Duality in quotient Banach spaces

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Let X be a Banach space. A linear manifold $M \subset X$ is called paraclosed if there exists a Banach space X_0 and a bounded operator $a : X_0 \to X$ such that $M = aX_0$. Clearly we can assume that the operator a is injective. Indeed, let $X_0' = X_0/\ker a$ and let $a' : X_0' \to X$ be the operator induced by a. Then a' is injective and $a'X_0' = aX_0 = M$. Equivalently, $M \subset X$ is paraclosed if there is a Banach space topology on M which is stronger then the topology inherited from X (such a topology is necessarily unique by the closed graph theorem).

For basic properties of paraclosed subspaces (called sometimes also operator ranges) see e.g. [2], [3], [4], [6]. Recall that intersection or sum of two paraclosed subspaces is again paraclosed. Although paraclosed subspaces exhibit many properties of closed subspaces, the lattice of all paraclosed subspaces of X is much richer than the lattice of all closed subspaces. For example, each bounded operator $S : X \to X$ has a nontrivial invariant paraclosed subspace.

Let X be a Banach space and $M \subset X$ a paraclosed subspace. Then the quotient X/M is no longer a topological vector space. However, the operators acting in X/M have still nice spectral properties. Since the results for operators in quotient Banach spaces have interesting consequences even in the spectral theory of bounded operators in Banach spaces, the quotients of Banach spaces (or of other more general types of spaces) attracted attention of several authors, see e.g. [8], [9], [10], [11], [13]. The spectrum of a morphism on a quotient Banach space X/aX_0 is always nonempty provided that $X \neq aX_0$ [11]. The joint spectrum and the corresponding analytic functional calculus for n-tuples of commuting morphisms were also defined [11], [12].

An important tool in the theory of operators in Banach spaces is the duality. The present paper is an attempt to introduce the duality for operators in quotient Banach spaces. The attempt is successful in an important partial case — for quotient Banach spaces X/M where M is a dense paraclosed subspace of X. Note that in the other extremal case — if M is a closed subspace of X — the quotient X/M is again a Banach space so that the duality theory is well-known.

We show below that each operator acting in X/M induces canonically two operators acting in $X/\overline{M}$ and $\overline{M}/M$, respectively. Thus these two extremal cases give information also about the general case.

The duality theory for operators in quotient Hilbert spaces was recently presented by Terescenco [5], [7].

All Banach spaces considered here are either real or complex. By an operator between two Banach spaces we mean always a bounded linear operator.

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The basic definitions and results concerning quotient Banach spaces can be found in [10], [1]. By a quotient Banach space we mean a linear vector space X/aX_0 where X and X_0 are Banach spaces and $a : X_0 \to X$ is an injective operator.

Let X/aX_0 and Y/bY_0 be quotient Banach spaces. A linear mapping $\phi : X/aX_0 \to Y/bY_0$ is called a morphism (in the category of quotient Banach spaces) if its lifted graph

$$G_0(\phi) = \{(x, y) \in X \oplus Y : \phi(x + aX_0) = y + bY_0\}$$

is a paraclosed subspace of $X \oplus Y$. Recall that the sum, as well as the composition of morphisms is also a morphism whenever these operations make sense, see for instance [1], Lemma I.9.3. A morphism that is injective and onto is called an isomorphism. By [10], the inverse of an isomorphism is also an isomorphism.

By a densely defined quotient Banach space we mean the quotient Banach space X/aX_0 such that aX_0 is dense in X.

Let X/aX_0 and Y/bY_0 be quotient Banach spaces and let $T : X \to Y$ be a bounded operator satisfying $T(aX_0) \subset bY_0$. Then T defines a morphism $\widehat{T} : X/aX_0 \to Y/bY_0$ by $\widehat{T}(x + aX_0) = Tx + bY_0$. Morphisms $\widehat{T}$ that arise in this way are called strict.

Set $T_0 = b^{-1}Ta : X_0 \to Y_0$. An easy application of the closed graph theorem implies that T_0 is a bounded operator. The following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle b} \\ X_0 & \xrightarrow{T_0} & Y_0 \end{array}$$

Clearly $\widehat{T} = 0$ if and only if $TX \subset bY_0$.

Let X/aX_0 be a densely defined quotient Banach space. Then $a^* : X^* \to X_0^*$ is also an injective operator. Define the dual of X/aX_0 by

$$(X/aX_0)^{\#} = X_0^*/a^*X^*. \tag{1}$$

Since a^*X^* is an operator range, it is paraclosed. Hence $(X/aX_0)^{\#}$ is also a quotient Banach space.

The duality defined by (1) has properties analogous to the duality of Banach spaces.

Let X/aX_0 and Y/bY_0 be quotient Banach spaces. Define their direct sum by

$$X/aX_0 \oplus Y/bY_0 = (X \oplus Y)/(a \oplus b)(X_0 \oplus Y_0).$$

By induction it is possible to define direct sums of a finite number of quotient Banach spaces.

Clearly the direct sum of a finite number of densely defined quotient Banach spaces is again a densely defined quotient Banach space.

Proposition 1. Let X_i/a_iX_{i0} $(i = 1, \dots, n)$ be densely defined quotient Banach space. Then

$$\Big(\bigoplus_i (X_i/a_iX_{i0})\Big)^{\#} \equiv \bigoplus_i (X_i/a_iX_{i0})^{\#},$$

where $\equiv$ denotes the isomorphism of quotient Banach spaces.

Proof. We prove the statement for two densely defined quotient Banach spaces X/aX_0 and Y/bY_0. For n-tuples of quotient Banach spaces the proof remains without any change. We have

$$((X/aX_0)\oplus(Y/bY_0))^{\#} = ((X\oplus Y)/(a\oplus b)(X_0\oplus Y_0))^{\#} = (X_0\oplus Y_0)^*/(a\oplus b)^*(X\times Y)^* \equiv$$

$$(X_0^*/a^*X^*) \oplus (Y_0^*/b^*Y^*) = (X/aX_0)^{\#} \oplus (Y/bY_0)^{\#}.$$

Proposition 2. Let X/aX_0 be a quotient Banach space. Let $J : X \to X^{**}$ and $J_0 : X_0 \to X_0^{**}$ be the canonical embeddings. Then
(i) $Ja = a^{**}J_0$,
(ii) $JaX_0 \subset a^{**}X_0^{**}$ so that $\widehat{J} : X/aX_0 \to X^{**}/a^{**}X_0^{**}$ is a strict morphism,
(iii) if X_0 is reflexive, then $\widehat{J}$ is injective,
(iv) if X is reflexive then $\widehat{J}$ is onto.

Proof.
(i) Let $x_0 \in X_0$ and $f \in X^*$. Then

$$\langle Jax_0, f\rangle = \langle ax_0, f\rangle = \langle x_0, a^*f\rangle = \langle J_0x_0, a^*f\rangle = \langle a^{**}J_0x_0, f\rangle.$$

Thus $Ja = a^{**}J_0$.
(ii) By (i), $JaX_0 \subset a^{**}X_0^{**}$.
(iii) Let X_0 be reflexive so that J_0 is onto. Let $x \in X$ and $Jx \in a^{**}X_0^{**}$, i.e., $Jx = a^{**}J_0x_0$ for some $x_0 \in X_0$. Then $Jx = Jax_0$ and $x = ax_0 \in aX_0$ since J is injective. Hence $\widehat{J} : X/aX_0 \to X^{**}/a^{**}X_0^{**}$ is injective.
(iv) Let X be reflexive so that J is onto. Then $\widehat{J}$ is also onto.

Our goal is to define the duality for morphisms between densely defined quotient Banach spaces. We start with strict morphisms.

Let X/aX_0 and Y/bY_0 be densely defined quotient Banach spaces, let $T : X \to Y$ be a bounded operator satisfying $T(aX_0) \subset bY_0$. Set $T_0 = b^{-1}Ta : X_0 \to Y_0$. Then the following diagram commutes

$$\begin{array}{ccc} X^* & \xleftarrow{T^*} & Y^* \\ \downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle b^*} \\ X_0^* & \xleftarrow{T_0^*} & Y_0^* \end{array}$$

so that $T_0^*(b^*Y^*) \subset a^*X^*$. Thus T_0^* defines a strict morphism $\widehat{T_0^*} : Y_0^*/b^*Y^* \to X_0^*/a^*X^*$. Define the dual of the strict morphism $\widehat{T} : X/aX_0 \to Y/bY_0$ by

$$(\widehat{T})^{\#} = \widehat{T_0^*} : (Y/bY_0)^{\#} \to (X/aX_0)^{\#}. \tag{2}$$

The duality between $\widehat{T}$ and $(\widehat{T})^{\#}$ has the expected properties. The next theorem shows also that the definition of the adjoint is correct.

Theorem 3. Let X/aX_0, Y/bY_0 and Z/cZ_0 be densely defined quotient Banach spaces, let $T, S : X \to Y$ and $U : Y \to Z$ be operators satisfying $T(aX_0) \subset bY_0$, $S(aX_0) \subset bY_0$ and $U(bY_0) \subset cZ_0$. Then

(i) if $\widehat{T} = 0$ then $\widehat{T}^{\#} = 0$,

(ii) $(\widehat{T} + \widehat{S})^{\#} = \widehat{T}^{\#} + \widehat{S}^{\#}$,

(iii) if $\alpha \in \mathbf{C}$ then $(\alpha\widehat{T})^{\#} = \alpha\widehat{T}^{\#}$,

(iv) $(\widehat{S}\widehat{T})^{\#} = \widehat{T}^{\#}\widehat{S}^{\#}$.

Proof. (i) Let $\widehat{T} = 0$ so that $TX \subset bY_0$. Define $R : X \to Y_0$ by $Rx = b^{-1}Tx \quad (x \in X)$. We show by the closed graph theorem that R is a bounded operator. Indeed, let $x_n \in X, x_n \to 0$ and $Rx_n \to y_0$. Then $Tx_n \to 0$ and in the same time $Tx_n = bRx_n \to by_0$. Since b is injective, we have $y_0 = 0$ and R is continuous.

Further $T_0 = Ra$ and $T_0^* = a^*R^*$ so that $T_0^*Y_0^* \subset a^*X_0^*$ and $\widehat{T}^{\#} = \widehat{T_0^*} = 0$. Thus the definition of $\widehat{T}^{\#}$ is correct.

The remaining statements are clear.

Let X/aX_0 and Y/bY_0 be densely defined quotient Banach spaces, let $T : X \to Y$ be an operator satisfying $T(aX_0) \subset bY_0$. Define operators $e_1 : X_0 \to X \oplus Y_0$ and $e_2 : X \oplus Y_0 \to Y$ by $e_1x_0 = dx_0 \oplus T_0x_0 \quad (x_0 \in X_0)$ and $e_2(x \oplus y_0) = Tx - dy_0 \quad (x \in X, y_0 \in Y_0)$, respectively. Clearly $e_2e_1 = 0$, e_1 is injective and e_2 has a dense range. The properties of the strict morphism $\widehat{T}$ are closely connected with the exactness properties of the complex

$$0 \longrightarrow X_0 \xrightarrow{e_1} X \oplus Y_0 \xrightarrow{e_2} Y \longrightarrow 0.$$

Note that the complex corresponding to the adjoint $\widehat{T}^{\#}$ is

$$0 \longrightarrow Y^* \xrightarrow{e_2^*} Y_0^* \oplus X^* \xrightarrow{e_1^*} X_0^* \longrightarrow 0.$$

Proposition 4.

(i) $\widehat{T}$ is onto $\Leftrightarrow$ e_2 is onto $\Leftrightarrow$ e_2 has closed range,

(ii) $\widehat{T}$ is injective $\Leftrightarrow \ker e_2 = \operatorname{Im} e_1 \Rightarrow \operatorname{Im} e_1$ is closed,

(iii) $\widehat{T}^{\#}$ is onto $\Leftrightarrow$ e_1^* is onto $\Leftrightarrow$ e_1 is bounded below $\Leftrightarrow \operatorname{Im} e_1$ is closed,

(iv) $\widehat{T}^{\#}$ is injective $\Leftrightarrow \ker e_1^* = \operatorname{Im} e_2^*$,

(v) if $\widehat{T}$ is injective then $\widehat{T}^{\#}$ is onto,

(vi) if $\widehat{T}^{\#}$ is injective then $\widehat{T}$ is onto,

(vii) $\widehat{T}^{\#}$ is an isomorphism $\Leftrightarrow$ $\widehat{T}$ is an isomorphism.

Proof. Straightforward.

Our aim now is to define the duality for non-strict morphisms.

Recall that a quotient Banach space X/aX_0 is called standard if X is an ℓ_1-space (over some set A). By [10], each morphism from a standard quotient Banach space to any quotient Banach space is strict. Further each quotient Banach space is isomorphic to a standard one. We repeat the proof of this fact $\big($(i)–(iv) of the following theorem$\big)$ since we need a little more detailed information.

Theorem 5. Let X/aX_0 be a quotient Banach space. Then there exists a standard quotient Banach space Z/cZ_0 and an operator $Q : Z \to X$ with the following properties:
(i) $QZ = X$ and $Q(cZ_0) \subset aX_0$. Thus $Q_0 = a^{-1}Qc : Z_0 \to X_0$ is a bounded operator,
(ii) $\widehat{Q} : Z/cZ_0 \to X/aX_0$ is a strict isomorphism,
(iii) $\ker Q \subset cZ_0$,
(iv) Q_0 is onto so that $Q(cZ_0) = aX_0$,
(v) $Q(\overline{cZ_0}) = \overline{aX_0}$.

Proof. (i) Let S_X be the unit sphere in X and let $Z = \ell_1(S_X)$. Let $Q : Z \to X$ be the canonically defined surjective operator.

Set $Z_0 = \{x_0 \oplus z \in X_0 \oplus Z : dx_0 = Qz\}$. Define $c : Z_0 \to Z$ by $c(x_0 \oplus z) = z$. Then c is injective. Indeed, suppose that $x_0 \oplus z \in Z_0$ and $c(x_0 \oplus z) = z = 0$. We have $ax_0 = Qz = 0$ so that $x_0 = 0$. Thus Z/cZ_0 is a standard quotient Banach space. Further $Q(cZ_0) \subset aX_0$. Indeed, let $x_0 \oplus z \in Z_0$ so that $ax_0 = Qz$. We have $Qc(x_0 \oplus z) = Qz = ax_0 \in aX_0$. Thus $\widehat{Q} : Z/cZ_0 \to X/aX_0$ is a strict morphism.

(ii) Since Q is onto, $\widehat{Q}$ is also onto. Suppose that $z \in Z$ and $Qz \in aX_0$. Then there exists $x_0 \in X_0$ such that $Qz = ax_0$ and $x_0 \oplus z \in Z_0$. Hence $z = c(x_0 \oplus z) \in cZ_0$ and $\widehat{Q}$ is injective by Proposition 4. Hence $\widehat{Q}$ is a strict isomorphism.

(iii) Let $z \in \ker Q$. Then $0 \oplus z \in Z_0$ so that $z = c(0 \oplus z) \in cZ_0$.

(iv) Let $x_0 \in X_0$. Since $Q : Z \to X$ is onto, there is $z \in Z$ such that $ax_0 = Qz$. Consequently $x_0 \oplus z \in Z_0$ and $ax_0 = Qz = Qc(x_0 \oplus z) \in Q(cZ_0)$. Hence $Q_0 : Z_0 \to X_0$ is onto.

(v) Since $Q(cZ_0) \subset aX_0$ and Q is continuous, we have also $Q(\overline{cZ_0}) \subset \overline{aX_0}$.

Conversely, let $x \in \overline{aX_0}$. Then there exists a sequence $x_n \in X_0$ such that $ax_n \to x$. For each n there is $z_n \in Z$ such that $ax_n = Qz_n$ and $z \in Z$ such that $Qz = x$. Thus $x_n \oplus z_n \in Z_0$, so that $z_n = c(x_n \oplus z_n) \in cZ_0$ and $Qz_n = ax_n \to x = Qz$. By the open mapping theorem there are vectors $u_n \in Z$ with $Qu_n = Q(z - z_n)$ and $u_n \to 0$. Thus $v_n = z_n - z - u_n \in \ker Q \subset cZ_0$ and $z - (z_n - v_n) = -u_n \to 0$ so that $z_n - v_n \to z \in \overline{cZ_0}$.

We define now adjoints to non-strict morphisms in densely defined quotient Banach spaces.

Let X/aX_0 and Y/bY_0 be densely defined quotient Banach spaces, let $\phi : X/aX_0 \to Y/bY_0$ be a morphism. By Theorem 5 there exist a canonically constructed standard quotient Banach space Z/cZ_0 and a surjective operator $Q : Z \to X$ such that $\widehat{Q} : Z/cZ_0 \to X/aX_0$ is a strict isomorphism. Thus $\phi\widehat{Q} : Z/cZ_0 \to Y/bY_0$ is strict and we define $\phi^\# : (Y/bY_0)^\# \to (X/aX_0)^\#$ by

$$\phi^\# = (\widehat{Q}^\#)^{-1} \cdot (\phi\widehat{Q})^\#. \tag{3}$$

Clearly $\phi^\#$ defined by (3) is a morphism. For strict morphisms the definition coincides with the original one. Indeed, let ϕ be strict, i.e., $\phi = \widehat{T}$ for some operator $T : X \to Y$. Then $\phi\widehat{Q} = \widehat{TQ}$ and the following diagram commutes

$$\begin{array}{ccccc} Z & \xrightarrow{Q} & X & \xrightarrow{T} & Y \\ \uparrow c & & \uparrow a & & \uparrow b \\ Z_0 & \xrightarrow{Q_0} & X_0 & \xrightarrow{T_0} & Y_0 \end{array}$$

We have $(\phi\widehat{Q})^{\#} = \widehat{Q_0^* T_0^*} = \widehat{Q_0^*}\widehat{T_0^*} = \widehat{Q}^{\#}\phi^{\#}$ and $\phi^{\#} = (\widehat{Q}^{\#})^{-1} \cdot (\phi\widehat{Q})^{\#}$. Thus the definition of the adjoint coincides with the definition for strict morphisms.

The properties of adjoints of strict morphisms can be easily generalized for non-strict morphisms.

Theorem 6. Let X/aX_0 and Y/bY_0 be densely defined quotient Banach spaces, let $\phi : X/aX_0 \to Y/bY_0$ be a morphism. Then

(i) if $\phi = 0$ then $\phi^{\#} = 0$,
(ii) if $\psi : X/aX_0 \to Y/bY_0$ is another morphism then $(\phi + \psi)^{\#} = \phi^{\#} + \psi^{\#}$,
(iii) if $\alpha \in \mathbf{C}$ then $(\alpha\phi)^{\#} = \alpha\phi^{\#}$,
(iv) if Z/cZ_0 is a densely defined quotient Banach space and $\psi : Y/bY_0 \to Z/cZ_0$ a morphism then $(\psi\phi)^{\#} = \phi^{\#}\psi^{\#}$,
(v) if ϕ is injective then $\phi^{\#}$ is onto; if $\phi^{\#}$ is injective then ϕ is onto,
(vi) $\phi^{\#}$ is an isomorphism $\Leftrightarrow$ ϕ is an isomorphism,
(vii) if ϕ is an isomorphism, then $(\phi^{-1})^{\#} = (\phi^{\#})^{-1}$.

Theorem 7. Let X/aX_0 and Y/bY_0 be quotient Banach spaces, let $\phi : X/aX_0 \to Y/bY_0$ be a morphism. Let $x \in \overline{aX_0}$. Then there exists $y \in \overline{bY_0}$ such that $\phi(x+aX_0) = y+bY_0$. In other words, the restriction of ϕ to $\overline{aX_0}/aX_0$ is a morphism from $\overline{aX_0}/aX_0$ to $\overline{bY_0}/bY_0$.

Proof. By Theorem 5, there is a standard quotient Banach space Z/cZ_0 and a strict morphism $\widehat{Q} : Z/cZ_0 \to X/aX_0$ (defined by an operator $Q : Z \to X$). Further $\phi\widehat{Q} : Z/cZ_0 \to Y/bY_0$ is strict so that $\phi\widehat{Q}$ is defined by an operator $S : Z \to Y$.

Let $x \in \overline{aX_0}$. By Theorem 5 (iv) there is $z \in \overline{cZ_0}$ such that $Qz = x$. Set $y = Sz$. We have

$$\phi(x + aX_0) = \phi(Qz + aX_0) = \phi\widehat{Q}(z + cZ_0) = Sz + cZ_0 = y + cZ_0.$$

Further $y = Sz \in \overline{bY_0}$.

Corollary 8. Let X/aX_0 and Y/bY_0 be quotient Banach spaces, let $\phi : X/aX_0 \to Y/bY_0$ be a morphism. Then ϕ induces an operator $T : X/\overline{aX_0} \to Y/\overline{bY_0}$ defined by $T(x + \overline{aX_0}) = y + \overline{bY_0}$ where $y + bY_0 = \phi(x + aX_0)$.

Theorem 9. Let X_j/a_jX_{j0}, $j = 1, \ldots, n$ and Y_i/b_iY_{i0}, $i = 1, \ldots, m$ be densely defined quotient Banach spaces. Let

$$\phi_{i,j} : X_j/a_jX_{j0} \to Y_i/b_iY_{i0},\ i = 1, \ldots, m,\ j = 1, \ldots, n$$

be morphisms. Define the linear mapping

$$\phi := \left[\phi_{i,j}\right]_{i,j} : \bigoplus_j (X_j/a_jX_{j0}) \to \bigoplus_i (Y_i/b_iY_{i0})$$

by $\phi(\widehat{x}_j) := \oplus_{i=1}^m \phi_{i,j}\widehat{x}_j \quad (x_j \in X_j, j = 1, \ldots, n)$, where $\widehat{x}$ denotes the class of x modulo the corresponding subspace. Then ϕ is a morphism and

$$\phi^{\#} = \left(\phi_{i,j}^{\#}\right)_{i,j}^t : \bigoplus_i (Y_{i0}^*/b_i^*Y_i^*) \to \bigoplus_j (X_{j0}^*/a_j^*X_j^*),$$

where $(\cdot)^t$ stands for the transposed matrix.

Proof. Write $X = \oplus_j X_j$ and $Y = \oplus_i Y_i$. For $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ denote by $\big[\psi^{(i,j)}\big]$ the $m \times n$ operator matrix whose entries at (i',j') are: $\phi_{i,j}$ if $(i',j') = (i,j)$ and zero otherwise.

We have

$$G_0(\big[\psi^{(i,j)}\big]) = G_0(\phi_{i,j}) + \Big(\bigoplus_{j' \neq j} X_{j'}\Big) \oplus \Big(\bigoplus_{i'=1}^{m} b_{i'} Y_{i'0}\Big)$$

where we identify $G_0(\phi_{i,j})$, $X_{j'}$ and $b_{i'}Y_{i'0}$ with the corresponding subspaces of $X \oplus Y$. Thus $G_0(\psi^{(i,j)})$ is a paraclosed subspace of $X \oplus Y$ and $\psi^{(i,j)}$ is a morphism.

Since $\phi = \sum_{i,j}\big[\psi^{(i,j)}\big]$ and the class of morphisms is closed under addition, ϕ is also a morphism.

To show the statement about the adjoints, let $\widehat{Q}_j$ and $\widehat{P}_i$ denote the strict morphisms that represent X_j/a_jX_{j0}, $(Y_i/b_iY_{i0}$, respectively) on standard quotient Banach spaces as stated in Theorem 5. Let $\theta_{i,j} := P_i^{-1}\phi_{i,j}Q_j$ and $\theta := \big[\theta_{i,j}\big]_{i,j} = P^{-1}\phi Q$, where $Q := \oplus_j Q_j$ and $P := \oplus_i P_i$. Then the desired conclusion $\phi^{\#} = \big[\phi_{i,j}^{\#}\big]_{i,j}^t$ is equivalent to the equality $\theta^{\#} = \big[\theta_{i,j}^{\#}\big]_{i,j}^t$. Thus we may suppose that all X_j and Y_i are standard. Hence we can assume that $\phi_{i,j} = \widehat{T}_{i,j}$ are strict morphisms. Set $T_0 := \big[T_{i,j,0}\big]_{i,j}$, where $T_{i,j,0} : a_jX_{j0} \to b_iY_{i0}$ is the restriction of $T_{i,j}$ to a_jX_{j0}.

Now the Banach space adjoint of the operator matrix T_0 is the transposed matrix of the adjoints $\big[T_{i,j,0}^*\big]_{j,i}$. By factoring through the subspaces $\oplus_j a_jX_{j0}$ and $\oplus_i b_i^*Y_i^*$ respectively, we obtain the equality in the statement.

Example 10. Let $B(H)$ be the set of all operators acting on a separable Hilbert space H. Denote by K the Banach space of all compact operators endowed with the uniform norm and by C_1 the space of all trace class operators on H endowed with the trace norm. Let $a : C_1 \to K$ be the natural inclusion. Then K/aC_1 is a densely defined quotient Banach space.

Further C_1^* can be identified with $B(H)$ and K^* with C_1; in both cases the duality is given by the formula $\langle T, S\rangle = \operatorname{tr} TS$. Thus $(K/aC_1)^{\#} \equiv B(H)/a^*C_1$ where $a^* : C_1 \to B(H)$ is also the natural inclusion.

For $A \in B(H)$ denote by L_A and R_A the operators of left (right) multiplication on $B(H)$ defined by $L_AS = AS$, $R_AS = SA$ $\big(S \in B(H)\big)$. Clearly L_A leaves invariant both K and C_1 so that the restriction $L_A|K$ defines a strict morphism acting on K/C_1. It is easy to verify that $\widehat{L_A|K}^{\#} = R_A$. Similarly $\widehat{R_A|K}^{\#} = L_A$.

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