The splitting spectrum differs from the Taylor spectrum

V. Müller*

Abstract. We construct a pair of commuting Banach space operators for which the splitting spectrum is different from the Taylor spectrum.

Let $A_1,\dots,A_n$ be mutually commuting operators in a Banach space X. The Koszul complex of the n-tuple $(A_1,\dots,A_n)$ is the complex

$$0 \longrightarrow \Lambda^0(X,e)\overset{\delta_0}{\longrightarrow}\Lambda^1(X,e)\overset{\delta_1}{\longrightarrow}\cdots\overset{\delta_{n-1}}{\longrightarrow}\Lambda^n(X,e)\longrightarrow 0$$

where $\Lambda^p(X,e)$ denotes the vector space of all forms of degree p in indeterminates $e_1,\dots,e_n$ with coefficients in X and the linear mappings $\delta_p:\Lambda^p(X,e)\to\Lambda^{p+1}(X,e)$ are defined by

$$\delta^p(xe_{i_1}\wedge\cdots\wedge e_{i_p})=\sum_{j=1}^n A_jxe_j\wedge e_{i_1}\wedge\cdots\wedge e_{i_p}.$$

It is well-known that $\delta_{p+1}\delta_p=0$ for every p. The Taylor spectrum $\sigma_T(A_1,\dots,A_n)$ is the set of all n-tuples $(\lambda_1,\dots\lambda_n)$ of complex numbers for which the Koszul complex of $(A_1-\lambda_1,\dots,A_n-\lambda_n)$ is not exact [5].

Instead of the Taylor spectrum it is sometimes useful to use the following variation of the Taylor spectrum, see e.g. [1], [3], [4]. We say that the n-tuple $(A_1,\dots,A_n)$ is splitting-regular if its Koszul complex is exact and the ranges of the operators δ_p are complemented in $\Lambda^{p+1}(X,e)$. Equivalently, there exists operators $\varepsilon_p:\Lambda^{p+1}(X,e)\to\Lambda^p(X,e)$ $(p=0,\dots,n-1)$ such that $\varepsilon_p\delta_p+\delta_{p-1}\varepsilon_{p-1}$ is the identity operator on $\Lambda^p(X,e)$ for $p=0,\dots,n$ (formally we set $\delta_{-1}=\delta_n=0$). The splitting spectrum $\sigma_S(A_1,\dots,A_n)$ is the set of all $(\lambda_1,\dots,\lambda_n)\in\mathbf{C}^n$ such that the n-tuple $(A_1-\lambda_1,\dots,A_n-\lambda_n)$ is not splitting-regular.

The splitting spectrum has similar properties as the Taylor spectrum. Clearly $\sigma_T(A_1,\dots,A_n)\subset\sigma_S(A_1,\dots,A_n)$. For Hilbert space operators these two spectra coincide and the same is true for n-tuples of operators in ℓ_1 or in ℓ_∞, [2]. Also for a single operator A_1 in an arbitrary Banach space $\sigma_T(A_1)=\sigma_S(A_1)$. Consequently the polynomially convex hulls of $\sigma_T(A_1,\dots,A_n)$ and of $\sigma_S(A_1,\dots,A_n)$ are equal.

It was generally expected that these two spectra are different for n-tuple of operators on a Banach space but no example was known and it was believed that such an example would be complicated [2]. The aim of this note is to fill this gap in the theory. Surprisingly, the constructed example is rather simple.

We denote by $R(T)$ and $N(T)$ the range and the kernel of an operator T. If X and Y are Banach spaces then $X\oplus Y$ denotes the direct sum endowed with the ℓ_1-norm, $\|(x,y)\|_{X\oplus Y}=\|x\|_X+\|y\|_Y \quad (x\in X, y\in Y)$. Similar convention we use for direct sums of more than two Banach spaces.

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Lemma 1. *There exists a Banach space Z and closed subspaces $Y_1, Y_2 \subset Z$ such that $Y_1 + Y_2 = Z$ and the subspace $\{(x, x) : x \in Y_1 \cap Y_2\}$ is not complemented in $Y_1 \oplus Y_2$.*

Proof. Fix a Banach space Y and a closed subspace $X \subset Y$ which is not complemented in Y.

Clearly $M = \{(x, -x) : x \in X\}$ is a closed subspace of $Y \oplus Y$. Denote $Z = (Y \oplus Y)/M$ and let $\pi : Y \oplus Y \to Z$ be the canonical projection. Clearly $\pi(x, 0) = \pi(0, x)$ for every $x \in X$. Consider operators $J_1, J_2 : Y \to Z$ defined by $J_1 y = \pi(y, 0)$ and $J_2 y = \pi(0, y) \quad (y \in Y)$. It is easy to check that J_1 and J_2 are isometries. Denote $Y_1 = J_1 Y$ and $Y_2 = J_2 Y$. Clearly $Z = Y_1 + Y_2$ and $Y_1 \cap Y_2 = \{\pi(x, 0) : x \in X\} = \{\pi(0, x) : x \in X\}$.

Suppose on the contrary that the space $D = \{\big(\pi(x, 0), \pi(0, x)\big) : x \in X\}$ is complemented in $Y_1 \oplus Y_2 = \{\big(\pi(y_1, 0), \pi(0, y_2)\big) : y_1, y_2 \in Y\}$. Then D is complemented also in the closed subspace

$$W = \{\big(\pi(y, 0), \pi(0, y)\big) : y \in Y\} = \{(J_1 y, J_2 y) : y \in Y\} \subset Y_1 \oplus Y_2.$$

Let $J : Y \to W$ be defined by $Jy = (\pi(\frac{y}{2}, 0), \pi(0, \frac{y}{2}))$. Clearly J is an isometry onto W and $JX = D$ so that X is complemented in Y, a contradiction.

Theorem 2. *There exist a Banach space W and commuting operators $A_1, A_2 \in \mathcal{L}(W)$ such that $\sigma_T(A_1, A_2) \neq \sigma_S(A_1, A_2)$.*

Proof. Let Z, Y_1 and Y_2 be the Banach spaces from the previous lemma. For $i, j \in \mathbf{Z}$ set

$$W_{ij} = \begin{cases} Z & (i, j \geq 1), \\ Y_1 & (i \geq 1, j \leq 0), \\ Y_2 & (i \leq 0, j \geq 1), \\ Y_1 \cap Y_2 & (i, j \leq 0). \end{cases}$$

Clearly $W_{ij} \subset W_{i+1,j}$ and $W_{ij} \subset W_{i,j+1}$. Set $W = \bigoplus_{i,j \in \mathbf{Z}} W_{ij}$ and let $A_1, A_2 \in \mathcal{L}(W)$ be the shift operators to the right and up,

$$A_1\Big(\bigoplus_{i,j} w_{ij}\Big) = \bigoplus_{i,j} w_{i-1,j}, \qquad A_2\Big(\bigoplus_{i,j} w_{ij}\Big) = \bigoplus_{i,j} w_{i,j-1}.$$

Clearly A_1 and A_2 are commuting isometries. Further $W_{ij} = W_{i+1,j} \cap Wi, j+1$ and $W_{ij} = W_{i-1,j} + Wi, j-1$ for all $i, j \in \mathbf{Z}$. So $R(A_1) + R(A_2) = W$.

The Koszul complex of the pair (A_1, A_2) is of the form

$$0 \longrightarrow W \xrightarrow{\delta_0} W \oplus W \xrightarrow{\delta_1} W \longrightarrow 0 \tag{1}$$

where $\delta_0 w = (A_1 w, A_2 w)$ and $\delta(w, z) = -A_2 w + A_1 z \quad (w, z \in W)$. Clearly δ_0 is bounded below and $R(\delta_1) = R(A_1) + R(A_2) = W$.

To show the exactness of the Koszul complex (1) it is sufficient to prove $N(\delta_1) \subset R(\delta_0)$. Let $\big(\bigoplus w_{ij}, \bigoplus z_{ij}\big) \in N(\delta_1)$ for some $w_{ij}, z_{ij} \in W_{ij}$. Then, for all $i, j \in \mathbf{Z}$, $w_{i,j-1} = z_{i-1,j}$ so that

$$w_{ij} = z_{i-1,j+1} \in W_{ij} \cap W_{i-1,j+1} = W_{i-1,j}$$

and

$$z_{ij} = w_{i+1,j-1} \in W_{ij} \cap W_{i+1,j-1} = W_{i,j-1}.$$

Set $u = \bigoplus w_{i+1,j} = \bigoplus z_{i,j+1}$. Then $\delta_1 u = (A_1 u, A_2 u) = (\bigoplus w_{ij}, \bigoplus z_{ij})$. Hence $N(\delta_1) = R(\delta_0)$, the Koszul complex (1) is exact and $(0,0) \notin \sigma_T(A_1, A_2)$.

We show that $R(\delta_0)$ is not complemented in $W \oplus W$. Suppose on the contrary that there exists a projection $P \in \mathcal{L}(W \oplus W)$ with range $R(\delta_0)$. Let $Q \in \mathcal{L}(W \oplus W)$ be defined by $Q(\bigoplus w_{ij}, \bigoplus z_{ij}) = (w_{1,0}, z_{0,1}) \in W_{1,0} \oplus W_{0,1}$. Clearly $Q^2 = Q$ and $PQP = QP$ so that $(QP)^2 = Q(PQP) = QP$ is also a projection with

$$R(QP) = \{(w_{1,0}, z_{0,1}) : w_{1,0} = z_{0,1} \in W_{0,0}\} = \{(x, x) : x \in Y_1 \cap Y_2\}.$$

Clearly $R(QP)$ is complemented also in $W_{1,0} \oplus W_{0,1} = Y_1 \oplus Y_2$ which is a contradiction with Lemma 1.

Hence $(0,0) \in \sigma_S(A_1, A_2)$ and $\sigma_S(A_1, A_2) \neq \sigma_T(A_1, A_2)$.

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Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1
Czech Republic
e-mail: vmuller@mbox.cesnet.cz