On smooth local resolvents

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Abstract. We exhibit an example of a bounded linear operator on a Banach space which admits an everywhere defined local resolvent with continuous derivatives of all orders.

Let T be a bounded linear operator acting on a complex Banach space X. It is well known that the resolvent $z \mapsto (T-z)^{-1}$ defined on the complement of the spectrum $\sigma(T)$ is unbounded. More precisely, $\|(T-z)^{-1}\| \to \infty$ whenever z approaches the spectrum $\sigma(T)$.

Let $x \in X$ be a nonzero vector. By a local resolvent of T at x we mean a function $f : U \to X$ defined on a set $U \supset \mathbb{C} \setminus \sigma(T)$ such that $(T-z)f(z) = x \quad (z \in U)$. Clearly the local resolvent is uniquely determined for all $z \in \mathbb{C} \setminus \sigma(T)$ and $f(z) = (T-z)^{-1}x$. Thus any local resolvent is analytic on the complement of the spectrum.

It was observed in [G] that a local resolvent can be bounded. Bounded local resolvents were further studied in [BG], [N], [BM] and it was shown that they are rather frequent. In [BM], an example of a continuous everywhere defined local resolvent was given (such a local resolvent is necessarily bounded since each local resolvent vanishes for $z \to \infty$).

The aim of this note is to exhibit an example of an everywhere defined C^∞ local resolvent. Note that by a basic result of local spectral theory there are no analytic everywhere defined local resolvents.

Denote by $\mathbb{C}, \mathbb{R}$ and $\mathbb{Z}_+$ the sets of all complex numbers, real numbers and non-negative integers, respectively.

Let X be a complex Banach space and $f : \mathbb{C} \to X$ a function. As usually we identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider f to be a function of two real variables x and y. We say that f is a C^∞-function if it has continuous derivatives $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}$ of all orders.

Theorem 1. There exist a Banach space X, an operator $T \in B(X)$, a nonzero vector $x \in X$ and a C^∞-function $f : \mathbb{C} \to X$ such that

$$(T-z)f(z) = x \qquad (z \in \mathbb{C}).$$

Proof. Let $\varphi : \mathbb{C} \to \langle 0, 1\rangle$ be a C^∞-function such that

$$\varphi(z) = \begin{cases} 1 & (|z| < 1/3), \\ 0 & (|z| > 2/3). \end{cases}$$

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Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = \frac{\varphi(z)-1}{z} \quad (z \neq 0)$, $g(0) = 0$. Clearly g is a C^∞-function and $g(z) = -\frac{1}{z}$ for $|z| > 2/3$.

Write $z = x + iy$ and let $g_{kl} = \frac{\partial^{k+l} g}{\partial x^k \partial y^l}$; formally we write $g_{00} = g$. It is easy to show by induction that

$$g_{kl}(z) = \Big(\frac{-1}{z}\Big)^{k+l+1} (k+l)!\, i^l$$

for all z, $|z| > 2/3$ and $k, l \in \mathbb{Z}_+$. In particular, all derivatives g_{kl} are bounded functions. For $n = 0, 1, \dots$ choose positive constants K_n such that $K_n \geq nK_{n-1} \quad (n \geq 1)$ and $\max\{|g_{kl}(z)| : z \in \mathbb{C}\} \leq K_{k+l}$ for all $k, l \geq 0$. Clearly $K_0 \geq 1$ and $K_n \geq n!$ for all n.

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc in the complex plane.

Let X_0 be the vector space with a Hamel basis formed by the vectors u and $u_\alpha^{kl} \quad (k, l \in \mathbb{Z}_+, \alpha \in \mathbb{D})$.

For $|\eta| = 1$ write for short $u_\eta^{kl} = \Big(-\frac{1}{\eta}\Big)^{k+l+1} (k+l)! i^l u$. Note that $u_\eta^{kl} = g_{kl}(\eta) u$.

Let $M \subset X_0$ be the subset formed by the following elements:

$$u,$$
$$\frac{1}{K_{k+l}}\, u_\alpha^{kl} \qquad (\alpha \in \mathbb{D}, k, l \in \mathbb{Z}_+),$$
$$\frac{1}{K_2(1-|\alpha|)^2}\,(\alpha u_\alpha^{00} + u) \qquad (\alpha \in \mathbb{D}, 2/3 < |\alpha|),$$
$$\frac{1}{t^2 K_{k+l+2}}\,\big(u_{\alpha+t}^{kl} - u_\alpha^{kl} - t u_\alpha^{k+1,l}\big) \qquad (\alpha \in \overline{\mathbb{D}}, k, l \in \mathbb{Z}_+, t \in \mathbb{R}, |t| < 1/3, \alpha + t \in \mathbb{D}),$$
$$\frac{1}{s^2 K_{k+l+2}}\,\big(u_{\alpha+is}^{kl} - u_\alpha^{kl} - s u_\alpha^{k,l+1}\big) \qquad (\alpha \in \overline{\mathbb{D}}, k, l \in \mathbb{Z}_+, s \in \mathbb{R}, |s| < 1/3, \alpha + is \in \mathbb{D}).$$

Let U be the absolutely convex hull of M. Clearly U is absorbing. Let $\|\cdot\|$ be the Minkowski seminorm determined by U, i.e., for $v \in X_0$ we have

$$\|v\| = \inf\Big\{\sum_{m \in M} |\gamma_m| : v = \sum_{m \in M} \gamma_m m\Big\},$$

where the coefficients γ_m are complex and only a finite number of them are nonzero.

We show first that $\|u\| = 1$. Clearly $\|u\| \leq 1$. Define the linear functional $h : X_0 \to \mathbb{C}$ by $h(u) = 1$ and $h(u_\alpha^{kl}) = g_{kl}(\alpha) \quad (\alpha \in \mathbb{D}, k, l \in \mathbb{Z}_+)$.

We show that $|h(v)| \leq \|v\|$ for all $v \in X_0$. To this end, it is sufficient to show that $|h(m)| \leq 1$ for all $m \in M$.

For $k, l \in \mathbb{Z}_+, \alpha \in \mathbb{D}$ we have $|h(u_\alpha^{kl})| = |g_{kl}(\alpha)| \leq K_{k+l}$, and for $\alpha \in \mathbb{D}, |\alpha| > 2/3$,

$$|h(\alpha u_\alpha^{00} + u)| = \alpha g(\alpha) + 1 = 0.$$

Further, for $k, l \in \mathbb{Z}_+, \alpha \in \overline{\mathbb{D}}, t, s \in \mathbb{R}, |t|, |s| < 1/3, \alpha + t, \alpha + is \in \mathbb{D}$ we have

$$\begin{aligned}\big|h(u_{\alpha+t}^{kl} - u_\alpha^{kl} - t u_\alpha^{k+1,l})\big| &= \big|g_{kl}(\alpha+t) - g_{kl}(\alpha) - t g_{k+1,l}(\alpha)\big| \\ &\leq t^2 \max\{|g_{k+2,l}(z)| : z \in \mathbb{C}\} \leq t^2 K_{k+l+2}\end{aligned}$$

and similarly,

$$\left|h(u^{kl}_{\alpha+is}-u^{kl}_\alpha-su^{k,l+1}_\alpha)\right|=\left|g_{kl}(\alpha+is)-g_{kl}(\alpha)-sg_{k,l+1}(\alpha)\right|\le s^2K_{k+l+2}.$$

Thus $|h(m)|\le 1$ for all $m\in M$ and consequently, $|h(v)|\le\|v\|$ for all $v\in X_0$. In particular, $\|u\|\ge|h(u)|=1$, and so $\|u\|=1$.

Define now the linear mapping $T_0:X_0\to X_0$ by

$$\begin{aligned}T_0u&=0,\\ T_0u^{00}_\alpha&=\alpha u^{00}_\alpha+u\qquad(\alpha\in\mathbb{D}),\\ T_0u^{kl}_\alpha&=\alpha u^{kl}_\alpha+ku^{k-1,l}_\alpha+ilu^{k,l-1}_\alpha\qquad(\alpha\in\mathbb{D},k,l\in\mathbb{Z}_+,k+l\ge 1)\end{aligned}$$

(here we set formally $u^{kl}_\alpha=0$ if either $k<0$ or $l<0$).

We show that $\|T_0v\|\le 4\|v\|$ for all $v\in X_0$. To this end, it is again sufficient to show that $\|T_0m\|\le 4$ for all $m\in M$.

We have

$$\|T_0u^{00}_\alpha\|=\|\alpha u^{00}_\alpha+u\|\le|\alpha|K_0+1\le 2K_0$$

and, for $k+l\ge 1$,

$$\|T_0u^{kl}_\alpha\|=\|\alpha u^{kl}_\alpha+ku^{k-1,l}_\alpha+ilu^{k,l-1}_\alpha\|\le|\alpha|K_{k+l}+kK_{k+l-1}+lK_{k+l-1}\le 2K_{k+l}.$$

For $2/3<|\alpha|<1$ we have

$$\|T_0(\alpha u^{00}_\alpha+u)\|=|\alpha|\cdot\|T_0u^{00}_\alpha\|\le\|\alpha u^{00}_\alpha+u\|.$$

Let $k,l\in\mathbb{Z}_+$, $\alpha\in\overline{\mathbb{D}}$, $t,s\in\mathbb{R}$, $|t|,|s|<1/3$, $\alpha+t\in\mathbb{D}$, $\alpha+is\in\mathbb{D}$. Then

$$\|u^{kl}_{\alpha+t}-u^{kl}_\alpha\|\le\left\|u^{kl}_{\alpha+t}-u^{kl}_\alpha-tu^{k+1,l}_\alpha\right\|+|t|\cdot\|u^{k+1,l}_\alpha\|\le t^2K_{k+l+2}+|t|\cdot K_{k+l+1}\quad(1)$$

and similarly,

$$\|u^{kl}_{\alpha+is}-u^{kl}_\alpha\|\le s^2K_{k+l+2}+|s|K_{k+l+1}.\quad(2)$$

For $\alpha\in\mathbb{D}$, $\alpha+t\in\mathbb{D}$ and $\alpha+is\in\mathbb{D}$ we have

$$\begin{aligned}&\left\|T_0(u^{00}_{\alpha+t}-u^{00}_\alpha-tu^{1,0}_\alpha)\right\|=\left\|(\alpha+t)u^{00}_{\alpha+t}-\alpha u^{00}_\alpha-\alpha tu^{1,0}_\alpha-tu^{00}_\alpha\right\|\\ \le&|\alpha|\cdot\left\|u^{00}_{\alpha+t}-u^{00}_\alpha-tu^{1,0}_\alpha\right\|+|t|\cdot\left\|u^{00}_{\alpha+t}-u^{00}_\alpha\right\|\le t^2K_2+|t|^3K_2+t^2K_1\le 3t^2K_2.\end{aligned}$$

Similarly,

$$\left\|T_0(u^{00}_{\alpha+is}-u^{00}_\alpha-su^{0,1}_\alpha)\right\|\le 3s^2K_2.$$

For $|\eta|=1$, $|t|<1/3$, $|\eta+t|<1$ we have

$$\left\|T_0(u^{00}_{\eta+t}-u^{00}_\eta-tu^{1,0}_\eta)\right\|=\|T_0u^{00}_{\eta+t}\|=\left\|(\eta+t)u^{00}_{\eta+t}+u\right\|\le(1-|\eta+t|)^2K_2\le t^2K_2$$

and analogous estimate holds for $\left\|T_0(u^{00}_{\eta+is}-u^{00}_\alpha-su^{0,1}_\alpha)\right\|$.

Let $k+l\geq 1$, $\alpha,\alpha+t,\alpha+is\in\mathbb{D}$. We have

$$\begin{aligned}
&\big\|T_0(u^{kl}_{\alpha+t}-u^{kl}_\alpha-tu^{k+1,l}_\alpha)\big\|\\
=&\big\|(\alpha+t)u^{kl}_{\alpha+t}+ku^{k-1,l}_{\alpha+t}+ilu^{k,l-1}_{\alpha+t}-\alpha u^{kl}_\alpha-ku^{k-1,l}_\alpha-ilu^{k,l-1}_\alpha\\
&\qquad\qquad-t\alpha u^{k+1,l}_\alpha-(k+1)tu^{k,l}_\alpha-itlu^{k+1,l-1}_\alpha\big\|\\
\leq&|\alpha|\cdot\big\|u^{kl}_{\alpha+t}-u^{kl}_\alpha-tu^{k+1,l}_\alpha\big\|+k\big\|u^{k-1,l}_{\alpha+t}-u^{k-1,l}_\alpha-tu^{k,l}_\alpha\big\|\\
&\qquad\qquad+l\big\|u^{k,l-1}_{\alpha+t}-u^{k,l-1}_\alpha-tu^{k+1,l-1}_\alpha\big\|+|t|\cdot\big\|u^{kl}_{\alpha+t}-u^{kl}_\alpha\big\|\\
\leq&t^2K_{k+l+2}+(k+l)t^2K_{k+l+1}+|t|^3K_{k+l+2}+t^2K_{k+l+1}\leq 4t^2K_{k+l+2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\big\|T_0(u^{kl}_{\alpha+is}-u^{kl}_\alpha-su^{k,l+1}_\alpha)\big\|\\
\leq&|\alpha|\cdot\big\|u^{kl}_{\alpha+is}-u^{kl}_\alpha-su^{k,l+1}_\alpha\big\|+k\big\|u^{k-1,l}_{\alpha+is}-u^{k-1,l}_\alpha-su^{k-1,l+1}_\alpha\big\|\\
&\qquad+l\big\|u^{k,l-1}_{\alpha+is}-u^{k,l-1}_\alpha-su^{k,l}_\alpha\big\|+|s|\cdot\big\|u^{kl}_{\alpha+is}-u^{kl}_\alpha\big\|\leq 4s^2K_{k+l+2}.
\end{aligned}$$

Let $k+l\geq 1$, $|\eta|=1$, $t\in\mathbb{R}$, $|t|<1/3$ and $\eta+t\in\mathbb{D}$. We have

$$\big\|u^{kl}_{\eta+t}-(u^{kl}_\eta+tu^{k+1,l}_\eta)\big\|\leq t^2K_{k+l+2}.$$

Recall that $u_\eta=\left(-\frac{1}{\eta}\right)^{k+l+1}(k+l)!i^lu$. Hence

$$\begin{aligned}
&\big\|T_0(u^{kl}_{\eta+t}-u^{kl}_\eta-tu^{k+1,l}_\eta)\big\|=\|T_0u^{kl}_{\eta+t}\|\\
=&\big\|(\eta+t)u^{kl}_{\eta+t}+ku^{k-1,l}_{\eta+t}+ilu^{k,l-1}_{\eta+t}\big\|\\
\leq&\big\|(\eta+t)u^{kl}_\eta+(\eta+t)tu^{k+1,l}_\eta+ku^{k-1,l}_\eta+ktu^{kl}_\eta+ilu^{k,l-1}_\eta+iltu^{k+1,l-1}_\eta\big\|+3t^2K_{k+l+2}\\
\leq&3t^2K_{k+l+2}+\left\|\left(\frac{-1}{\eta}\right)^{k+l}(k+l-1)!i^{l-1}u\right\|\\
&\cdot\left|(\eta+t)\frac{-i}{\eta}(k+l)+\frac{(\eta+t)ti}{\eta^2}(k+l)(k+l+1)+ki-\frac{kti}{\eta}(k+l)+il-\frac{lti}{\eta}(k+l)\right|\\
=&3t^2K_{k+l+2}+\left\|\left(\frac{-1}{\eta}\right)^{k+l+1}(k+l)!i^lu\right\|\cdot\left|\eta+t-\frac{(\eta+t)t}{\eta}(k+l+1)-\eta+kt+lt\right|\\
=&3t^2K_{k+l+2}+(k+l)!|t|\cdot\left|1-(k+l+1)-\frac{t}{\eta}(k+l+1)+k+l\right|\\
=&3^2K_{k+l+2}+(k+l+1)!t^2\leq 4t^2K_{k+l+2}.
\end{aligned}$$

Similarly, for $|\eta|=1$, $s\in\mathbb{R}$, $|s|<1/3$ and $\eta+is\in\mathbb{D}$ we have

$$\big\|T_0(u^{kl}_{\eta+is}-u^{kl}_\eta-su^{k,l+1}_\eta)\big\|\leq 4K_{k+l+2}.$$

Hence $\|T_0m\|\leq 4$ for all $m\in M$ and consequently, $\|T_0v\|\leq 4\|v\|$ for all $v\in X_0$.

Let $X_1=\{v\in X_0:\|v\|=0\}$. Then $T_0X_1\subset X_1$. Let X be the completion of X_0/X_1. Then T_0 induces the operator $T:X\to X$ and $\|T\|\leq 4$.

Define the function $f : \mathbb{C} \to X$ by

$$f(z) = \begin{cases} u_z^{00} + X_1 & (|z| < 1), \\ -\frac{u}{z} + X_1 & (|z| \geq 1). \end{cases}$$

Clearly $(T - z)f(z) = u + X_1$ for all $z \in \mathbb{C}$ and $u + X_1 \neq 0$.

It remains to show that f is infinitely differentiable.

Clearly f is even analytic for $|z| > 1$. For $|z| < 1$ we can show by induction that $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}(z) = u_z^{kl} + X_1$. Indeed, this follows from the estimates

$$\lim_{t\to 0} \left\| \frac{u_{z+t}^{k,l} - u_z^{k,l}}{t} - u_z^{k+1,l} \right\| \leq \lim_{t\to 0} |t| K_{k+l+2} = 0$$

and similarly,

$$\lim_{s\to 0} \left\| \frac{u_{z+is}^{k,l} - u_z^{k,l}}{s} - u_z^{k,l+1} \right\| = 0$$

Finally, for $|z| = 1$ we show by induction that $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}(z) = u_z^{kl} + X_1$. Clearly

$$\lim_{\substack{t\to z \\ |z+t|>1}} \frac{u_{z+t}^{kl} - u_z^{kl}}{t} = \lim_{\substack{t\to z \\ |z+t|>1}} \frac{g_{kl}(z+t) - g_{kl}(z)}{t} u = g_{k+1,l}(z)u = u_z^{k+1,l}$$

and

$$\lim_{\substack{t\to z \\ |z+t|<1}} \left\| \frac{u_{z+t}^{kl} - u_z^{kl}}{t} - u_z^{k+1,l} \right\| = 0.$$

Similar statements hold for derivatives in the imaginary direction y. Thus for all $k, l \in \mathbb{Z}_+$ and $|z| = 1$ we have $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}(z) = u_z^{kl} + X_1$.

By (1) and (2), all the derivatives $z \mapsto u_z^{kl}$ are continuous.

Remark 2. In [BM], there was constructed an example of an operator $T \in B(X)$ and $x \neq 0$ such that $\operatorname{int} \sigma(T) = \emptyset$ and there is a continuous local resolvent $f : \mathbb{C} \to X$ satisfying $(T - z)f(z) = x \quad (z \in \mathbb{C})$. It was raised a question whether there is a similar example with smooth local resolvent.

As it was observed by J. Kolář, such an example with C_1- local resolvent cannot exist. Indeed, each local resolvent is analytic on the complement of the spectrum. Therefore it satisfies the Cauchy-Riemann conditions on $\mathbb{C} \setminus \sigma(T)$. If the local resolvent is C_1 and $\operatorname{int} \sigma(T) = \emptyset$, then the local resolvent satisfies the Cauchy-Riemann conditions everywhere, and so it is an entire function. It is well known (and an easy consequence of the Liouville theorem) that such a local resolvent cannot exist.

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