# The distance from the Apostol spectrum

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**Abstract.** *If $T$ is an s-regular operator in a Banach space (i.e. $T$ has closed range and $N(T) \subset R^\infty(T)$) and $\gamma(T)$ is the Kato reduced minimum modulus, then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \sup\{r : T - \lambda \text{ is s} - \text{regular for } |\lambda| < r\}.$$

Let $x$ be an element of a Banach algebra $A$. The spectral radius of $x$ is given by the well-known spectral radius formula: $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

There are a number generalizations of this formula. If we denote $d(x) = \inf\{\|xy\| : y \in A, \|y\| = 1\}$ and by $\tau_l(x) = \{\lambda \in \mathbf{C} : d(x - \lambda) = 0\}$ the left approximate point spectrum of $x$ then dist $\{0, \tau_l(x)\} = \lim_{n\to\infty} d(x^n)^{1/n}$, see [13], [9]. In particular in the algebra $B(X)$ of all bounded linear operators in a Banach space $X$ this gives formulas for radii of boundedness below or surjectivity:

$$\sup\{r : T - \lambda \text{ is bounded below for } |\lambda| < r\} = \lim_{n\to\infty} j(T^n)^{1/n}$$

and

$$\sup\{r : T - \lambda \text{ is onto for } |\lambda| < r\} = \lim_{n\to\infty} k(T^n)^{1/n},$$

where $j(T)$ and $k(T)$ are the moduli of injectivity and surjectivity of $T$:

$$j(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}$$

and

$$k(T) = \sup\{r : TU_X \supset rU_X\},$$

where $U_X$ is the closed unit ball in $X$.

For a bounded linear operator $T$ in a Banach space $X$ denote by $N(T)$ and $R(T)$ its kernel and range, respectively. Denote further $R^\infty(T) = \bigcap_{n=1}^\infty R(T^n)$ and $N^\infty(T) = \bigcup_{n=1}^\infty N(T^n)$.

The injectivity and surjectivity moduli of an operator which is bounded below (onto) are special cases of the Kato reduced minimum modulus [7]

$$\gamma(T) = \inf\Big\{\frac{\|Tx\|}{\text{dist}\,\{x, N(T)\}} : x \in X \backslash N(T)\Big\}$$

(for $T = 0$ we define formally $\gamma(T) = \infty$).

The existence and the meaning of the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ in a more general setting was studied by Apostol [1] and Mbekhta [10].

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**Definition.** *Let $T \in B(X)$. We say that $T$ is s-regular (= semi-regular) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$.*

The s-regular operators and closely related classes of operators were studied (under various names) by many authors, see [3], [4], [5], [6], [8], [16]. We list some of the most important equivalent conditions for s-regular operators, see [11], [12].

**Theorem.** *Let $T \in B(X)$ be an operator with a close range. The following conditions are equivalent:*

(1) *$T$ is s-regular,*
(2) *the function $\lambda \mapsto R(T-\lambda)$ is continuous at $0$ in the gap topology,*
(3) *the function $\lambda \mapsto N(T-\lambda)$ is continuous at $0$ in the gap topology,*
(4) *the function $\lambda \mapsto \gamma(T-\lambda)$ is continuous at $0$,*
(5) $\liminf_{\lambda\to 0} \gamma(T-\lambda) > 0$,
(6) $N^\infty(T) \subset R(T)$,
(7) $N^\infty(T) \subset R^\infty(T)$.

Denote further $\sigma_\gamma(T) = \{\lambda \in \mathbf{C} : T-\lambda \text{ is not s}-\text{regular}\}$. The set $\sigma_\gamma(T)$ was studied by Apostol [1], Rakočevič [15], Mbekhta and Ouahab [11], [12] and Mbekhta [10]. The terminology is not unified; we suggest to call $\sigma_\gamma(T)$ the Apostol spectrum of $T$.

The Apostol spectrum $\sigma_\gamma(T)$ is always a non-empty compact subset of the complex plane, $\partial\sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T)$ and $\sigma_\gamma f(T) = f\sigma_\gamma(T)$ for any function $f$ analytic in a neighbourhood of $\sigma(T)$.

If $T$ is an s-regular operator in a Hilbert space then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \text{dist}\,\{0, \sigma_\gamma(T)\} = \sup\{r : T-\lambda \text{ is s}-\text{regular for } |\lambda| < r\}, \tag{1}$$

see [1], Theorem 3.2 or [10], Theorem 3.1.

The aim of this paper is to prove equality (1) for operators in Banach spaces. This gives a positive answer to the conjecture of Rakočevič [15] and Mbekhta and generalizes the above mentioned results for radii of injectivity and surjectivity.

Further we study the essential version of this result.

If $T$ is a semi-Fredholm operator then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists by [2] and it is equal to the semi-Fredholm radius of $T$:

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \sup\{r : T-\lambda \text{ is semi}-\text{Fredholm for } |\lambda| < r\},$$

see [17] and [2].

We prove a similar formula for essentially s-regular operators which generalizes the semi-Fredholm case.

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**Lemma 1.** *$T \in B(X)$ is s-regular if and only if there exists a closed subspace $M \subset X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by $T$ is bounded below.*

**Proof.** If $T$ is s-regular then set $M = R^\infty(T)$. It is well-known that $M$ is closed and (see e.g. [4],Theorem 3.4) that $TM = M$ and $\tilde{T} : X/M \to X/M$ is bounded from bellow.

Conversely, let $M$ be the subspace of $X$ with the required properties. Then $TM = M$ implies $M \subset R^\infty(T)$. If $Tx = 0$ then $\tilde{T}(x + M) = 0$ and the injectivity of $\tilde{T}$ implies $x \in M$. Thus $N(T) \subset M \subset R^\infty(T)$.

It remains to prove that $T$ has closed range. Let $\pi : X \to X/M$ be the canonical projection. We show $R(T) = \pi^{-1}R(\tilde{T})$. If $y \in R(T), y = Tx$ for some $x \in X$ then $\pi y = Tx + M = \tilde{T}(x + M) \in R(\tilde{T})$ so that $R(T) \subset \pi^{-1}R(\tilde{T})$. If $y \in X$ and $\pi y \in R(\tilde{T})$, i.e. $y + M = Tx + M$ for some $x \in X$ then $y \in R(T)$ since $M \subset R(T)$. Thus $R(T) = \pi^{-1}R(\tilde{T})$ which is closed since $R(\tilde{T})$ is closed and $\pi$ continuous.

**Lemma 2.** *Let $T \in B(X)$ and let $M$ be a closed subspace of $X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by $T$ is bounded below. Denote by $T_1 : M \to M$ the restriction of $T$ to $M$. Then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \min\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\}.$$

**Proof.** The limits on the right hand side exist by [17]. If $T^n x = 0$ then $\tilde{T}^n(x+M) = 0$, i.e. $x \in M$. Thus $N(T^n) \subset M$ and $N(T_1^n) = N(T^n)$. We have

$$\begin{aligned}\gamma(T_1^n) &= \inf\Big\{\frac{\|T_1^n x\|}{\text{dist}\,\{x, N(T_1^n)\}} : x \in M\backslash N(T_1^n)\Big\}\\ &= \inf\Big\{\frac{\|T^n x\|}{\text{dist}\,\{x, N(T^n)\}} : x \in M\backslash N(T^n)\Big\} \ge \gamma(T^n).\end{aligned}$$

Further, since $TM = M$,

$$\begin{aligned}\gamma(\tilde{T}^n) &= \inf\Big\{\frac{\|\tilde{T}^n(x+M)\|}{\|x+M\|} : x \notin M\Big\} = \inf\Big\{\frac{\|T^n x + M\|}{\text{dist}\,\{x, M\}} : x \notin M\Big\}\\ &\ge \inf\Big\{\frac{\|T^n x\|}{\text{dist}\,\{x, M\}} : x \notin M\Big\} \ge \inf\Big\{\frac{\|T^n x\|}{\text{dist}\,\{x, N(T^n)\}} : x \notin M\Big\} \ge \gamma(T^n).\end{aligned}$$

Thus $\gamma(T^n) \le \min\{\gamma(T_1^n), \gamma(\tilde{T}^n)\}$ and

$$\limsup_{n\to\infty} \gamma(T^n)^{1/n} \le \min\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\}.$$

Denote by

$$s = \min\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\}.$$

We prove $\liminf_{n\to\infty} \gamma(T^n)^{1/n} \ge s$.

Let $n \ge 1$, $x = x_0 \in R(T^n)$, $\|x\| = 1$ and let $s > \varepsilon > 0$. Then $x + M \in R(\tilde{T}^n)$ and

$$\|\tilde{T}^{-i}(x + M)\| \le \gamma(\tilde{T}^i)^{-1}\|x + M\| \le \gamma(\tilde{T}^i)^{-1} \qquad (i = 1, \dots, n).$$

Thus there exist vectors $x_i \in \tilde{T}^{-i}(x+M)$ such that

$$\|x_i\| \leq \gamma(\tilde{T}^i)^{-1}(1+\varepsilon) \qquad (i=1,\dots,n).$$

Denote by $m_i = Tx_{i+1} - x_i \qquad (i=0,\dots,n-1)$. Then

$$\|m_i\| \leq \|T\|\|x_{i+1}\| + \|x_i\| \leq (1+\varepsilon)\big[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\big] \qquad (i=0,\dots,n-1).$$

Further $\tilde{T}^i(m_i+M) = T^{i+1}x_{i+1} - T^i x_i + M = M$ so that $m_i \in M$ for each $i$. We have

$$\sum_{i=0}^{n-1} T^i m_i = (T^n x_n - T^{n-1}x_{n-1}) + (T^{n-1}x_{n-1} - T^{n-2}x_{n-2}) + \cdots + (Tx_1 - x_0) = T^n x_n - x.$$

Since $T_1 M \to M$ is onto, there exist vectors $m_i' \in M$ such that $T^{n-i}m_i' = m_i$ and $\|m_i'\| \leq (1+\varepsilon)\gamma(T_1^{n-i})^{-1}\|m_i\|$. Thus

$$T^n\big(x_n - \sum_{i=0}^{n-1} m_i'\big) = T^n x_n - \sum_{i=0}^{n-1} T^i m_i = x$$

and

$$\Big\|x_n - \sum_{i=0}^{n-1} m_i'\Big\| \leq (1+\varepsilon)\gamma(\tilde{T}^n)^{-1} + \sum_{i=0}^{n-1}(1+\varepsilon)^2\gamma(T_1^{n-i})^{-1}\Big[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\Big].$$

Thus

$$\gamma(T^n)^{-1} \leq (1+\varepsilon)\gamma(\tilde{T}^n)^{-1} + \sum_{i=0}^{n-1}(1+\varepsilon)^2\gamma(T_1^{n-i})^{-1}\Big[\|T\|\gamma(\tilde{T}^{i+1})^{-1} + \gamma(\tilde{T}^i)^{-1}\Big].$$

Find $n_0$ such that

$$\gamma(T_1^i) \geq (s-\varepsilon)^i, \quad \gamma(\tilde{T}^i) \geq (s-\varepsilon)^i \qquad (i \geq n_0).$$

Denote by

$$K = \max_{1\leq i\leq n_0+1} \max\big\{\gamma(T_1^i)^{-1}, \gamma(\tilde{T}^i)^{-1}, (s-\varepsilon)^{-i}\big\}.$$

For $n$ large enough we have

$$\begin{aligned}
\gamma(T^n)^{-1} &\leq (1+\varepsilon)^2\Big[(s-\varepsilon)^{-n} + \sum_{i=n_0}^{n-n_0-1}(s-\varepsilon)^{i-n}\big(\|T\|(s-\varepsilon)^{-i-1} + (s-\varepsilon)^{-i}\big) \\
&+ \sum_{i=0}^{n_0-1}(s-\varepsilon)^{i-n}\big(\|T\|\cdot K + K\big) + \sum_{i=n-n_0}^{n-1} K\big(\|T\|(s-\varepsilon)^{-i-1} + (s-\varepsilon)^{-i}\big)\Big] \\
&\leq (1+\varepsilon)^2(s-\varepsilon)^{n_0-n}\Big[K + (n-2n_0)(K\cdot\|T\| + K) + 2n_0K(\|T\|\cdot K + K) + \Big] \\
&\leq (1+\varepsilon)^2(s-\varepsilon)^{n_0-n} n\cdot K',
\end{aligned}$$

where $K'$ is a constant independent of $n$. Hence

$$\liminf_{n\to\infty} \gamma(T^n)^{1/n} \geq \liminf_{n\to\infty} (s-\varepsilon)^{\frac{n-n_0}{n}} = s-\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\liminf_{n\to\infty} \gamma(T^n)^{1/n} \geq s$, so that

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = s.$$

**Theorem 3.** *Let $T \in B(X)$ be s-regular. Then*

$$\operatorname{dist}\{0, \sigma_\gamma(T)\} = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

**Proof.** Denote $r = \operatorname{dist}\{0, \sigma_\gamma(T)\}$. Let $M = R^\infty(T), T_1 = T|M$ and let $\tilde{T} : X/M \to X/M$ be the operator induced by $T$. If $\lambda$ is a complex number satisfying

$$|\lambda| < \lim_{n\to\infty} \gamma(T^n)^{1/n} = \min\{\lim_{n\to\infty} \gamma(T_1^n)^{1/n}, \lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n}\},$$

then $T_1 - \lambda$ is onto and $\tilde{T} - \lambda$ is bounded below. Thus $T - \lambda$ is s-regular by Lemma 1 and $\lim_{n\to\infty} \gamma(T^n)^{1/n} \leq r$.

Conversely, it is well-known (see e.g. [15], Theorem 5.2) that $R^\infty(T-\lambda)$ is constant on the component of $\mathbf{C}\setminus\sigma_\gamma(T)$ containing 0, in particular $R^\infty(T-\lambda) = M$ for $|\lambda| < r$. If $|\lambda| < r$ then $(T-\lambda)M = M$ and $\tilde{T} - \lambda = \widetilde{T-\lambda} : X/M \to X/M$ is bounded below. Thus $\lim_{n\to\infty} \gamma(T_1^n)^{1/n} \geq r$ and $\lim_{n\to\infty} \gamma(\tilde{T}^n)^{1/n} \geq r$. Hence $\lim_{n\to\infty} \gamma(T^n)^{1/n} \geq r$ by Lemma 2.

**Remark.** It is possible to deduce the inequality $\operatorname{dist}\{0, \sigma_\gamma(T)\} \geq \lim_{n\to\infty} \gamma(T^n)^{1/n}$ from [11], Theorem 2.10. We have obtained a new direct proof of this result.

**Definition.** *$T \in B(X)$ is called essentially s-regular if $R(T)$ is closed and there exists a finite dimensional subspace $F \subset X$ such that $N(T) \subset R^\infty(T) + F$.*

*Define further $\sigma_{e\gamma}(T) = \{\lambda \in \mathbf{C} : T - \lambda$ is not essentially s-regular $\}$.*

For properties of essentially s-regular operators and the set $\sigma_{e\gamma}(T)$ see [14],[15].

**Theorem 4.** *Let $T \in B(X)$ be essentially s-regular . Then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \max\{r : T - \lambda \text{ is s-regular for } 0 < |\lambda| < r\} = \operatorname{dist}\{0, \sigma_\gamma(T)\setminus\{0\}\}.$$

**Proof.** By [14],Theorem 3.1 or [15], Theorem 2.1 there exist subspaces $X_1, X_2 \subset X$ such that $X = X_1 \oplus X_2, \dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|X_1$ si nilpotent and $T_2 = T|X_2$ is s-regular (the Kato decomposition). By the previous theorem $\operatorname{dist}\{0, \sigma_\gamma(T_2)\} = \lim_{n\to\infty} \gamma(T_2^n)^{1/n}$. For $n \geq \dim X_1$ we have $T_1^n = 0$ so that

$N(T^n) = X_1 \oplus N(T_2^n)$. Let $P$ be the projection with $R(P) = X_2$ and $N(P) = X_1$. Let $x_2 \in X_2$. We have

$$\begin{aligned}
&\operatorname{dist}\{x_2, N(T_2^n)\} = \inf\{\|x_2 - y_2\| : y_2 \in X_2, T_2^n y_2 = 0\} \\
\le &\|P\| \inf\{\|y_1 \oplus (x_2 - y_2)\| : y_1 \in X_1, y_2 \in X_2, T_2^n y_2 = 0\} \\
= &\|P\| \operatorname{dist}\{x_2, N(T^n)\} \le \|P\| \operatorname{dist}\{x_2, N(T_2^n)\}.
\end{aligned}$$

Then

$$\begin{aligned}
\gamma(T_2^n) &= \inf\Big\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T_2^n)\}} : x_2 \in X_2 \backslash N(T_2^n)\Big\} \\
&\le \inf\Big\{\frac{\|T^n x_2\|}{\operatorname{dist}\{x_2, N(T^n)\}} : x_2 \in X_2 \backslash N(T^n)\Big\} \\
&= \inf\Big\{\frac{\|T^n (x_1 \oplus x_2)\|}{\operatorname{dist}\{x_1 \oplus x_2, N(T^n)\}} : x_1 \oplus x_2 \in X \backslash N(T^n)\Big\} = \gamma(T^n)
\end{aligned}$$

and

$$\begin{aligned}
\gamma(T^n) &\le \inf\Big\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T^n)\}} : x_2 \in X_2 \backslash N(T_2^n)\Big\} \\
&\le \|P\| \inf\Big\{\frac{\|T_2^n x_2\|}{\operatorname{dist}\{x_2, N(T_2^n)\}} : x_2 \in X_2 \backslash N(T_2^n)\Big\} = \|P\| \gamma(T_2^n).
\end{aligned}$$

Hence $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \lim_{n\to\infty} \gamma(T_2^n)^{1/n}$.

If $\lambda \neq 0$ then $T - \lambda$ is s-regular if and only if $T_2 - \lambda$ is s-regular. Then

$$\max\{r : T - \lambda \text{ is s} - \text{regular for } 0 < |\lambda| < r\} = \operatorname{dist}\{0, \sigma_\gamma(T_2)\} = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

The following lemma is an analog of Lemma 1 for essentially s-regular operators:

**Lemma 5.** *$T \in B(X)$ is essentially s-regular if and only if there exists a closed subspace $M \subset X$ such that $TM = M$ and the operator $\tilde{T} : X/M \to X/M$ induced by $T$ is upper semi-Fredholm.*

**Proof.** If $T$ is essentially s-regular then set $M = R^\infty(T)$. If $X = X_1 \oplus X_2$ is the Kato decomposition ($\dim X_1 < \infty, TX_1 \subset X_1, TX_2 \subset X_2, T_1 = T|X$ nilpotent and $T_2 = T|X_2$ s-regular) then $M = R^\infty(T_2) \subset X_2$ and $TM = T_2 M = M$. If $x = x_1 \oplus x_2$ satisfies $Tx \in M$ then $T_2 x_2 \in M$ so that $x_2 \in M$. Thus $x \in X_1 + M$ and $N(\tilde{T}) \subset X_1 + M$. Hence $\dim N(\tilde{T}) < \infty$.

Let $\pi : X \to X/M$ be the canonical projection. Since $M \subset R(T)$ and $R(\tilde{T}) = \{Tx + M : x \in X\} = \pi R(T)$ the range of $\tilde{T}$ is closed. Thus $\tilde{T}$ is upper semi-Fredholm.

Conversely, let $M$ be a subspace of $X$ with the required properties. We can prove that $R(T)$ is closed in exactly the same way as in Lemma 1.

Further $M \subset R^\infty(T)$. If $Tx = 0$ then $\tilde{T}(x + M) = 0$, i.e. $\pi x \in N(\tilde{T})$. Thus $N(T) \subset \pi^{-1} N(\tilde{T}) \subset M + F \subset R^\infty(T) + F$ for a finite dimensional subspace $F \subset X$.

**Theorem 6.** *Let $T, A \in B(X), TA = AT$, let $A$ be a quasinilpotent. Then*
(1) $\sigma_\gamma(T+A) = \sigma_\gamma(T)$,
(2) $\sigma_{\gamma e}(T+A) = \sigma_{\gamma e}(T)$.

**Proof.** Let $T$ be an essentially s-regular operator and let $A$ be a quasinilpotent commuting with $T$. Denote $M = R^\infty(T), T_1 = T|M$ and let $\tilde{T} : X/M \to X/M$ be the operator induced by $T$. Clearly $AM \subset M$ so that we can define operators $A_1 = A|M$ and $\tilde{A} : X/M \to X/M$ induced by $A$. Clearly $r(A_1) = \lim_{n\to\infty} \|A_1^n\|^{1/n} \leq \lim_{n\to\infty} \|A^n\|^{1/n} = 0$ and $r(\tilde{A}) = \lim_{n\to\infty} \|\tilde{A}^n\|^{1/n} \leq \lim_{n\to\infty} \|A^n\|^{1/n} = 0$ so that $\sigma(A_1) = \{0\}$ and $\sigma(\tilde{A}) = \{0\}$. Denote by

$$\begin{aligned} \sigma_\delta(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not onto}\}, \\ \sigma_\pi(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not bounded below}\} \quad \text{and} \\ \sigma_{\pi e}(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not upper semi} - \text{Fredholm}\} \end{aligned}$$

the defect spectrum, the approximate point spectrum and the essential approximate point spectrum, respectively.

By the spectral mapping property for these spectra we have

$$\begin{aligned} \sigma_\delta(T+A) &= \sigma_\delta(T), \\ \sigma_\pi(\tilde{T}+\tilde{A}) &= \sigma_\pi(\tilde{T}), \\ \sigma_{\pi e}(\tilde{T}+\tilde{A}) &= \sigma_{\pi e}(\tilde{T}). \end{aligned}$$

Thus $0 \notin \sigma_\delta(T+A)$, i.e. $(T+A)M = M$. Similarly $0 \notin \sigma_{\pi e}(\tilde{T}+\tilde{A})$, i.e. $\tilde{T}+\tilde{A}$ is upper semi-Fredholm. By the previous lemma $T+A$ is essentially s-regular. This proves (2).

If $T$ is s-regular and $A$ an quasinilpotent commuting with $T$ then in the same way $(T+A)M = M$ and $\tilde{T}+\tilde{A}$ is bounded below. Hence $T+A$ is s-regular by Lemma 1.

**Remark.** Statement (1) for Hilbert space operators was proved in [10], Theorem 4.8. The second statement gives a positive answer to question 3 of [15].

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