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Integral Equations and Operator Theory

Compressions of stable contractions

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Abstract. The stability of compressions of stable contractions is studied and a sufficient orbit condition is given. On the other hand, it is shown that there are non-stable compressions of the 1-dimensional backward shift and a complete characterization of weighted unilateral shifts with this property is provided. Dilations of bilateral weighted shifts to backward shifts are also considered.

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1. Introduction

Let H be a complex Hilbert space, and let $L(H)$ denote the C^*-algebra of all bounded linear operators acting on H. An operator $T \in L(H)$ is called *stable*, if its positive powers converge to zero in the strong operator topology, that is when $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in H$. The Banach–Steinhaus Theorem shows that each stable operator T is power bounded, which means the boundedness of the norm-sequence $\{\|T^n\|\}_{n\in\mathbb{N}}$, indexed by the set $\mathbb{N}$ of positive integers.

Let $P(H)$ stand for the set of all orthogonal projections in $L(H)$. We are interested in the question whether the stability of $T \in L(H)$ implies the stability of the operator $T_P := PTP \in L(H)$, for a projection $P \in P(H)$. Let $R(P)$ denote the range of P. The operator $PT|R(P) \in L(R(P))$ is called the *compression* of T to the subspace $R(P)$. The equations $T_P^n = P(TP)^n$, $(PT)^n = T_P^{n-1}T$ and $(TP)^n = T(PT)^{n-1}P$ $(n \in \mathbb{N})$ show that the operators T_P, PT, TP and the compression of T to the subspace $R(P)$ are stable at the same time.

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If the Hilbert space H is non-separable, then it can be decomposed into an orthogonal sum of separable subspaces, which are reducing for both T and P. Hence, we can and shall assume that H is separable.

Let $P(T)$ be the set of all projections in $P(H)$, whose range is an invariant subspace of T. For $P \in P(T)$, the operator $T|R(P) \in L(R(P))$ is called the *restriction* of T to its invariant subspace $R(P)$. The elements of the set $P_s(T) := \{P_1 - P_2 : P_1, P_2 \in P(T), P_1 \geq P_2\}$ are the projections whose ranges are semiinvariant subspaces of T. It can be easily seen that, for any $P \in P_s(T)$, the equality $T_P^n = PT^nP$ is true for every $n \in \mathbb{N}$. Thus, the stability of T is inherited by T_P in that case.

Changing the viewpoint, passing from the compression to the operator on the larger space, another terminology is also in use. Let F and G be Hilbert spaces. We say that an operator $A \in L(F)$ can be *dilated* to an operator $B \in L(G)$, in notation: $A \overset{\mathrm{d}}{\prec} B$, if there exists an isometry $Z \in L(F, G)$ such that $A = Z^*BZ$. This happens precisely when A is unitarily equivalent to a compression of B to a subspace of G. The operator A can be *power dilated* to B, in notation: $A \overset{\mathrm{pd}}{\prec} B$, if there exists an isometry $Z \in L(F, G)$ such that $A^n = Z^*B^nZ$ for every $n \in \mathbb{N}$. It is known that $A \overset{\mathrm{pd}}{\prec} B$ if and only if A is unitarily equivalent to the compression of B to a subspace semiinvariant with respect to B (see [S, Lemma 0]). If B is stable and A can be power dilated to B, then A is clearly stable. The question is whether the stability of B implies the stability of A, if A can be only dilated to B. We give a simple example which shows that the answer is negative in such a generality.

An operator T is called *uniformly stable*, if $\lim_{n\to\infty} \|T^n\| = 0$. This happens if and only if its spectral radius $r(T)$ is less than 1. In general, even the uniform stability of T does not imply the stability of T_P. Indeed, let (e_1, e_2) be an orthonormal basis in the Hilbert space H, and let $T \in L(H)$ be defined by $Te_1 := 2e_2,\ Te_2 := 0$. Then $T^2 = 0$, and so T is uniformly stable. On the other hand, if $P \in P(H)$ is the projection onto the 1-dimensional subspace spanned by the vector $e_1 + e_2$, then $T_P(e_1 + e_2) = PT(e_1 + e_2) = P(2e_2) = e_1 + e_2$, and so T_P is not stable. Therefore, the stability of T_P could be expected only under additional conditions.

It is natural to make the assumption that $T \in L(H)$ is a contraction: $\|T\| \leq 1$. Actually, the question whether compressions of stable contractions are also stable was posed to the first named author by Rongwei Yang. If T is a strict contraction then the answer is obviously positive, since $\|T_P\| \leq \|T\| < 1$ implies the uniform stability of T_P. It is also easy to verify that the stability of the contraction T is inherited by T_P if the projection P has finite rank. Indeed, it is enough to show that $r(T_P) < 1$. Assuming $r(T_P) = 1$, there exist $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $0 \neq x \in H$ such that $T_P x = \lambda x$. Since $\|x\| = \|T_P x\| \leq \|TPx\| \leq \|Px\| \leq \|x\|$, we infer that $Tx = \lambda x$, which contradicts to the stability of T.

By a well-known theorem of C. Foias, restrictions of the infinite dimensional backward shift provide all stable contractions. To be more precise, for any $1 \leq n \leq \infty$ $(:= \aleph_0)$ fix an n-dimensional Hilbert space E_n, and let us consider the corresponding Hardy space $H^2(E_n)$. The operator $S_n \in L(H^2(E_n))$ of multiplication by the identical function $\chi(z) = z$ is the n-dimensional unilateral shift, and its adjoint $B_n := S_n^* \in L(H^2(E_n))$ is the n-dimensional backward shift. We recall that the defect operators of a contraction $T \in L(H)$ are defined by $D_T := (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} := (I - TT^*)^{\frac{1}{2}}$. The defect spaces of T are the closures of the ranges of the defect operators: $\mathcal{D}_T := (D_T H)^-$, $\mathcal{D}_{T^*} := (D_{T^*} H)^-$, and $d_T := \dim \mathcal{D}_T$, $d_{T^*} := \dim \mathcal{D}_{T^*}$ are the defect numbers of T. (For more information on the role of these objects in the study of Hilbert space contractions, we refer to the monograph [NF].) Let S_T denote the operator of multiplication by χ on $H^2(\mathcal{D}_T)$. The adjoint $B_T = S_T^*$ is unitarily equivalent to B_n, where $n = d_T$. If the contraction T is stable, then the transformation $Z_T \colon H \to H^2(\mathcal{D}_T)$, $h \mapsto \sum_{n=0}^{\infty} \chi^n D_T T^n h$ is an isometry, whose range is invariant for B_T. Since $T = Z_T^* B_T Z_T$, we can see that T is unitarily equivalent to a restriction of B_∞. Taking into account that compressions of restrictions of B_∞ are compressions of B_∞, we obtain that Yang's question is equivalent to the problem whether all compressions of the infinite dimensional backward shift B_∞ are stable. (We mention also that by a recent result of J.-C. Bourin in [B], for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of strict contractions with $\sup_n \|A_n\| < 1$, there exists a decomposition $H^2(E_\infty) = \sum_{n=1}^{\infty} \oplus M_n$ such that A_n is unitarily equivalent to the compression of B_∞ to M_n for all n.)

In [TW] K. Takahashi and P.Y. Wu studied the question which contractions can be dilated to a unilateral shift. They proved that if at least one of the defect indices of the contraction T is finite, and if T can be dilated to B_∞, then T is stable. Another result due to C. Benhida and D. Timotin states that if $T \in L(H)$ is a stable contraction with $d_{T^*} < \infty$, and if for $P \in P(H)$ the projection $I - P$ has finite rank, then the operator T_P is also stable (see [BT, Lemma 3.3]). In Section 2 we give an orbit condition yielding the stability for compressions of B_∞.

In view of a general theorem on contractions, it can be easily justified that if a non-stable contraction T can be dilated to B_∞, then contractions similar to the unilateral shift S_1 can also be dilated to B_∞. Indeed, T is necessarily completely non-unitary, and its residual set $\rho(T)$ is of positive Lebesgue measure. (We recall that the Borel subset $\rho(T)$ of the unit circle $\mathbb{T}$ is the support of the spectral measure of the canonical unitary operator associated with T; for its detailed study we refer to [K2].) Choosing an appropriate sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ on $\mathbb{T}$, it can be attained that $\rho(T_1) = \mathbb{T}$ holds for the orthogonal sum $T_1 = \sum_{n=1}^{\infty} \oplus\, \alpha_n T$. Then, by [K1, Theorem 3] there exists a subspace M_1, invariant for T_1, such that the restriction $T_2 := T_1|M_1$ is similar to S_1. Taking into account that $\alpha_1 B_\infty \oplus \alpha_2 B_\infty \oplus \cdots$ is unitarily equivalent to B_∞, we infer that T_2 can be dilated to B_∞.

It is proved in Section 3 that there are indeed non-stable unilateral weighted shifts, similar to S_1, which can be dilated even to B_1, and so the answer to Yang's question is negative. Actually, a complete characterization of such unilateral weighted shifts is given. Finally, in Section 4, dilations of bilateral weighted shifts into backward shifts are studied.

2. Orbit condition

We are going to show that a contraction, which is close to an isometry regarding the behaviour of the orbit of a vector, cannot be dilated to the infinite dimensional backward shift $B_\infty \in L(H^2(E_\infty))$. The proof relies on some elementary inequalities.

Let us fix a projection $P \in P(H^2(E_\infty))$, and let us consider the operator $B_P := (B_\infty)_P = PB_\infty P \in L(H^2(E_\infty))$. We shall examine the orbit $\{B_P^n u\}_{n\in\mathbb{N}}$ of an arbitrarily chosen vector $u \in H^2(E_\infty)$ under the action of B_P.

Let $\mathbb{Z}_+$ denote the set of non-negative integers. For any $k \in \mathbb{Z}_+$, let $E_k \in P(H^2(E_\infty))$ be the projection onto the subspace $S_\infty^k E_\infty$. (Here E_∞ is identified with the set of constant functions in $H^2(E_\infty)$.) The projections $\{E_k\}_{k=0}^{\infty}$ are pairwise orthogonal, and the series $\sum_{k=0}^{\infty} E_k$ converges to the identity operator I in the strong operator topology.

Lemma 1. *The constant components of the orbit vectors converge to zero*:

$$\lim_{n\to\infty} \|E_0 B_P^n u\| = 0.$$

Proof. The equation $B_\infty E_0 = 0$ implies that $B_P^{n+1} u = PB_\infty (I - E_0) B_P^n u$ holds, for every $n \in \mathbb{N}$. Thus $\|B_P^{n+1} u\|^2 \le \|(I - E_0) B_P^n u\|^2 = \|B_P^n u\|^2 - \|E_0 B_P^n u\|^2$, whence $\|E_0 B_P^n u\|^2 \le \|B_P^n u\|^2 - \|B_P^{n+1} u\|^2$ follows. Since B_P is a contraction, the sequence $\{\|B_P^n u\|\}_{n\in\mathbb{N}}$ converges decreasingly to a non-negative number. Hence the dominating sequence $\{\|B_P^n u\|^2 - \|B_P^{n+1} u\|^2\}_{n\in\mathbb{N}}$ tends to zero, which yields the statement. Q.E.D.

By the next lemma each Fourier coefficient of the vectors in the orbit converges to zero.

Lemma 2.
(a) *For every* $n \in \mathbb{N}$, $\|(I-P)B_\infty B_P^n u\|^2 \leq \|B_P^n u\|^2 - \|B_P^{n+1}u\|^2$.
(b) *For every* $n \in \mathbb{N}, k \in \mathbb{Z}_+$, *we have*

$$\|E_{k+1}B_P^n u\| \leq \|E_k B_P^{n+1}u\| + \left(\|B_P^n u\|^2 - \|B_P^{n+1}u\|^2\right)^{\frac{1}{2}}.$$

(c) *For every* $k \in \mathbb{Z}_+$, $\lim_{n\to\infty} \|E_k B_P^n u\| = 0$ *is true.*

Proof. (a): It is immediate that $\|(I-P)B_\infty B_P^n u\|^2 = \|B_\infty B_P^n u\|^2 - \|B_P^{n+1}u\|^2 \leq \|B_P^n u\|^2 - \|B_P^{n+1}u\|^2$.
(b): Taking into account that $E_k B_\infty = B_\infty E_{k+1}$, we can write $B_\infty E_{k+1} B_P^n = E_k B_\infty B_P^n = E_k B_P^{n+1} + E_k(I-P)B_\infty B_P^n$. Hence

$$\|E_{k+1}B_P^n u\| = \|B_\infty E_{k+1} B_P^n u\| \leq \|E_k B_P^{n+1} u\| + \|(I-P)B_\infty B_P^n u\|$$

is true, and an application of (a) yields the requested inequality.
(c): This statement follows by induction on k, relying on (b) and Lemma 1. Q.E.D.

For any $k \in \mathbb{Z}_+$, let us consider the projection $Q_k := \sum_{j=k}^\infty E_j \in P(H^2(E_\infty))$.

Lemma 3. *For every* $n, l \in \mathbb{N}$ *and* $k \in \mathbb{Z}_+$, *we have*

$$\|Q_k B_P^{n+l} u\| \leq \|Q_{k+l} B_P^n u\| + \sum_{j=n}^{n+l-1} \left(\|B_P^j u\|^2 - \|B_P^{j+1}u\|^2\right)^{\frac{1}{2}}.$$

Proof. Since $Q_k B_\infty = B_\infty Q_{k+1}$, we infer that

$$Q_k B_P^{n+1} = Q_k B_\infty B_P^n - Q_k(I-P)B_\infty B_P^n = B_\infty Q_{k+1} B_P^n - Q_k(I-P)B_\infty B_P^n.$$

Hence $\|Q_k B_P^{n+1}u\| \leq \|Q_{k+1}B_P^n u\| + \|(I-P)B_\infty B_P^n u\|$, and so Lemma 2.(a) yields the required inequality for $l = 1$. Then the statement can be verified by induction on l. Q.E.D.

Now, we are ready to prove our theorem.

Theorem 4. *If* $\sum_{n=1}^\infty \left(\|B_P^n u\|^2 - \|B_P^{n+1}u\|^2\right)^{\frac{1}{2}} < \infty$, *then* $\lim_{n\to\infty} \|B_P^n u\| = 0$.

Proof. In view of Lemma 3, the inequality

$$\|B_P^{n+l}u\| \leq \sum_{j=0}^{k-1} \|E_j B_P^{n+l} u\| + \|Q_{k+l} B_P^n u\| + \sum_{j=n}^{\infty} \left(\|B_P^j u\|^2 - \|B_P^{j+1}u\|^2\right)^{\frac{1}{2}}$$

holds, for any $k, l, n \in \mathbb{N}$. Given a positive ε, let us choose $n_0 \in \mathbb{N}$ so that

$$\sum_{j=n_0}^{\infty} \left(\|B_P^j u\|^2 - \|B_P^{j+1}u\|^2\right)^{\frac{1}{2}} < \varepsilon/3.$$

Since $\lim_{k\to\infty} \|Q_k B_P^{n_0} u\| = 0$, we can find $k_0 \in \mathbb{N}$ such that $\|Q_{k_0+l} B_P^{n_0} u\| < \varepsilon/3$ is true for every $l \in \mathbb{N}$. Finally, by Lemma 2.(c) there exists $l_0 \in \mathbb{N}$ such that $\sum_{j=0}^{k_0-1} \|E_j B_P^{n_0+l} u\| < \varepsilon/3$ is valid, for every $l \geq l_0$. Then $\|B_P^{n_0+l} u\| < \varepsilon$ is fulfilled for $l \geq l_0$, which proves the statement. Q.E.D.

By the aforementioned theorem of Foias we obtain the following immediate consequence of Theorem 4.

Corollary 5. *Let $T \in L(H)$ be a stable contraction, and let $P \in P(H)$, $x \in H$ be given. If $\sum_{n=1}^{\infty} \left(\|T_P^n x\|^2 - \|T_P^{n+1} x\|^2\right)^{\frac{1}{2}} < \infty$, then $\lim_{n\to\infty} \|T_P^n x\| = 0$.*

3. Dilation of unilateral weighted shifts

The simplest examples for contractions similar to S_1 can be found in the class of unilateral weighted shifts.

Let $\{v_k\}_{k\in\mathbb{N}}$ be an orthonormal basis in the Hilbert space K. Given any bounded sequence $\{w_k\}_{k\in\mathbb{N}}$ of complex numbers, let us consider the operator $W \in L(K)$, defined by $Wv_k := w_k v_{k+1}$ $(k \in \mathbb{N})$. The unilateral weighted shift W is a contraction precisely when $|w_k| \leq 1$ for every $k \in \mathbb{N}$. All such contractions are obtained up to unitary equivalence assuming that w_k belongs to the closed interval $[0, 1]$, for every $k \in \mathbb{N}$. Therefore we can assume that $w_k \in [0, 1]$ for all $k \in \mathbb{N}$.

It is easy to verify that W is non-stable if and only if $\prod_{k=k_0}^{\infty} w_k > 0$ for some $k_0 \in \mathbb{N}$, which happens exactly when $\sum_{k=1}^{\infty}(1 - w_k) < \infty$. Furthermore, this condition is equivalent to the decomposability of W in the form $W = W_0 \oplus W_1$, where W_0 is a nilpotent operator on a finite dimensional space, and W_1 is a unilateral weighted shift similar to S_1.

Let us assume that W is a non-stable contraction (that is $\{w_k\}_{k\in\mathbb{N}} \subset [0, 1]$ and $\sum_{k=1}^{\infty}(1 - w_k) < \infty$), and that W can be dilated to the infinite dimensional backward shift B_∞. There exists $k_0 \in \mathbb{N}$ such that $\prod_{k=k_0}^{\infty} w_k > 0$. Since $\lim_{n\to\infty} \|W^n v_{k_0}\| = \lim_{n\to\infty} \prod_{k=k_0}^{k_0+n-1} w_k > 0$, we infer by Theorem 4 that

$$\sum_{n=1}^{\infty} \left(\|W^n v_{k_0}\|^2 - \|W^{n+1} v_{k_0}\|^2\right)^{\frac{1}{2}} = \infty.$$

Taking into account that, for every $n \in \mathbb{N}$,

$$\left(\|W^n v_{k_0}\|^2 - \|W^{n+1} v_{k_0}\|^2\right)^{\frac{1}{2}} = \left(\prod_{k=k_0}^{k_0+n-1} w_k\right) (1 - w_{k_0+n}^2)^{\frac{1}{2}} \leq 2(1 - w_{k_0+n})^{\frac{1}{2}},$$

we conclude that $\sum_{k=1}^{\infty}(1 - w_k)^{\frac{1}{2}} = \infty$.

We shall show that under these conditions W can be really dilated to B_∞, even more, it can be dilated to the 1-dimensional backward shift B_1. Namely, we are going to prove the following theorem.

Theorem 6. *Let $W \in L(K)$ be the unilateral weighted shift corresponding to the weight sequence $\{w_k\}_{k\in\mathbb{N}} \subset [0,1]$, satisfying the condition $\sum_{k=1}^\infty (1-w_k) < \infty$. The non-stable contraction W can be dilated to B_∞ if and only if it can be dilated to B_1, which happens exactly when $\sum_{k=1}^\infty (1-w_k)^{\frac{1}{2}} = \infty$.*

It is easy to find sequences satisfying the previous conditions. For example, these are fulfilled if $w_k = 1 - \varepsilon k^{-p}$ $(k \in \mathbb{N})$ with $1 < p \le 2$ and $0 < \varepsilon < 1$. Therefore, the answer for Yang's question is negative: there are stable contractions having non-stable compressions. We note also that if ε is small, then the similarity constant $s(W, S_1) := \inf\{\|Q\| \cdot \|Q^{-1}\| : QW = S_1 Q\}$ can be arbitrarily close to 1.

Proof. Let $\{w_k\}_{k\in\mathbb{N}} \subset [0,1]$ be a weight sequence satisfying the conditions $\sum_{k=1}^\infty (1-w_k) < \infty$ and $\sum_{k=1}^\infty (1-w_k)^{\frac{1}{2}} = \infty$. We have to show that the corresponding unilateral weighted shift $W \in L(K)$, $Wv_k := w_k v_{k+1}$ $(k \in \mathbb{N})$, can be dilated to B_1.

For any $k \in \mathbb{N}$, let $\alpha(k) \in [0, \frac{\pi}{2}]$ be defined by $\cos\alpha(k) = w_k$. The assumption $\sum_{k=1}^\infty (1-w_k) < \infty$ yields that $\lim_{k\to\infty} \alpha(k) = 0$. Taking into account that $(1-w_k)^{\frac{1}{2}} \le (1-w_k^2)^{\frac{1}{2}} = \sin\alpha(k) \le 2(1-w_k)^{\frac{1}{2}}$ and $\frac{2}{\pi}\alpha(k) \le \sin\alpha(k) \le \alpha(k)$ $(k \in \mathbb{N})$, the assumption $\sum_{k=1}^\infty (1-w_k)^{\frac{1}{2}} = \infty$ can be equivalently expressed as $\sum_{k=1}^\infty \alpha(k) = \infty$. For any $i, j \in \mathbb{N}$, $i \le j$, let us use the notation $\alpha(i,j) := \sum_{k=i}^{j} \alpha(k)$.

With $\{\alpha(k)\}_{k\in\mathbb{N}}$ we associate three sequences: $\{k_j\}_{j=0}^\infty \subset \mathbb{Z}_+$, $\{\widetilde{\alpha}(j)\}_{j=0}^\infty \subset [0, \frac{\pi}{2}]$ and $\{r_j\}_{j=0}^\infty \subset \mathbb{N}$ in the following way. Setting $k_0 := 0$ and $\widetilde{\alpha}(0) := 0$, let us assume that $\{k_i\}_{i=0}^j$ and $\{\widetilde{\alpha}(i)\}_{i=0}^j$ have already been defined, for $j \in \mathbb{Z}_+$. Then k_{j+1} is defined as the minimum of the integers k satisfying the conditions $k > k_j$ and $\widetilde{\alpha}(j) + \alpha(k_j+1, k) > \frac{5\pi}{2}$. (The assumption $\sum_{i=1}^\infty \alpha(i) = \infty$ ensures the existence of such a k. Clearly, $k_{j+1} > k_j + 4$.) Since $0 \le \frac{5\pi}{2} - (\widetilde{\alpha}(j) + \alpha(k_j+1, k_{j+1}-1)) < \alpha(k_{j+1}) \le \frac{\pi}{2}$, we infer that

$$\begin{aligned}\sin\left(\widetilde{\alpha}(j) + \alpha(k_j+1, k_{j+1}-1)\right) &= \cos\left(\frac{5\pi}{2} - (\widetilde{\alpha}(j) + \alpha(k_j+1, k_{j+1}-1))\right) \\ &> \cos\alpha(k_{j+1}) \ge 0,\end{aligned}$$

and so there exists a unique $\widetilde{\alpha}(j+1) \in [0, \frac{\pi}{2}]$ such that

$$\cos\widetilde{\alpha}(j+1) = \cos\alpha(k_{j+1}) \left(\sin(\widetilde{\alpha}(j) + \alpha(k_j+1, k_{j+1}-1))\right)^{-1}.$$

The sequence $\{r_j\}_{j=0}^\infty$ is defined by $r_j := k_{j+1} - k_j$ $(j \in \mathbb{Z}_+)$. Note that $r_j > 4$.

Let us choose a sequence $\{n_j\}_{j=0}^{\infty}$ of positive integers satisfying the conditions $n_0 > r_0$, $n_1 > n_0 + r_1$, and $n_j > n_{j-1} + r_j + r_{j-2}$ for every $j \geq 2$.

Fixing a unit vector $e_0 \in E_1$, let us consider the orthonormal basis $\{e(n) := S_1^n e_0\}_{n=0}^{\infty}$ in the Hardy space $H^2(E_1)$.

For any $j \in \mathbb{Z}_+$, let

$$u(k_j) := (\cos \widetilde{\alpha}(j)) e(n_j) + (\sin \widetilde{\alpha}(j)) e(n_{j+1} + r_j),$$

and, for any $1 \leq i < r_j$, let

$$\begin{aligned} u(k_j + i) \quad := & \quad (\cos(\widetilde{\alpha}(j) + \alpha(k_j + 1, k_j + i)))\, e(n_j - i) \\ + & \quad (\sin(\widetilde{\alpha}(j) + \alpha(k_j + 1, k_j + i)))\, e(n_{j+1} + r_j - i). \end{aligned}$$

The assumptions made at the choice of $\{n_j\}_{j=0}^{\infty}$ ensure that the resulting sequence $\{u(k)\}_{k=0}^{\infty}$ is orthonormal.

Exploiting the fact that

$$\langle (\cos \varphi) f + (\sin \varphi) g, (\cos \psi) f + (\sin \psi) g \rangle = \cos(\psi - \varphi)$$

is valid whenever (f, g) forms an orthonormal system, we infer that

$$\langle B_1 u(k_j + i - 1), u(k_j + i) \rangle = \cos \alpha(k_j + i)$$

holds, for every $j \in \mathbb{Z}_+$ and $1 \leq i < r_j$. Furthermore, it is easy to see that

$$\langle B_1 u(k_{j+1} - 1), u(k_{j+1}) \rangle = \sin(\widetilde{\alpha}(j) + \alpha(k_j + 1, k_{j+1} - 1)) \cos \widetilde{\alpha}(j+1) = \cos \alpha(k_{j+1})$$

is true, for every $j \in \mathbb{Z}_+$. Thus, we have obtained that the equation

$$\langle B_1 u(k-1), u(k) \rangle = \cos \alpha(k) = w_k$$

is fulfilled, for every $k \in \mathbb{N}$.

Taking into account that the vector $B_1 u(k-1)$ is orthogonal to $u(l)$ whenever $l \neq k$ ($k \in \mathbb{N}$), we conclude that the compression of B_1 to the subspace M, spanned by the vectors $\{u(k)\}_{k=0}^{\infty}$, is unitarily equivalent to the unilateral weighted shift W. Q.E.D.

4. Dilation of bilateral weighted shifts

Let us consider the Hilbert space $L^2(E_n)$ of vector-valued functions, defined with respect to the Lebesgue measure μ on $\mathbb{T}$, where E_n is an n-dimensional Hilbert space. The operator $\check{S}_n \in L(L^2(E_n))$ of multiplication by the identical function χ is the n-dimensional bilateral shift.

Let $\{v_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis in the Hilbert space $\check{K}$, indexed by the set $\mathbb{Z}$ of all integers. Given a bounded sequence $\{w_k\}_{k \in \mathbb{Z}}$ of complex numbers, let

$\check{W} \in L(\check{K})$ be defined by $\check{W} v_k := w_k v_{k+1}$ $(k \in \mathbb{Z})$. The bilateral weighted shift $\check{W}$ is a contraction precisely when $|w_k| \leq 1$ holds, for every $k \in \mathbb{Z}$. We may assume without loss of generality that $w_k \in [0,1]$ $(k \in \mathbb{Z})$.

We note that the contraction $\check{W}$ is similar to the unitary operator $\check{S}_1$ if and only if $w_k > 0$ is true for every $k \in \mathbb{Z}$, and $\sum_{k=1}^{\infty}(1-w_k) < \infty$, $\sum_{k=1}^{\infty}(1-w_{-k}) < \infty$ are valid. The following theorem shows that there are operators in the similarity class of unitaries, which can be dilated to B_1.

Theorem 7. *Let $\check{W} \in L(\check{K})$ be the bilateral weighted shift corresponding to the weight sequence $\{w_k\}_{k\in\mathbb{Z}} \subset [0,1]$. If $\sum_{k=1}^{\infty}(1-w_k)^{\frac{1}{2}} = \sum_{k=1}^{\infty}(1-w_{-k})^{\frac{1}{2}} = \infty$, then $\check{W}$ can be dilated to the 1-dimensional backward shift B_1.*

Proof. For any $k \in \mathbb{Z}$, let $\alpha(k) \in [0, \frac{\pi}{2}]$ be defined by $\cos\alpha(k) = w_k$. Let us consider the sequences $\{k_j\}_{j=0}^{\infty}$, $\{\widetilde{\alpha}(j)\}_{j=0}^{\infty}$ and $\{r_j\}_{j=0}^{\infty}$, associated with $\{\alpha(k)\}_{k=1}^{\infty}$ according to the proof of Theorem 6, with initial data $k_0 = 0$ and $\widetilde{\alpha}(0) = 0$. Furthermore, let $\{k_{-j}\}_{j=0}^{\infty}$, $\{\widetilde{\alpha}(-j)\}_{j=0}^{\infty}$ and $\{r_{-j}\}_{j=0}^{\infty}$ be the sequences associated with $\{\alpha(-k)\}_{k=1}^{\infty}$, with initial data $k_{-0} = 0$ and $\widetilde{\alpha}(-0) = \alpha(0)$. (Here we make difference between the indices 0 and -0.) The positive integers $\{n_{\pm j}\}_{j=0}^{\infty}$ are chosen in the following way. We set $n_0 > r_0$ and $n_{-0} := n_0 + 1$. Assuming that $\{n_{\pm i}\}_{i=0}^{j}$ have already been defined, for $j \in \mathbb{Z}_+$, let $n_{j+1} > n_{-j} + r_{-j} + r_{j+1} + 2$ and $n_{-(j+1)} := n_{j+1} + r_j + r_{-j} + 2$.

The vectors $\{u(k)\}_{k=0}^{\infty}$ are defined as in the proof of Theorem 6. On the other hand, for any $j \in \mathbb{Z}_+$, let

$$u(-k_{-j}) := (\cos\widetilde{\alpha}(-j))e(n_{-j}) + (\sin\widetilde{\alpha}(-j))e(n_{-(j+1)} - r_{-j}),$$

and, for any $1 \leq i < r_{-j}$, let

$$\begin{aligned} u(-k_{-j} - i) \quad &:= \quad (\cos(\widetilde{\alpha}(-j) + \alpha(-k_{-j} - 1, -k_{-j} - i))\, e(n_{-j} + i) \\ &+ \quad (\sin(\widetilde{\alpha}(-j) + \alpha(-k_{-j} - 1, -k_{-j} - i))\, e(n_{-(j+1)} - r_{-j} + i). \end{aligned}$$

(For $k, l \in \mathbb{Z}_+, k \leq l,\ \alpha(-k, -l) := \sum_{s=k}^{l} \alpha(-s)$.) The resulting set $\{u(\pm k)\}_{k=0}^{\infty}$ is an orthonormal system in $H^2(E_1)$. Let us consider the subspace $M := M_- \oplus M_+$, where $M_+ := \vee\{u(k)\}_{k=0}^{\infty}$ and $M_- := \vee\{u(-k)\}_{k=0}^{\infty}$.

It is easy to verify that $P_M B_1 u(k-1) = P_{M_+} B_1 u(k-1) = w_k u(k)$ $(k \in \mathbb{N})$ is true. We obtain by symmetry that $P_M S_1 u(-(k-1)) = P_{M_-} S_1 u(-(k-1)) = w_{-k} u(-k)$ $(k \in \mathbb{N})$. Since, for any $l \in \mathbb{Z}$, we have

$$\langle P_M B_1 u(-k), u(l)\rangle = \langle u(-k), P_M S_1 u(l)\rangle = \delta(l, -(k-1)) w_{-k},$$

where $\delta(i,j) := 1$ if $i = j$ and $\delta(i,j) := 0$ otherwise, it follows that $P_M B_1 u(-k) = w_{-k} u(-(k-1))$ $(k \in \mathbb{N})$. Furthermore, the relation $B_1 e(-0) = e(0)$ implies that

$$\begin{aligned} P_M B_1 u(-0) &= P_M B_1 \left((\cos\alpha(0)) e(n_{-0}) + (\sin\alpha(0)) e(n_{-1} - r_{-0})\right) \\ &= (\cos\alpha(0)) e(n_0) = w_0 u(0). \end{aligned}$$

Therefore, the compression of B_1 to M is unitarily equivalent to the bilateral weighted shift $\check{W}$. Q.E.D.

Keeping the previous notation, for any $j \in \mathbb{Z}_+$, let $a_j := n_j - r_j$. Let us also introduce the notation $s_0 := r_0 + r_{-0} + 1$, and $s_j := r_j + r_{-j} + r_{j-1} + r_{-(j-1)} + 2$ for $j \in \mathbb{N}$. We can see that M is included in the subspace $M_H := \vee\{e(n)\}_{n \in H}$, where $H = \mathbb{N} \cap \left(\cup_{j=0}^{\infty} [a_j, a_j + s_j]\right)$. In view of this observation, we can strengthen the statement of the previous theorem.

Corollary 8. *For any $i \in \mathbb{N}$, let $\check{W}_i \in L(\check{K})$ be the bilateral weighted shift corresponding to the weight sequence $\{w_{i,k}\}_{k \in \mathbb{Z}} \subset [0,1]$. If $\sum_{k=1}^{\infty} (1 - w_{i,k})^{\frac{1}{2}} = \infty$ and $\sum_{k=1}^{\infty} (1 - w_{i,-k})^{\frac{1}{2}} = \infty$ for every $i \in \mathbb{N}$, then the orthogonal sum $\sum_{i=1}^{\infty} \oplus \check{W}_i$ can be dilated to B_1.*

Proof. For every $i \in \mathbb{N}$, let $\{r_{i,\pm j}\}_{j=0}^{\infty}$ and $\{s_{i,j}\}_{j=0}^{\infty}$ be the sequences corresponding to the weight sequence $\{w_{i,k}\}_{k \in \mathbb{Z}}$. Let $\tau\colon \mathbb{N} \to \mathbb{N} \times \mathbb{Z}_+$ be a bijection. We set $a_{\tau(1)} \in \mathbb{N}$ arbitrarily. Assuming that $\{a_{\tau(l)}\}_{l=1}^{m}$ have already been defined, for $m \in \mathbb{N}$, let us choose $a_{\tau(m+1)} \in \mathbb{N}$ so that $a_{\tau(m+1)} > a_{\tau(m)} + s_{\tau(m)} + 2$ hold. Having introduced the positive integers $\{a_{i,j} : i \in \mathbb{N}, j \in \mathbb{Z}_+\}$, we can define the numbers $\{n_{i,\pm j} : i \in \mathbb{N}, j \in \mathbb{Z}_+\}$ as follows: $n_{i,j} := a_{i,j} + r_{i,j}$ $(j \in \mathbb{Z}_+)$, $n_{i,-0} := n_{i,0} + 1$ and $n_{i,-j} := n_{i,j} + r_{i,j-1} + r_{i,-(j-1)} + 2$ $(j \in \mathbb{N})$. For every $i \in \mathbb{N}$, let M_i be the subspace of $H^2(E_1)$ constructed with these data in the way described in the proof of Theorem 7. If $i_1 \neq i_2$, then the subspaces M_{i_1} and $B_1 M_{i_1}$ are orthogonal to M_{i_2}. Thus, taking the orthogonal sum $M := \sum_{i=1}^{\infty} \oplus M_i$, we conclude that $\sum_{i=1}^{\infty} \oplus \check{W}_i$ is unitarily equivalent to the compression of B_1 to M. Q.E.D.

Since the unilateral weighted shifts are restrictions of bilateral weighted shifts, an analogous extension of Theorem 6 is also valid.

The assumption $\sum_{k=1}^{\infty} (1 - w_{-k})^{\frac{1}{2}} = \infty$ in Theorem 7 was made for technical reasons. It can be dropped if we increase the dimension of the backward shift.

Theorem 9. *Let $\check{W} \in L(\check{K})$ be the bilateral weighted shift corresponding to the weight sequence $\{w_k\}_{k \in \mathbb{Z}} \subset [0,1]$. If $\sum_{k=1}^{\infty} (1 - w_k)^{\frac{1}{2}} = \infty$, then $\check{W}$ can be dilated to the 3-dimensional backward shift B_3.*

Proof. Fixing an orthonormal basis (e_1, e_2, e_3) in E_3, the system $\{e_i(n) := S_3^n e_i : 1 \leq i \leq 3, n \in \mathbb{Z}_+\}$ will be an orthonormal basis in $H^2(E_3)$. For any $k \in \mathbb{Z}$, let $\alpha(k) \in [0, \frac{\pi}{2}]$ be defined by $\cos\alpha(k) = w_k$. Let us consider the sequences $\{k_j\}_{j=0}^{\infty}$, $\{\widetilde{\alpha}(j)\}_{j=0}^{\infty}$, $\{r_j\}_{j=0}^{\infty}$ and $\{n_j\}_{j=0}^{\infty}$ associated with $\{\alpha(k)\}_{k\in\mathbb{N}}$ in the proof of Theorem 6, with $k_0 = 0$ and $\widetilde{\alpha}(0) = 0$.

The orthonormal sequence $\{u(k)\}_{k\in\mathbb{Z}}$ is defined as follows. Let $u(0) := e_1(n_0)$, and for any $1 \leq i < r_0$, let

$$u(i) := (\cos\alpha(1,i))e_1(n_0 - i) + (\sin\alpha(1,i))e_3(n_1 + r_0 - i).$$

For any $j \in \mathbb{N}$, let

$$u(k_j) := (\cos\widetilde{\alpha}(j))e_3(n_j) + (\sin\widetilde{\alpha}(j))e_3(n_{j+1} + r_j),$$

and, for any $1 \leq i < r_j$, let

$$\begin{aligned} u(k_j + i) \quad := & \quad (\cos(\widetilde{\alpha}(j) + \alpha(k_j + 1, k_j + i)))\, e_3(n_j - i) \\ + & \quad (\sin(\widetilde{\alpha}(j) + \alpha(k_j + 1, k_j + i)))\, e_3(n_{j+1} + r_j - i). \end{aligned}$$

Finally, for any $k \in \mathbb{N}$, let

$$u(-k) := (\cos\alpha(0, -(k-1))) \cdot e_1(n_0 + k) + (\sin\alpha(0, -(k-1))) \cdot e_2(n_0 + k).$$

It is easy to verify that the compression of B_3 to the subspace $M := \vee\{u(k)\}_{k\in\mathbb{Z}}$ is unitarily equivalent to the bilateral weighted shift $\check{W}$. Q.E.D.

In the light of the previous theorems a transparent characterization of all contractions, which can be dilated to B_∞, seems to be out of reach.

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