On algebras without generalized topological divisors of zero

H. Arizmendi and V Müller*

Abstract

We exhibit an example of a non-commutative B_0-algebra without generalized topological divisors of zero. This gives an answer to Problems 1 and 2 of Żelazko, [5].

One of basic concepts of the theory of Banach algebras is the concept of topological divisors of zero. A topological divisor of zero in a Banach algebra A is a non-zero element $x \in A$ such that there exists a sequence of elements $x_n \in A, \|x_n\| = 1$ with $\lim_{n\to\infty} x_n x = 0$. By a well-known result of Shilov a complex Banach algebra either possesses a topological divisor of zero or is isomorphic to the field of complex numbers. An analogous result fails for general locally convex algebras and therefore Żelazko [3] introduced the concept of generalized topological divisors of zero.

A topological algebra A is said to possess generalized topological divisors of zero if there are sets $S_1, S_2 \subset A$ such that $0 \notin \overline{S_1}, 0 \notin \overline{S_2}$ but $0 \in \overline{S_1 S_2}$. This is equivalent to the condition that there exists a neighbourhood U of zero such that $0 \in \overline{(A \setminus U)^2}$.

In [4] it was proved that a complex m-convex algebra either possesses generalized topological divisors of zero or is isomorphic to the field of complex numbers and conjectured that this is also true for an arbitrary topological algebra. This conjecture was disproved in [1] where a commutative B_0-algebra possessing no generalized topological divisors of zero was constructed.

A B_0-algebra is a completely metrizable locally convex algebra. The topology of a B_0-algebra can be given by means of a sequence of seminorms $\|\cdot\|_n, n = 1, 2, \dots$ satisfying

$$\|x\|_1 \le \|x\|_2 \le \cdots \qquad (x \in A) \tag{1}$$

and

$$\|xy\|_n \le \|x\|_{n+1} \cdot \|y\|_{n+1} \tag{2}$$

for all $x, y \in A$ and $n = 1, 2, \dots$.

It is easy to see (cf. [5]) that a B_0-algebra A does not possess generalized topological divisors of zero if and only if its topology can be given by a sequence of seminorms $\|\cdot\|_n, n = 1, 2, \dots$ satisfying (1), (2) and, for some positive constants c_n,

$$\|x\|_n \cdot \|y\|_n \le c_n \|xy\|_{n+1} \qquad (x, y \in A, n = 1, 2, \dots). \tag{3}$$

In this paper we construct a complex non-commutative B_0-algebra without generalized topological divisors of zero. This gives a positive answer to Problem 1 of [5].

The constructed algebra A has also the property that, for all nets $(x_\alpha), (y_\alpha)$ of elements of A,

$$x_\alpha y_\alpha \to 0 \Longleftrightarrow y_\alpha x_\alpha \to 0. \tag{4}$$

---

* The paper was written during the second author's stay at the Instituto de Matemáticas, Universidad Nacional Autonoma de México.

For Banach algebras this is equivalent to the condition

$$\|xy\| \geq k\|yx\| \qquad (x,y \in A)$$

for some positive constant k. By a result of Le Page [2] such an algebra is necessarily commutative. By using a similar argument it is possible to show that an m-convex algebra satisfies (4) if and only if it is commutative, cf. [5].

The present example shows that for B_0-algebras condition (4) does not imply commutativity. This gives a negative answer to Problem 2 of [5].

Let A be the algebra of all polynomials in two non-commuting variables x_1, x_2. If $p \in A$ then p can be written as $p = \sum_{k=0}^{n} p_k$, where $n = \deg p$ and the p_k's are homogeneous polynomials of degree k.

For a homogeneous polynomial

$$p_k = \sum_{i_1,\dots,i_k=1}^{2} c_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

with complex coefficients $c_{i_1,\dots,i_k}$ we denote by

$$\|p_k\| = \sum_{i_1,\dots,i_k=1}^{2} |c_{i_1,\dots,i_k}|.$$

Lemma 1. Let $\{\alpha_k\}_{k=0}^{\infty}$ be a sequence of positive numbers, $1 \leq \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$. Then there exists a sequence $\{\beta_k\}_{k=0}^{\infty}$, $1 \leq \beta_0 \leq \beta_1 \leq \cdots$ such that $\beta_k \geq \alpha_{2k} \quad (k = 1,2,\dots)$ and

$$\sum_{k=0}^{\infty} \beta_k \|(pq)_k\| \geq \left(\sum_{i=0}^{\infty} \alpha_i \|p_i\|\right) \cdot \left(\sum_{j=0}^{\infty} \alpha_j \|q_j\|\right) \tag{5}$$

for all polynomials $p, q \in A$.

Proof. Set $\beta_0 = 2\alpha_0^2$ and define inductively

$$\beta_k = \max\Big\{\alpha_{2k}, 2^{2k+3}\alpha_k^2\big(2\alpha_k^2 + k\beta_{k-1}^2\big)^2\Big\} \qquad (k = 1,2,\dots).$$

We prove by induction on n that

$$\sum_{k=0}^{n} \beta_k \|(pq)_k\| \geq \left(1 + \frac{1}{2^n}\right)\left(\sum_{i=0}^{\deg p} \alpha_i \|p_i\|\right) \cdot \left(\sum_{j=0}^{\deg q} \alpha_j \|q_j\|\right) \tag{6}$$

for all polynomials $p, q \in A$ with $\deg p + \deg q = n$ and $\sum_{i=0}^{\deg p} \|p_i\| = 1 = \sum_{j=0}^{\deg q} \|q_j\|$. This will clearly imply (5).

Suppose (6) is true for all polynomials $p, q \in A$ with $\deg p + \deg q \leq n$ and

$$\sum_{i=0}^{\deg p} \|p_i\| = 1 = \sum_{j=0}^{\deg q} \|q_j\|.$$

This implies that (6) is true for all polynomials $p, q \in A$ with $\deg p + \deg q \le n$ and either $\sum_{i=0}^{\deg p} \|p_i\| = 1$ or $\sum_{j=0}^{\deg q} \|q_j\| = 1$.

Let $p = \sum_{i=0}^{l} p_i, q = \sum_{j=0}^{m} q_j$ be polynomials where $l = \deg p$, $m = \deg q$, $l + m = n + 1$ and $\sum_{i=0}^{l} \|p_i\| = 1 = \sum_{j=0}^{m} \|q_j\|$. Set

$$\varepsilon = \frac{1}{2^{n+1}\big(2\alpha_{n+1}^2 + (n+1)\beta_n\big)}.$$

We distinguish three cases:

1) Let $\|p_l\| < \varepsilon$. Denote by $\tilde{p} = \sum_{i=0}^{l-1} p_i$. Then

$$(pq)_k = (\tilde{p}q)_k \qquad (k \le l-1)$$

and

$$(pq)_k = (\tilde{p}q)_k + p_l q_{k-l} \qquad (k \ge l),$$

so that

$$\|(\tilde{p}q)_k\| \le (pq)_k + \|p_l\| \cdot \|q_{k-l}\| \le \|(pq)_k\| + \varepsilon \qquad (k = 0, 1, \dots, n).$$

Further

$$\sum_{k=0}^{n} \beta_k \|(\tilde{p}q)_k\| \ge \Big(1 + \frac{1}{2^n}\Big)\Big(\sum_{i=0}^{l-1} \alpha_i \|p_i\|\Big) \cdot \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big)$$

by the induction assumption. Thus we have

$$\sum_{k=0}^{n+1} \beta_k \|(pq)_k\| \ge \sum_{k=0}^{n} \beta_k (\|(\tilde{p}q)_k\| - \varepsilon)$$

$$\ge \Big(1 + \frac{1}{2^n}\Big)\Big(\sum_{i=0}^{l-1} \alpha_i \|p_i\|\Big) \cdot \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big) - \varepsilon \sum_{k=0}^{n} \beta_k$$

$$\ge \Big(1 + \frac{1}{2^n}\Big)\Big(\sum_{i=0}^{l} \alpha_i \|p_i\|\Big)\Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big) - \Big(1 + \frac{1}{2^n}\Big)\alpha_l \|p_l\| \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big) - \varepsilon(n+1)\beta_n$$

$$\ge \Big(1 + \frac{1}{2^{n+1}}\Big)\Big(\sum_{i=0}^{l} \alpha_i \|p_i\|\Big) \cdot \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big) + \frac{1}{2^{n+1}} - 2\alpha_{n+1}^2 \varepsilon - \varepsilon(n+1)\beta_n$$

$$\ge \Big(1 + \frac{1}{2^{n+1}}\Big)\Big(\sum_{i=0}^{l} \alpha_i \|p_i\|\Big) \cdot \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big).$$

2) If $\|q_m\| < \varepsilon$ then we can get (6) analogously.

3) Suppose $\|p_l\| \ge \varepsilon, \|q_m\| \ge \varepsilon$. Then

$$\sum_{k=0}^{n+1} \beta_k \|(pq)_k\| \ge \beta_{n+1} \|(pq)_{n+1}\| \ge \beta_{n+1} \varepsilon^2$$

$$\ge 2\alpha_{n+1}^2 \ge \Big(1 + \frac{1}{2^{n+1}}\Big)\Big(\sum_{i=0}^{l} \alpha_i \|p_i\|\Big) \cdot \Big(\sum_{j=0}^{m} \alpha_j \|q_j\|\Big).$$

Thus we have proved (6) for all polynomials $p, q \in A$ with $\sum_{i=0}^{\infty} \|p_i\| = \sum_{j=0}^{\infty} \|q_j\| = 1$ and hence (5).

Theorem 2. There exists a non-commutative B_0-algebra without generalized topological divisors of zero.

Proof. Consider the algebra A from the previous lemma. Set $\alpha_{1,k} = 1 \quad (k = 0, 1, \ldots)$ and find positive numbers $\alpha_{n,k} \quad (n = 2, 3, \ldots,\ k = 0, 1, \ldots)$ inductively on n such that $1 \leq \alpha_{n,0} \leq \alpha_{n,1} \leq \cdots,\ \alpha_{n+1,k} \geq \alpha_{n,2k}$ and

$$\sum_{k=0}^{\infty} \alpha_{n+1,k} \|(pq)_k\| \geq \left(\sum_{i=0}^{\infty} \alpha_{n,i} \|p_i\| \right) \cdot \left(\sum_{j=0}^{\infty} \alpha_{n,j} \|q_j\| \right)$$

for all polynomials $p, q \in A$ and for $n = 1, 2, \ldots$. Define seminorms $\| \cdot \|_n$ on $A \quad (n = 1, 2, \ldots)$ by

$$\|p\|_n = \sum_{i=0}^{\infty} \alpha_{n,i} \|p_i\| \qquad (p \in A).$$

Then we have

$$\|pq\|_{n+1} \geq \|p\|_n \cdot \|q\|_n \qquad (p, q \in A).$$

Further, for $i \leq j$, we have

$$\alpha_{n+1,i} \cdot \alpha_{n+1,j} \geq \alpha_{n+1,j} \geq \alpha_{n,2j} \geq \alpha_{n,i+j}$$

and thus

$$\|pq\|_n \leq \|p\|_{n+1} \cdot \|q\|_{n+1} \qquad (p, q \in A).$$

Thus the completion of A with the topology given by the seminorms $\|\cdot\|_n$ is a B_0-algebra without generalized topological divisors of zero.

Theorem 3. There exists a non-commutative B_0-algebra C such that $x_\alpha y_\alpha \to 0$ if and only if $y_\alpha x_\alpha \to 0$ for all pairs of nets $(x_\alpha), (y_\alpha)$ of elements of C.

Proof. Let C be the algebra from the previous example. Suppose $(x_\alpha), (y_\alpha)$ are nets of elements of C and $x_\alpha y_\alpha \to 0$. Then, for $n = 1, 2, \ldots$, we have

$$\|y_\alpha x_\alpha\|_n \leq \|y_\alpha\|_{n+1} \cdot \|x_\alpha\|_{n+1} \leq \|x_\alpha y_\alpha\|_{n+2} \to 0,$$

hence $y_\alpha x_\alpha \to 0$.

References

[1] H. Arizmendi, W. Żelazko, A B_0-algebra without generalized topological divisors of zero, Studia Math. 82 (1985), 191–198.

[2] C. Le Page, Sur quelques conditions entraînant la commutativité dans les algèbres de Banach, C. R. Acad. Sci. Paris 265 (1967), 235–237.

[3] W. Żelazko, Metric generalizations of Banach algebras, Dissertationes Math. (Rozprawy Mat.) 47 (1965).

[4] W. Żelazko, On generalized topological divisors of zero in m-convex locally convex algebras, Studia Math. 28 (1966), 9–16.

[5] W. Żelazko, On generalized topological divisors of zero, Studia Math. 85 (1987), 137–148.

Instituto de Matemáticas
Universidad Nacional Autonoma de México
Cuidad Universitaria, Mexico D.F. 04510
México

Institute of Mathematics
Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1
Czech Republic