Open set of eigenvalues and SVEP

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By definition, every operator without SVEP has a nonempty open set consisting of eigenvalues. On the other hand, it is easy to construct an operator T with SVEP such that the interior of the point spectrum $\sigma_p(T)$ is nonempty. As a simple example, see [LN], p. 15, let $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$ be the open unit disc, let X be the Banach space of all bounded complex-valued functions on $\mathbb{D}$ with the supremum norm and let $T \in B(X)$ be the operator of multiplication by the independent variable z. It is easy to see that T has SVEP and $\sigma_p(T) = \mathbb{D}$. Moreover, T is decomposable.

Note that the Banach space in the last example is non-separable. At first glance it seems that a separable Banach space is too "small" for the existence of an operator T with SVEP and with nonempty interior of the point spectrum. This motivates the following question which was raised at the Workshop on Operator Theory in Warsaw, 2004.

Problem. Do there exist a separable Banach space and a bounded linear operator with SVEP on it such that the interior of the point spectrum of the operator is not empty?

The aim of this note is to give a positive answer to this question.

Theorem. There exist a separable Banach space X and a bounded linear operator $T \in B(X)$ with SVEP such that the interior of the point spectrum of T is not empty.

Proof. Let m be the restriction of the planar Lebesgue measure to the disc $\mathbb{D}$, let $L^1(m)$ be the Banach space of all complex valued functions on $\mathbb{D}$ that are absolutely integrable with respect to m, and let $A^1(m)$ be the Bergman space of all functions in $L^1(m)$ that are analytic on $\mathbb{D}$. It is well known that $A^1(m)$ is the closure of all polynomials [HKZ], p. 4.

Define a linear operator M on $L^1(m)$ by $(Mf)(z) := zf(z) \quad (f \in L^1(m))$. It is easy to see that M is a bounded linear operator on $L^1(m)$ with $\|M\| \leq 1$. Let $X = L^1(m)/A^1(m)$ and let T be the quotient operator induced by M on X. We claim that T is an operator with SVEP and that $\operatorname{Int} \sigma_p(T) \neq \emptyset$.

Let us check the second assertion first. Choose and fix $\lambda \in \mathbb{D}$. Since the constant function 1 is not in $(z-\lambda)A^1(m)$ the function $f_\lambda(z) := (z-\lambda)^{-1}$ cannot be in $A^1(m)$. On the other hand, it is straightforward that $f_\lambda \in L^1(m)$. Since

$$(T-\lambda)\big(f_\lambda + A^1(m)\big) = 1 + A^1(m) = 0$$

in X, we conclude that $f_\lambda + A^1(m) \in X$ is a nonzero eigenvector of T at λ. This proves the inclusion $\mathbb{D} \subseteq \operatorname{Int} \sigma_p(T)$. Since $\|T\| \leq 1$, we have $\sigma(T) = \overline{\mathbb{D}}$.

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Suppose that T does not have SVEP. Then there exist $\lambda \in \mathbb{D}$, an open disc $U \subseteq \mathbb{D}$ centred at λ, and a nonzero analytic function $F: U \to X$ such that $(T-z)F(z)=0$ for all $z \in U$. Let $F(z)=\sum_{n=0}^{\infty} x_n(z-\lambda)^n$ be the Taylor expansion of F in U. Then the following relations hold between the coefficients

$$(T-\lambda)x_0 = 0 \qquad \text{and} \qquad (T-\lambda)x_n = x_{n-1} \quad (n \geq 1).$$

Since F is nontrivial, some of the coefficients are nonzero; without loss of generality we may assume that $x_0 \neq 0$. Thus,

$$(T-\lambda)^2 x_1 = (T-\lambda)x_0 = 0 \qquad \text{and} \qquad (T-\lambda)x_1 = x_0 \neq 0.$$

Let $f \in L^1(m)$ be such that $x_1 = f + A^1(m)$. Then there are $\alpha \in \mathbb{C}$ and $h \in A^1(m)$ that satisfy the equality

$$(z-\lambda)^2 f(z) = \alpha + (z-\lambda)h(z)$$

for almost all $z \in \mathbb{D}$. It follows

$$f(z) = \alpha(z-\lambda)^{-2} + (z-\lambda)^{-1}h(z),$$

which is possible only if $\alpha = 0$ since $f(z)$ and $(z-\lambda)^{-1}h(z)$ are in $L^1(m)$ and $(z-\lambda)^{-2}$ is not in $L^1(m)$. We get

$$(T-\lambda)x_1 = (T-\lambda)\big(f + A^1(m)\big) = h + A^1(m) = 0$$

in X, a contradiction. Therefore T has SVEP. $\square$

Note that the operator M from the above proof is decomposable. Therefore T has the decomposition property (δ), which means that T is in a some sense close to the class of decomposable operators. However, it is probably not decomposable. It is for sure not spectral because, as the following well known argument shows, there do not exist spectral operators on separable Banach spaces with nonempty interior of the point spectrum at all. Namely, assume that S is a spectral operator on a separable Banach space X with the spectral measure E which is bounded by a constant $c > 0$. If $\lambda \neq \mu$ are eigenvalues of S and x and y are the corresponding eigenvectors, then $E(\{\lambda\})(x-y) = x$ and therefore $\|x\| \leq c\|x-y\|$. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a dense subset that exists by separability. It follows that in any ball centred at x_n and with radius $1/2c$ there exists at most one eigenvector with norm 1, i.e. there are at most countably many eigenvalues of S.

References

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