k for all k, and therefore also to all products $A_k B_k, B_k A_k$. Thus

$$\sigma_T(AB) = \bigcup_{j=1}^{n} \sigma_T(AB|H_j) \qquad \text{and} \qquad \sigma_T(BA) = \bigcup_{j=1}^{n} \sigma_T(BA|H_j).$$

Since $A_j|H_j$ is invertible for all j, by [ChCH], Theorem 3.3 we have $\sigma_T(AB|H_j) = \sigma_T(BA|H_j)$. Hence $\sigma_T(AB) = \sigma_T(BA)$. □

Acknowledgment. The paper was written during the second author's stay at the Kanagawa University. The author would like to thank for perfect working conditions and warm hospitality there.

References

[ChCH] M. Chō, R.E. Curto, T. Huruya, n-Tuples of operators satisfying $\sigma_T(AB) = \sigma_T(BA)$, Lin. Alg. Appl. 341 (2002), 291–298.

[H] R. Harte, On criss-cross commutativity, J. Operator theory 37 (1997), 303–309.

[L1] S. Li, On the commuting properties of Taylor's spectrum, Chinese Sci. Bull. 37 (1992), 1849–1852.

[L2] S. Li, Taylor spectral invariance for crisscross commuting pairs on Banach spaces, Proc. Amer. Math. Soc. 124 (1996), 2069–2071.

[P] C.R. Putnam, Normal operators and strong limit approximations, Indiana Univ. Math. J. 32 (1983), 377–379.

[W] G. Weiss, The Fuglede commutativity theorem modulo the Hilbert-Schmidt class and generating functions for matrix operators II, J. Operator Theory 5 (1981), 3–16.

Department of Mathematics
Kanagawa University
Yokohama 221-8686 Japan
chiyom01@kanagawa-u.ac.jp

Institute of Mathematics AV ČR
Zitna 25, 115 67 Prague 1
Czech Republic
muller@math.cas.cz