On the Taylor functional calculus

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Abstract. We give a Martinelli-Vasilescu type formula for the Taylor functional calculus and a simple proof of its basic properties.

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Let $A = (A_1, \dots, A_n)$ be an n-tuple of mutually commuting operators acting on a Banach space X. The existence of the Taylor functional calculus [18], [19], for simpler versions see [10], [8], [3], [4], [5] and [15], is one of the most important results of spectral theory. However, the formula defining $f(A)$ for a function f analytic on a neighbourhood of the Taylor spectrum has some drawbacks. The operator $f(A)$ is defined locally, the formula gives only $f(A)x$ for each $x \in X$. Therefore it is not easy to see that $f(A)$ is bounded. Moreover, the formula is rather inexplicit and it is quite difficult to prove even the basic properties of the calculus.

The situation is better for Hilbert space operators. In [20] and [21], Vasilescu gave an explicit Martinelli-type formula defining $f(A)$ which is much easier to handle.

The ideas of Vasilescu were used in [9] to prove a similar formula for Banach space operators. The method works, however, only for functions analytic on a neighbourhood of the split-spectrum which is in general bigger than the Taylor spectrum. The main tool is the existence of generalized inverses for operators that appear in the Koszul complex. For similar ideas see also [1].

In this paper we obtain a similar formula for the general Taylor functional calculus. The main innovation is the use of non-linear (but continuous) general inverses. In this way we obtain a formula that defines $f(A)$ globally, and so the continuity of $f(A)$ and the continuity of the functional calculus become clear. The formula is more explicit, and so it is possible to avoid some technical difficulties in the proof of the basic properties of the calculus. The cohomogical methods are avoided and the proofs are based only on the Stokes and the Bartle-Graves theorems.

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All Banach spaces in this paper are complex. Denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on a Banach space X.

Definition 1. Let X, Y be Banach spaces. Denote by $\mathcal{H}(X, Y)$ the set of all continuous mappings $f : X \to Y$ that are homogeneous (i.e., $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{C}$ and $x \in X$).

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If $f \in \mathcal{H}(X,Y)$ then $\sup\{\|f(x)\| : x \in X, \|x\| = 1\} < \infty$. Clearly $\mathcal{H}(X,Y)$ with this norm is a Banach space. Write for short $\mathcal{H}(X)$ instead of $\mathcal{H}(X,X)$. Clearly $\mathcal{B}(X) \subset \mathcal{H}(X)$.

Theorem 2. (Bartle-Graves, see [2], Proposition 5.9) *Let M be a closed subspace of a Banach space X and let $\varepsilon > 0$. Then there exists $h \in \mathcal{H}(X/M, X)$ such that $\|h\| < 1+\varepsilon$ and $h(x+M) \in x+M$ for each class $x+M \in X/M$.*

Lemma 3. *Let X,Y be Banach spaces and let $T : X \to Y$ be a bounded linear operator with closed range. Let $f \in \mathcal{H}(Y)$ satisfy $f(Y) \subset \operatorname{Im} T$. Then there exists $g \in \mathcal{H}(Y,X)$ such that $f = Tg$.*

Proof. Let $h : X/\operatorname{Ker} T \to X$ be the selection given by the Bartle-Graves theorem. Let $T_0 : X/\operatorname{Ker} T \to \operatorname{Im} T$ be the operator induced by T. Set $g = hT_0^{-1}f$. For $y \in Y$ we have $Tgy = ThT_0^{-1}fy = fy$, and so $Tg = f$. Q.E.D.

Proposition 4. *Let $X_0, \dots, X_n$ be Banach spaces, let $\delta_j : X_j \to X_{j+1}$ $(j = 0, \dots, n-1)$ be bounded linear operators and suppose that the sequence*

$$0 \longrightarrow X_0 \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} X_n \longrightarrow 0$$

is exact. Let $g_j \in \mathcal{H}(X_j)$ $(j = 0, \dots, n)$. The following statements are equivalent:
(i) $\delta_j g_j = g_{j+1}\delta_j$ $(j = 0, \dots, n-1)$;
(ii) there exist mappings $V_j \in \mathcal{H}(X_{j+1}, X_j)$ $(j = 0, \dots, n-1)$ such that

$$\begin{aligned} &V_0\delta_0 = g_0, \\ &V_j\delta_j + \delta_{j-1}V_{j-1} = g_j \quad (j = 1, \dots, n-1), \\ &\delta_{n-1}V_{n-1} = g_n. \end{aligned}$$

Proof. (ii)⇒(i): Suppose that the mappings V_j satisfy (ii). We have

$$\delta_j g_j = \delta_j(V_j\delta_j + \delta_{j-1}V_{j-1}) = \delta_j V_j \delta_j$$

and

$$g_{j+1}\delta_j = (V_{j+1}\delta_{j+1} + \delta_j V_j)\delta_j = \delta_j V_j \delta_j$$

(note that the same relations are true also for $j = 0$ and $j = n-1$). Thus $\delta_j g_j = g_{j+1}\delta_j$ for all j.

(i)⇒(ii): Since δ_{n-1} is onto, there exists V_{n-1} such that $\delta_{n-1}V_{n-1} = g_n$.

We construct mappings V_j inductively. Suppose that $1 \le j \le n-1$ and that $V_j \in \mathcal{H}(X_{j+1}, X_j)$ satisfies $V_{j+1}\delta_{j+1} + \delta_j V_j = g_{j+1}$ (for $j = n-1$ set formally $V_n = 0$ and $\delta_n = 0$). We have

$$\delta_j(g_j - V_j\delta_j) = g_{j+1}\delta_j - \delta_j V_j \delta_j = g_{j+1}\delta_j - (g_{j+1} - V_{j+1}\delta_{j+1})\delta_j = 0.$$

Thus $(g_j - V_j\delta_j)(X_j) \subset \operatorname{Ker}\delta_j = \operatorname{Im}\delta_{j-1}$ and there exists $V_{j-1} \in \mathcal{H}(X_j, X_{j-1})$ such that $\delta_{j-1}V_{j-1} = g_j - V_j\delta_j$. Thus $V_j\delta_j + \delta_{j-1}V_{j-1} = g_j$.

At the end, suppose that $V_0 \in \mathcal{H}(X_1, X_0)$ satisfies $g_1 = V_1\delta_1 + \delta_0 V_0$. Then $\delta_0 V_0 \delta_0 = (g_1 - V_1\delta_1)\delta_0 = g_1\delta_0 = \delta_0 g_0$. Since δ_0 is one-to-one, we have $V_0\delta_0 = g_0$. This finishes the proof. Q.E.D.

We recall now the basic notations of Taylor [18].

Denote by $\Lambda[s]$ the complex exterior algebra generated by the indeterminates $s = (s_1, \ldots, s_n)$. Then

$$\Lambda[s] = \bigoplus_{p=0}^{n} \Lambda^p[s],$$

where $\Lambda^p[s]$ is the set of all elements of degree p in $\Lambda[s]$. Thus the elements of $\Lambda^p[s]$ are of form

$$\sum_{1 \le i_1 < \cdots < i_p \le n} \alpha_{i_1,\ldots,i_p} s_{i_1} \wedge \cdots \wedge s_{i_p}$$

where $\alpha_{i_1,\ldots,i_p}$ are complex numbers. The multiplication operation $\wedge$ is anticommutative, $s_i \wedge s_j = -s_j \wedge s_i$ for all i, j. In particular $s_i \wedge s_i = 0$. Clearly $\dim \Lambda^p[s] = \binom{n}{p}$ and $\dim \Lambda[s] = 2^n$.

Let X be a Banach space. Then we write $\Lambda[s, X] = X \otimes \Lambda[s]$ and $\Lambda^p[s, X] = X \otimes \Lambda^p[s]$. Thus the elements of $\Lambda^p[s, X]$ are of form

$$\sum_{1 \le i_1 < \cdots < i_p \le n} x_{i_1,\ldots,i_p} s_{i_1} \wedge \cdots \wedge s_{i_p}$$

where $x_{i_1,\ldots,i_p} \in X$ (the symbol $\otimes$ is omitted in order to simplify the notation).

Let $A = (A_1, \ldots, A_n)$ be an $n-$tuple of mutually commuting operators in X. Define operator $\delta_A : \Lambda[s, X] \to \Lambda[s, X]$ by

$$\delta_A(x s_{i_1} \wedge \cdots \wedge s_{i_p}) = \sum_{j=1}^{n} (A_j x) s_j \wedge s_{i_1} \wedge \cdots \wedge s_{i_p}.$$

Write $\delta_A^p = \delta_A | \Lambda^p[s, X]$. The Koszul complex $\mathcal{K}(A)$ is the sequence

$$0 \longrightarrow \Lambda^0[s, X] \xrightarrow{\delta_A^0} \Lambda^1[s, X] \xrightarrow{\delta_A^1} \cdots \xrightarrow{\delta_A^{n-1}} \Lambda^n[s, X] \longrightarrow 0.$$

Then $(\delta_A)^2 = 0$, i.e., $\delta_A^p \delta_A^{p-1} = 0$ for all p. It is convenient to set formally $\Lambda^{-1}[s, X] = \Lambda^{n+1}[s, X] = 0$; similarly let δ_A^{-1} and δ_A^n be the zero operators.

We say that the $n-$tuple $A = (A_1, \ldots, A_n)$ is Taylor-regular if the Koszul complex $\mathcal{K}(A)$ is exact (i.e., $\operatorname{Im} \delta_A = \operatorname{Ker} \delta_A$). The Taylor spectrum $\sigma_T(A)$ is the set of all $n-$tuples $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $A - \lambda = (A_1 - \lambda_1, \ldots, A_n - \lambda_n)$ is not Taylor-regular. It is well-known that $\sigma_T(A)$ is a nonempty compact subset of $\mathbb{C}^n$. Further, the Taylor spectrum satisfies the projection property, see [18], [16].

Let $A = (A_1, \ldots, A_n)$ be a Taylor-regular n-tuple of operators. By Proposition 4, there are "generalized inverses" $V_j \in \mathcal{H}\big(\Lambda^{j+1}[s, X], \Lambda^j[s, X]\big)$ such that $\delta_A^{j-1} V_{j-1} + V_j \delta_A^j = I_{\Lambda^j[s,X]}$. In a simpler form, we have $\delta_A V + V\delta_A = I_{\Lambda[s,X]}$ where $V \in \mathcal{H}(\Lambda[s, X])$ is defined by $V(\oplus_{j=0}^n \psi_j) = \oplus_{j=1}^n V_{j-1}\psi_j \quad (\psi_j \in \Lambda^j[s, X])$.

Our first goal is to show that it is possible to find such generalized inverses depending smoothly on $z \in \mathbb{C}^n \setminus \sigma_T(A)$.

Proposition 5. *Let $A = (A_1, \dots, A_n)$ be an $n-$tuple of mutually commuting operators on a Banach space X. Let $G = \mathbb{C}^n \setminus \sigma_T(A)$. Then there exists a C^∞- function $V : G \to \mathcal{H}(\Lambda[s, X])$ such that $\delta_{A-z} V(z) + V(z)\delta_{A-z} = I_{\Lambda[s,X]}$, and*

$$V(z)\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X] \qquad (z \in G, p = 0, \dots, n).$$

Proof. Consider Banach spaces

$$M_1 = \bigoplus_{j=0}^{n-1} \mathcal{H}\big(\Lambda^{j+1}[s, X], \Lambda^j[s, X]\big),$$

$$M_2 = \bigoplus_{j=0}^{n} \mathcal{H}\big(\Lambda^j[s, X]\big) \text{ and}$$

$$M_3 = \bigoplus_{j=0}^{n-1} \mathcal{H}\big(\Lambda^j[s, X], \Lambda^{j+1}[s, X]\big).$$

For $z \in G$ define mappings $\Phi(z) : M_1 \to M_2$ and $\Psi(z) : M_2 \to M_3$ by

$$\Phi(z)\Big(\bigoplus_{j=0}^{n-1} V_j\Big) = V_0\delta^0_{A-z} \oplus \bigoplus_{j=1}^{n-1}\big(V_j\delta^j_{A-z} + \delta^{j-1}_{A-z}V_{j-1}\big) \oplus \delta^{n-1}_{A-z}V_{n-1}$$

and

$$\Psi(z)\Big(\bigoplus_{j=0}^{n} g_j\Big) = \bigoplus_{j=0}^{n-1}\big(\delta^j_{A-z}g_j - g_{j+1}\delta^j_{A-z}\big).$$

Clearly $\Phi(z)$ and $\Psi(z)$ are bounded linear operators depending analytically on $z \in G$ and, by Proposition 4, $\operatorname{Im}\Phi(z) = \operatorname{Ker}\Psi(z)$. Further $I_{\Lambda[s,X]} = \oplus I_{\Lambda^i[s,X]} \in \operatorname{Im}\Phi(z)$ for all $z \in G$.

Let $\lambda \in G$. By [18], Lemma 2.2, cf. also [17], there is a neighbourhood U_λ of λ and an analytic function $V_\lambda : U_\lambda \to M_1$ such that $\Phi(z)V_\lambda(z) = I \quad (z \in U_\lambda)$.

Let $\{\varphi_i\}_{i=1}^\infty$ be a $C^\infty-$partition of unity subordinated to the cover $\{U_\lambda, \lambda \in G\}$ of G, i.e. φ_i's are C^∞-functions, $0 \le \varphi_i \le 1$, $\operatorname{supp}\varphi_i \subset U_{\lambda_i}$ for some $\lambda_i \in G$, for each $\lambda \in G$ there exists a neighbourhood U of λ such that all but finitely many of φ_i's are 0 on U and $\sum_{i=1}^\infty \varphi_i(z) = 1$ for each $z \in G$.

For $z \in G$ set $V(z) = \sum_{i=1}^\infty \varphi_i(z)V_{\lambda_i}(z)$. Clearly V is a C^∞-function satisfying $V(z)\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X]$ and $\Phi(z)V(z) = I$ for all $z \in G$. Q.E.D.

Remark 6. (i) Function Φ is regular in G (i.e., $\operatorname{Im}\Phi(z)$ changes continuously). The existence of a C^∞-function V satisfying $\Phi(z)V(z) = I$ follows also directly from a deep result of Mantlik [11]. The present argument, however, is more elementary.

(ii) It is possible to require also that $V(z)^2 = 0$ and $V(z)\delta_{A-z}V(z) = V(z)$ for all $z \in G$. In particular, $V(z)$ is a generalized inverse of δ_{A-z}.

Indeed, let $V : G \to \mathcal{H}(\Lambda[s, X])$ be the function constructed in Proposition 5, i.e., $\delta_{A-z}V(z) + V(z)\delta_{A-z} = I$ and $V(z)\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X]$.

Clearly $\delta_{A-z}V(z)\delta_{A-z} = \delta_{A-z}$. Set $V'(z) = V(z)\delta_{A-z}V(z)$. Then

$$\delta_{A-z}V'(z)\delta_{A-z} = \delta_{A-z}V(z)\delta_{A-z}V(z)\delta_{A-z} = \delta_{A-z}$$

and

$$V'(z)\delta_{A-z}V'(z) = V(z)\delta_{A-z}V(z)\delta_{A-z}V(z)\delta_{A-z}V(z) = V(z)\delta_{A-z}V(z) = V'(z).$$

Further

$$\begin{aligned}&\delta_{A-z}V'(z) + V'(z)\delta_{A-z} = \delta_{A-z}V(z)\delta_{A-z}V(z) + V(z)\delta_{A-z}V(z)\delta_{A-z}\\ &= \delta_{A-z}V(z) + V(z)\delta_{A-z} = I.\end{aligned}$$

Finally we have

$$V'(z) = \big(V'(z)\delta_{A-z} + \delta_{A-z}V'(z)\big)V'(z) = V'(z) + \delta_{A-z}V'(z)^2,$$

and so $\delta_{A-z}V'(z)^2 = 0$. Thus $V'(z)^2 = \big(V'(z)\delta_{A-z} + \delta_{A-z}V'(z)\big)V'(z)^2 = 0$.

These additional properties of the generalized inverse V, however, are not essential for our purpose and we are not going to use them in the sequel.

In the following we fix a commuting n-tuple $A = (A_1, \dots, A_n)$ of bounded linear operators on a Banach space X, the set $G = \mathbb{C}^n \setminus \sigma_T(A)$ and a C^∞-function $V : G \to \mathcal{H}(\Lambda[s, X])$ with the properties of Proposition 5.

Consider the space $C^\infty(G, \Lambda[s, X])$. Clearly this space can be identified with the set $\Lambda[s, C^\infty(G, X)]$.

Function $V : G \to \mathcal{H}(\Lambda[s, X])$ induces naturally the operator (denoted by the same symbol) $V : C^\infty(G, \Lambda[s, X]) \to C^\infty(G, \Lambda[s, X])$ by

$$(Vy)(z) = V(z)y(z) \qquad \big(z \in G, y \in C^\infty(G, \Lambda[s, X])\big).$$

Similarly we define operator δ_{A-z} (or δ for short if no ambiguity can arise) acting in $C^\infty(G, \Lambda[s, X])$ by

$$(\delta y)(z) = \delta_{A-z}\, y(z) \qquad \big(z \in G, y \in C^\infty(G, \Lambda[s, X])\big).$$

Clearly $\delta^2 = 0$, $V\delta + \delta V = I_{\Lambda[s, C^\infty(G,X)]}$ and both V and δ are "graded", i.e.

$$\begin{aligned}&V\Lambda^p[s, C^\infty(G, X)] \subset \Lambda^{p-1}[s, C^\infty(G, X)] \qquad \text{and}\\ &\delta\Lambda^p[s, C^\infty(G, X)] \subset \Lambda^{p+1}[s, C^\infty(G, X)].\end{aligned}$$

Consider now another indeterminates $\mathrm{d}\bar{z} = (\mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_n)$ and the space $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$. Define the linear operator

$$\bar{\partial} : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$$

by

$$\bar{\partial} f s_{i_1} \wedge \dots \wedge s_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{j_q} = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} \mathrm{d}\bar{z}_k \wedge s_{i_1} \wedge \dots \wedge s_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{j_q}.$$

Clearly $\bar{\partial}^2 = 0$.

Operators V and δ can be lifted from $\Lambda[s, C^\infty(G,X)]$ to $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)]$ by

$$V(\psi \wedge \mathrm{d}\bar{z}_{i_1} \wedge \dots \mathrm{d}\bar{z}_{i_p}) = (V\psi) \wedge \mathrm{d}\bar{z}_{i_1} \wedge \dots \mathrm{d}\bar{z}_{i_p} \qquad \text{and}$$
$$\delta(\psi \wedge \mathrm{d}\bar{z}_{i_1} \wedge \dots \mathrm{d}\bar{z}_{i_p}) = (\delta\psi) \wedge \mathrm{d}\bar{z}_{i_1} \wedge \dots \mathrm{d}\bar{z}_{i_p}$$

for all $\psi \in \Lambda[s, C^\infty(G,X)]$. Clearly the properties of V and δ are preserved: $\delta^2 = 0$, $V\delta + \delta V = I$ and both V and δ are graded. Note also that $\delta\bar{\partial} = -\bar{\partial}\delta$ and $(\bar{\partial} + \delta)^2 = 0$.

Let $W : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)]$ be the mapping defined in the following way: if $\psi \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)]$, $\psi = \psi_0 + \dots + \psi_n$ where ψ_j is the part of ψ of degree j in $\mathrm{d}\bar{z}$, then set $W\psi = \eta_0 + \dots + \eta_n$ where

$$\begin{aligned}
\eta_0 &= V\psi_0, \\
\eta_1 &= V(\psi_1 - \bar{\partial}\eta_0), \\
&\vdots \\
\eta_n &= V(\psi_n - \bar{\partial}\eta_{n-1}).
\end{aligned} \tag{1}$$

Note that η_j is the part of $W\psi$ of degree j in $\mathrm{d}\bar{z}$.

Lemma 7. *Let $W : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)]$ be the mapping defined above. Then:*
(i) $\operatorname{supp} W\psi \subset \operatorname{supp}\psi$ *for all* ψ*;*
(ii) if G' is an open subset of G and $\psi \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G,X)]$ satisfies $(\bar{\partial} + \delta)\psi = 0$ on G', then $(\bar{\partial} + \delta)W\psi = \psi$ on G';
(iii) $(\bar{\partial} + \delta)W(\bar{\partial} + \delta) = \bar{\partial} + \delta$.

Proof. (i) Clear.

(ii) Let $\psi = \psi_0 + \dots + \psi_n$ where ψ_j is the part of ψ of degree j in $\mathrm{d}\bar{z}$. Condition $(\bar{\partial} + \delta)\psi = 0$ on G' can be rewritten as

$$\begin{aligned}
\delta\psi_0 &= 0, \\
\bar{\partial}\psi_0 + \delta\psi_1 &= 0, \\
&\vdots \\
\bar{\partial}\psi_{n-1} + \delta\psi_n &= 0
\end{aligned} \tag{2}$$

(condition $\bar{\partial}\psi_n = 0$ is satisfied automatically).

Let $W\psi = \eta_0 + \dots + \eta_n$ where η_j are defined by (1). The required condition $(\bar{\partial} + \delta)W\psi = \psi$ then becomes

$$\begin{aligned}
\delta\eta_0 &= \psi_0, \\
\bar{\partial}\eta_0 + \delta\eta_1 &= \psi_1, \\
&\vdots \\
\bar{\partial}\eta_{n-1} + \delta\eta_n &= \psi_n
\end{aligned} \tag{3}$$

on G' (again, $\bar\partial\eta_n = 0$ automatically).

By (1) and (2), we have $\delta\eta_0 = \delta V\psi_0 = (\delta V + V\delta)\psi_0 = \psi_0$ and $\bar\partial\eta_0 + \delta\eta_1 = \bar\partial\eta_0 + \delta V(\psi_1 - \bar\partial\eta_0) = \bar\partial\eta_0 + (I - V\delta)(\psi_1 - \bar\partial\eta_0)\bar\partial\eta_0 = \psi_1 - V\delta(\psi_1 - \bar\partial\eta_0) = \psi_1$ since $\delta(\psi_1 - \bar\partial\eta_0) = \delta\psi_1 + \bar\partial\delta\eta_0 = \delta\psi_1 + \bar\partial\psi_0 = 0$.

We prove (3) by induction. Suppose that $\bar\partial\eta_{j-1} + \delta\eta_j = \psi_j$ for some $j \geq 1$. Then $\delta(\psi_{j+1} - \bar\partial\eta_j) = \delta\psi_{j+1} + \bar\partial\delta\eta_j = \delta\psi_{j+1} + \bar\partial\psi_j = 0$ by the induction assumption, and $\bar\partial\eta_j + \delta\eta_{j+1} = \bar\partial\eta_j + \delta V(\psi_{j+1} - \bar\partial\eta_j) = \bar\partial\eta_j + (I - V\delta)(\psi_{j+1} - \bar\partial\eta_j) = \psi_{j+1}$.

(iii) Since $(\bar\partial + \delta)^2 = 0$, the statement follows from (ii). Q.E.D.

Remark 8. Without any change it is possible to prove the preceding theorem in a more general form. Let $z \mapsto A(z)$ be an analytic function defined on an open subset $G \subset \mathbb{C}^n$ such that the values $A(z)$ are Taylor regular n-tuples of operators on X for all $z \in G$. Let $\psi \in \Lambda[s, \mathrm{d}\bar z, C^\infty(G, X)]$ satisfy $(\bar\partial + \delta_{A(z)})\psi = 0$. Then there exists a form $\theta \in \Lambda[s, \mathrm{d}\bar z, C^\infty(G, X)]$ with $\operatorname{supp}\theta \subset \operatorname{supp}\psi$ and $\psi = (\bar\partial + \delta_{A(z)})\theta$.

We interpret the differential form

$$(2i)^{-n}\mathrm{d}\bar z_1 \wedge \cdots \wedge \mathrm{d}\bar z_n \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{4}$$

as the Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$.

Let P be the natural projection $P : \Lambda[s, \mathrm{d}\bar z, C^\infty(G, X)] \to \Lambda[\mathrm{d}\bar z, C^\infty(G, X)]$ that annihilates all terms containing at least one of the indeterminates $s_1, \ldots, s_n$ and leaves invariant all the remaining terms.

The following simple lemma will be used frequently.

Proposition 9. *Let $\eta \in \Lambda^n[s, d\bar z, C^\infty(G, X)]$ be a differential form with a compact support disjoint with $\sigma_T(A)$ such that $(\bar\partial + \delta)\eta = 0$. Then*

$$\int_{\mathbb{C}^n} P\eta \wedge \mathrm{d}z = 0$$

where $\mathrm{d}z$ *stands for* $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$.

Proof. We have

$$P\eta = P(\bar\partial + \delta)W\eta = P\bar\partial W\eta = \bar\partial PW\eta$$

where $PW\eta$ has a compact support. By the Stokes theorem, we have

$$\int_{\mathbb{C}^n} P\eta \wedge \mathrm{d}z = \int_{\mathbb{C}^n} \bar\partial PW\eta \wedge \mathrm{d}z = 0.$$

Q.E.D.

Let U be a neighbourhood of $\sigma_T(A)$. It is possible to find a compact neighbourhood Δ of $\sigma_T(A)$ such that $\Delta \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Let f be a function analytic in U. Define operator $f(A)$ by

$$f(A)x = \frac{-1}{(2\pi i)^n}\int_{\partial\Delta} PWf(z)xs \wedge \mathrm{d}z \qquad (x \in X), \tag{5}$$

where $\mathrm{d}z$ stands for $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and $s = s_1 \wedge \cdots \wedge s_n$. By the Stokes formula,

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_\Delta \bar{\partial}\varphi PWf(z)xs \wedge \mathrm{d}z$$

where φ is a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ and to 1 on $\mathbb{C}^n \setminus \Delta$ (consequently, $\varphi = 1$ also on $\partial\Delta$).

On $\mathbb{C}^n \setminus \Delta$ we have

$$\bar{\partial}\varphi PWfxs = P(\bar{\partial} + \delta)Wfxs = Pfxs = 0.$$

Thus we can write

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi PWf(z)xs \wedge \mathrm{d}z. \tag{6}$$

It is clear from the Stokes theorem that the definition of $f(A)x$ does not depend on the choice of function φ and, by (6), it is independent of Δ.

We show that $f(A)$ does not depend on the choice of the generalized inverse V which determines W.

Suppose that W_1, W_2 are two operators satisfying

$$(\bar{\partial} + \delta)W_i f(z)xs = f(z)xs \qquad (i = 1, 2).$$

For those z where $\varphi \equiv 1$ we have

$$(\bar{\partial} + \delta)\varphi(W_1 - W_2)f(z)xs = 0,$$

and so the form $(\bar{\partial} + \delta)\varphi(W_1 - W_2)f(z)xs$ satisfies the conditions of Proposition 9. Hence

$$0 = \int_{\mathbb{C}^n} P(\bar{\partial} + \delta)\varphi(W_1 - W_2)f(z)xs \wedge \mathrm{d}z = \int_{\mathbb{C}^n} P\bar{\partial}\varphi(W_1 - W_2)f(z)xs \wedge \mathrm{d}z =$$

$$= \int_{\mathbb{C}^n} \bar{\partial}\varphi PW_1 f(z)xs \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar{\partial}\varphi PW_2 f(z)xs \wedge \mathrm{d}z.$$

It is possible to express the mapping PW that appears in the definition of the functional calculus more explicitly. By the definition of W, we have

$$PWxs = (-1)^{n-1}V(\bar{\partial}V)^{n-1}xs = (-1)^{n-1}V_0\bar{\partial}V_1\bar{\partial}\cdots\bar{\partial}V_{n-1}xs.$$

Since $\Lambda[s, X]$ is a direct sum of 2^n copies of X, we can express $V(z) : \Lambda[s, X] \to \Lambda[s, X]$ in the matrix form whose entries are elements of $\mathcal{H}(X)$ depending smoothly on $z \in G$.

Clearly we can write $PWxs = \sum_{i=1}^n M^{(i)}x\mathrm{d}\bar{z}_1 \wedge \cdots \widehat{\mathrm{d}\bar{z}_i} \cdots \wedge \mathrm{d}\bar{z}_n$ for certain functions $M^{(i)} \in C^\infty\big(G, \mathcal{H}(X)\big)$ where the hat denotes the omitted term.

Thus we can write formulas (5) and (6) also globally:

$$\begin{aligned} f(A) &= \frac{-1}{(2\pi i)^n} \int_{\partial\Delta} PWf(z)Is \wedge \mathrm{d}z = \frac{-1}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi PWf(z)Is \wedge \mathrm{d}z \\ &= \frac{(-1)^n}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi V(\bar{\partial}V)^{n-1} f(z)Is \wedge \mathrm{d}z \end{aligned} \tag{7}$$

where $I = I_X$ is the identity operator on X. The coefficients of forms in (7) are $\mathcal{H}(X)$-valued C^∞-functions. Therefore $f(A) \in \mathcal{H}(X)$.

Lemma 10. *$f(A)$ is a bounded linear operator.*

Proof. Since $f(A) \in \mathcal{H}(X)$, it is sufficient to show only the additivity. Let $x, y \in X$. Let φ be a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ such that $\operatorname{supp}(1-\varphi)$ is compact. Then

$$\begin{aligned}
&-(2\pi i)^n \big(f(A)(x+y) - f(A)x - f(A)y\big) \\
&= \int_{\mathbb{C}^n} \bar\partial\varphi PWf \cdot (x+y)s \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar\partial\varphi PWfxs \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar\partial\varphi PWfys \wedge \mathrm{d}z \\
&= \int_{\mathbb{C}^n} P\eta \wedge \mathrm{d}z
\end{aligned}$$

where

$$\eta = (\bar\partial + \delta)\varphi Wf \cdot (x+y)s - (\bar\partial+\delta)\varphi Wfxs - (\bar\partial+\delta)\varphi Wfys.$$

Clearly η has a compact support disjoint with $\sigma_T(A)$ and $(\bar\partial+\delta)\eta = 0$. By Proposition 9, $\int P\eta \wedge \mathrm{d}z = 0$ and $f(A)(x+y) = f(A)x + f(A)y$. Q.E.D.

Proposition 11. *For $n = 1$ the functional calculus defined by (7) coincides with the classical functional calculus given by the Cauchy formula.*

Proof. Let $A \in \mathcal{B}(X)$ and let f be a function analytic on a neighbourhood of $\sigma(A)$. Then $Wxs = Vxs = (A-z)^{-1}x$. Thus, for a suitable contour Σ surrounding $\sigma(A)$, we have

$$f(A) = \frac{-1}{2\pi i}\int_\Sigma PWfIs \wedge \mathrm{d}z = \frac{-1}{2\pi i}\int_\Sigma (A-z)^{-1} f(z) I \mathrm{d}z = \frac{1}{2\pi i}\int_\Sigma f(z)(z-A)^{-1}\mathrm{d}z,$$

which is the Cauchy formula. Q.E.D.

Proposition 12. *Let f be a function analytic on a neighbourhood of $\sigma_T(A)$, $1 \le j \le n$ and $g(z) = z_j f(z)$. Then $g(A) = A_j f(A)$.*

Proof. The statement is well-known for $n = 1$. Suppose that $n \ge 2$. Then

$$\begin{aligned}
-(2\pi i)^n \big(A_j f(A) - g(A)\big) &= A_j \int_{\mathbb{C}^n} \bar\partial\varphi PWfIs \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar\partial\varphi PWz_j fIs \wedge \mathrm{d}z \\
&= \int_{\mathbb{C}^n} \bar\partial\varphi f \cdot (A_j - z_j) PWIs \wedge \mathrm{d}z.
\end{aligned}$$

For $F \subset \{1, \dots, n\}$, $F = \{i_1, \dots, i_p\}$ where $i_1 < i_2 < \cdots < i_p$ write $s_F = s_{i_1} \wedge \cdots \wedge s_{i_p}$. Express $WIs \in \Lambda^{n-1}\big[s, \mathrm{d}\bar z, C^\infty(G, X)\big]$ as

$$WIs = \sum_{F \subset \{1,\dots,n\}} s_F \wedge \xi_F$$

where ξ_F contains no variable from $s_1, \dots, s_n$. Since $(\bar\partial + \delta_{A-z})WIs = Is$, for each $F \ne \{1, \dots, n\}$ we have

$$\bar\partial \xi_F + \sum_{k \in F} (-1)^{\operatorname{card}\{k' \in F : k' < k\}} (A_k - z_k)\xi_{F \setminus \{k\}} = 0.$$

In particular, for $F=\{j\}$ we have

$$(A_j-z_j)PWIs=(A_j-z_j)\xi_\emptyset=-\bar\partial\xi_{\{j\}}.$$

Thus

$$\int_{\mathbb{C}^n}\bar\partial\varphi f\cdot(A_j-z_j)PWIs\wedge \mathrm{d}z=-\int_{\mathbb{C}^n}\bar\partial\varphi f\bar\partial\xi_{\{j\}}\wedge \mathrm{d}z$$
$$=-\int_{\mathbb{C}^n}\bar\partial\big(\varphi\bar\partial f\xi_{\{j\}}-\bar\partial\varphi f\xi_{\{j\}}\big)\wedge \mathrm{d}z=0$$

by the Stokes theorem. Hence $g(A)=A_jf(A)$. Q.E.D.

Proposition 13. *Let $A=(A_1,\dots,A_n)\in\mathcal{B}(X)^n$, $B=(B_1,\dots,B_m)\in\mathcal{B}(X)^m$. Suppose that $(A,B)=(A_1,\dots,A_n,B_1,\dots,B_m)$ is a commuting $(n+m)$-tuple and let f and g be functions analytic on a neighbourhood of $\sigma_T(A)$ and $\sigma_T(B)$, respectively. Define function h by $h(z,w)=f(z)\cdot g(w)$. Then $h(A,B)=g(B)f(A)$.*

Proof. Write $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_m)$. Denote by $\bar\partial_z$, $\bar\partial_w$ and $\bar\partial_{z,w}$ the $\bar\partial$ operator corresponding to z,w and (z,w), respectively. We associate with B another system $t=(t_1,\dots,t_m)$ of exterior indeterminates when defining the operator δ_{B-w}.

Choose mappings W_A,W_B and $W_{A,B}$ corresponding to the tuples A,B and (A,B). Let Δ' and Δ'' be compact neighbourhoods of $\sigma_T(A)$ and $\sigma_T(B)$ contained in the domains of definition of f and g, respectively. Let φ,ψ and χ be C^∞-functions equal to 0 on a neighbourhood of $\sigma_T(A)$ $\big(\sigma_T(B)$ and $\sigma_T(A,B)\big)$, and to 1 on $\mathbb{C}^n\setminus\Delta'$ $\big(\mathbb{C}^m\setminus\Delta''$ and $\mathbb{C}^{n+m}\setminus\Delta'\times\Delta''$, respectively$\big)$.

Denote by P_s and P_t the projections which annihilate all terms containing at least one of variables $s_1,\dots,s_n$ ($t_1,\dots,t_m$, respectively) and leave invariant the remaining terms. Set $P=P_sP_t$.

Let $x\in X$. We have

$$f(A)x=\frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n}\bar\partial_z\varphi P_sW_Afxs\wedge\mathrm{d}z=\frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n}P_s\xi\wedge\mathrm{d}z$$

where $\xi=(\bar\partial_z+\delta_{A-z})\varphi W_Afxs-fxs$. On $\mathbb{C}^n\setminus\Delta'$ we have $\varphi\equiv1$ and so $\xi\equiv0$. Thus $\operatorname{supp}\xi$ is compact. Further

$$g(B)f(A)x=\frac{1}{(2\pi i)^{n+m}}\int_{\mathbb{C}^m}P_t(\bar\partial_w+\delta_{B-w})\psi W_Bg\Big(\int_{\mathbb{C}^n}P_s\xi\wedge\mathrm{d}z\Big)t\wedge\mathrm{d}w.$$

Since W_B is not linear, we cannot interchange it with the inner integral. However, consider the form

$$\eta=(\bar\partial_w+\delta_{B-w})\psi W_Bg\Big(\int_{\mathbb{C}^n}P_s\xi\wedge\mathrm{d}z\Big)t-(\bar\partial_w+\delta_{B-w})\psi\int_{\mathbb{C}^n}W_B\big(P_sg\xi\wedge\mathrm{d}z\wedge t\big)$$

where W_B is extended to $\Lambda\big[\mathrm{d}\bar z,\mathrm{d}z,t,\mathrm{d}\bar w,C^\infty(\mathbb{C}^n\times(\mathbb{C}^m\setminus\sigma_T(B)),X)\big]$ in the obvious way. Clearly $(\bar\partial_w+\delta_{B-w})\eta=0$ and $\operatorname{supp}\eta$ is disjoint with $\sigma_T(B)$. On $\mathbb{C}^m\setminus\Delta''$ we have $\psi\equiv1$ and $\eta\equiv0$ since $\operatorname{supp}W_B(P_sg\xi\wedge\mathrm{d}z\wedge t)\subset\operatorname{supp}\xi\times\mathbb{C}^m$ and we can interchange $\bar\partial_w$ with the second integral. Thus $\int_{\mathbb{C}^m}P_t\eta\wedge\mathrm{d}w=0$ and we have

$$(2\pi i)^{n+m}g(B)f(A)x=\int_{\mathbb{C}^m}P_t(\bar\partial_w+\delta_{B-w})\psi\int_{\mathbb{C}^n}W_B\big(P_sg\xi\wedge\mathrm{d}z\wedge t\big)\wedge\mathrm{d}w.\tag{8}$$

On the other hand, $-(2\pi i)^{m+n}h(A,B)x = \int P\eta_1 \wedge \mathrm{d}z \wedge \mathrm{d}w$ where

$$\eta_1 = (\bar\partial_{z,w} + \delta_{A-z,B-w})\chi W_{A,B} hxs \wedge t - hxs \wedge t.$$

Clearly $\operatorname{supp}\eta_1$ is compact.

Set

$$\eta_2 = (\bar\partial_{z,w} + \delta_{A-z,B-w})\psi W_{A,B} g\xi \wedge t - g\xi \wedge t.$$

Clearly $\operatorname{supp}\eta_2 \subset \operatorname{supp}\xi \times \mathbb{C}^m$. Moreover, if $\psi \equiv 1$ then $\eta_2 \equiv 0$, so $\operatorname{supp}\eta_2$ is compact. On a neighbourhood of $\sigma_T(A,B)$ we have $\eta_2 = -g\xi \wedge t = fgxs \wedge t = -\eta_1$. By Proposition 9, we have $\int P(\eta_1 + \eta_2) \wedge \mathrm{d}z \wedge \mathrm{d}w = 0$ and so

$$\begin{aligned}(2\pi i)^{m+n}h(A,B)x &= \int_{\mathbb{C}^{n+m}} P\eta_2 \wedge \mathrm{d}z \wedge \mathrm{d}w \\ &= (-1)^{mn}\int_{\mathbb{C}^m}\Big(\int_{\mathbb{C}^n} P_t(\bar\partial_{z,w} + \delta_{B-w})\psi P_s W_{A,B} g\xi \wedge t \wedge \mathrm{d}z\Big) \wedge \mathrm{d}w\end{aligned}$$

by the Fubini theorem (the factor $(-1)^{mn}$ is caused by convention (4) defining the Lebesgue measures in $\mathbb{C}^n$, $\mathbb{C}^m$ and $\mathbb{C}^{m+n}$, respectively). By the Stokes theorem we have

$$(2\pi i)^{m+n}h(A,B)x = \int_{\mathbb{C}^m} P_t(\bar\partial_w + \delta_{B-w})\psi\Big(\int_{\mathbb{C}^n} P_s W_{A,B} g\xi \wedge \mathrm{d}z \wedge t\Big) \wedge \mathrm{d}w.$$

Consider the form

$$\eta_3 = (\bar\partial_w + \delta_{B-w})\psi \int_{\mathbb{C}^n} P_s W_{A,B} g\xi \wedge \mathrm{d}z \wedge t - (\bar\partial_w + \delta_{B-w})\psi \int_{\mathbb{C}^n} W_B(P_s g\xi \wedge \mathrm{d}z \wedge t).$$

Clearly $\operatorname{supp}\eta_3 \cap \sigma_T(B) = \emptyset$ and $(\bar\partial_w + \delta_{B-w})\eta_3 = 0$. If $\psi \equiv 1$ then, by the Stokes theorem,

$$\begin{aligned}\eta_3 = &\int_{\mathbb{C}^n} P_s(\bar\partial_{z,w} + \delta_{A-z,B-w})W_{A,B} g\xi \wedge \mathrm{d}z \wedge t - \int_{\mathbb{C}^n} \bar\partial_z P_s W_{A,B} g\xi \wedge \mathrm{d}z \wedge t \\ &- \int_{\mathbb{C}^n} P_s g\xi \wedge \mathrm{d}z \wedge t = \int_{\mathbb{C}^n} P_s g\xi \wedge \mathrm{d}z \wedge t - \int_{\mathbb{C}^n} P_s g\xi \wedge \mathrm{d}z \wedge t = 0.\end{aligned}$$

Thus $\int P_t\eta_3 \wedge \mathrm{d}w = 0$ and

$$\begin{aligned}(2\pi i)^{n+m}h(A,B)x &= \int_{\mathbb{C}^m} P_t(\bar\partial_w + \delta_{B-w})\psi \int_{\mathbb{C}^n} W_B(P_s g\xi \mathrm{d}z \wedge t) \wedge \mathrm{d}w \\ &= (2\pi i)^{m+n} g(B)f(A)x\end{aligned}$$

by (8). Hence $h(A,B) = g(B)f(A)$. Q.E.D.

We shall use the following simple lemma:

Lemma 14. *Let K be a compact subset of $\mathbb{C}^n$ and let f be a function analytic on an open neighbourhood of K. Then there are functions $h_j \quad (j = 1, \dots, n)$ analytic on a neighbourhood of the set $D = \{(z, z) : z \in K\}$ such that*

$$f(z) - f(w) = \sum_{j=1}^{n} (z_j - w_j) \cdot h_j(z, w).$$

Proof. For $j = 1, \dots, n$ define g_j by

$$g_j(z_1, \dots, z_n, w_1, \dots, w_n) = f(z_1, \dots, z_j, w_{j+1}, \dots, w_n) - f(z_1, \dots, z_{j-1}, w_j, \dots, w_n).$$

It is easy to see that g_j is defined and analytic on a neighbourhood of D.

Let $h_j(z, w) = \frac{g_j(z,w)}{z_j - w_j}$. Clearly h_j is analytic at each point (z, w) with $z_j \neq w_j$. By the Weierstrass division theorem (see [7], p. 70), h_j can be defined and is analytic also on a neighbourhood of each point (z, w) with $z_j = w_j$. Thus h_j is analytic on a neighbourhood of D. Clearly

$$\sum_{j=1}^{n} (z_j - w_j) \cdot h_j(z, w) = \sum_{j=1}^{n} g_j(z, w) = f(z) - f(w).$$

Q.E.D.

Denote by $\mathcal{A}_K$ the algebra of all functions analytic on a neighbourhood of a compact set $K \subset \mathbb{C}^n$ (more precisely, the algebra of all germs of functions analytic on a neighbourhood of K).

Theorem 15. *Let $A = (A_1, \dots, A_n)$ be an n-tuple of mutually commuting operators on X. Then:*

(i) the mapping $f \mapsto f(A)$ is linear and multiplicative, i.e., the Taylor functional calculus is a homomorphism from $\mathcal{A}_{\sigma_T(A)}$ to $\mathcal{B}(X)$;

(ii) if p is a polynomial, $p(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$ then $p(A) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha A^\alpha$;

(iii) if $f_n \to f$ uniformly on a compact neighbourhood of $\sigma_T(A)$ then $f_n(A) \to f(A)$ in the norm topology;

(iv) $f(A) \in (A)''$ for each $f \in A_{\sigma_T(A)}$.

Proof. (i) The linearity of the mapping $f \mapsto f(A)$ is clear. Let f and g be functions analytic on a neighbourhood of $\sigma_T(A)$. Consider the $(2n)$-tuple (A, A). It is easy to see that $\sigma_T(A, A) = \{(z, z) : z \in \sigma_T(A)\}$. Define functions $h_1(z, w) = f(z)g(w)$ and $h_2(z, w) = f(z)g(z)$. By Lemma 14, we can write $g(z) - g(w) = \sum_{i=1}^{n} (z_i - w_i) q_i(z, w)$ for some functions $q_1, \dots, q_n$ analytic on a neighbourhood of $\sigma_T(A, A)$. By Proposition 13, we have $h_1(A, A) = f(A)g(A)$ and $h_2(A, A) = (fg)(A)$. Thus, by Proposition 12,

$$(fg)(A) - f(A)g(A) = h_2(A, A) - h_1(A, A) = \sum_{i=1}^{n} (A_i - A_i)(fq_i)(A, A) = 0.$$

Hence $(fg)(A) = f(A)g(A)$.

(ii) The statement follows from Propositions 11 and 13.

(iii) follows from the definition.

(iv) Let $S \in \mathcal{B}(X)$ be an operator commuting with $A_1, \dots, A_n$. By Proposition 13, it is possible to consider $f(A)$ to be a function of the $(n+1)$-tuple $(A_1, \dots, A_n, S)$. Therefore $f(A)$ commutes with its argument S. Hence $f(A) \in (A)''$. Q.E.D.

It follows from the general theory [23] that the Taylor spectrum satisfies the spectral mapping property for all polynomials (and consequently, for all functions that can be approximated by polynomials uniformly on a neighbourhood of the Taylor spectrum). In fact the spectral mapping property is true for all analytic functions. To show this, we need the following lemma:

Lemma 16. *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of operators on X, let $c = (c_1, \dots, c_n) \in \sigma_T(A)$ and let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Consider exterior indeterminates $t = (t_1, \dots, t_n)$ and operator $\delta_{A-c,t} : \Lambda[t, X] \to \Lambda[t, X]$ defined by $\delta_{A-c,t}\psi = \sum_{j=1}^{n}(A_j - c_j)t_j \wedge \psi$ $(\psi \in \Lambda[t, X])$. Let $\eta_0 \in \operatorname{Ker} \delta_{A-c,t}$. Then $(f(A) - f(c))\eta_0 \in \delta_{A-c,t}\Lambda[t, X]$.*

Proof. Without loss of generality we can assume that η_0 is homogeneous of degree p, $0 \le p \le n$.

To define $f(A)$, consider exterior indeterminates $s = (s_1, \dots, s_n)$, the mapping δ_{A-z} acting on $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$ defined by $\delta_{A-z}\psi = \sum_{j=1}^{n}(A_j - z_j)s_j \wedge \psi$ and the mapping W_A corresponding to A. We can lift δ_{A-z} and W_A to the space $\Lambda[s, t, \mathrm{d}\bar{z}, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$ in the natural way. Note that δ_{A-z} and W_A are connected with variables s; the mapping $\delta_{A-c,t}$ is related to variables t.

Set $\eta = f\eta_0 \wedge s$ and $\xi_1 = \sum_{k=0}^{n}(-1)^k W_A(\delta_{A-c,t}W_A)^k\eta$. We show by induction that $(\bar\partial + \delta_{A-z})(\delta_{A-c,t}W_A)^k\eta = 0$ for all k. This is clear for $k = 0$; for $k \ge 1$ we have

$$\begin{aligned}(\bar\partial + \delta_{A-z})(\delta_{A-c,t}W_A)^k\eta &= -\delta_{A-c,t}(\bar\partial + \delta_{A-z})W_A(\delta_{A-c,t}W_A)^{k-1}\eta \\ &= -\delta_{A-c,t}(\delta_{A-c,t}W_A)^{k-1}\eta = 0.\end{aligned}$$

Hence

$$\begin{aligned}(\bar\partial + \delta_{A-z} + \delta_{A-c,t})\xi_1 &= (\bar\partial + \delta_{A-z})\xi_1 + \delta_{A-c,t}\xi_1 \\ &= \sum_{k=0}^{n}(-1)^k(\delta_{A-c,t}W_A)^k\eta + \sum_{k=0}^{n}(-1)^k(\delta_{A-c,t}W_A)^{k+1}\eta = \eta\end{aligned}$$

since $(\delta_{A-c,t}W_A)^{n+1} = 0$. Let φ be a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ such that $\operatorname{supp}(1 - \varphi)$ is compact. Let P_s be the projection annihilating all terms that contain at least one of the variables $s_1, \dots, s_n$ and leaving invariant all other terms.

Consider the integral

$$\int (\bar\partial + \delta_{A-c,t})P_s\varphi\xi_1 \wedge \mathrm{d}z = \int (\bar\partial + \delta_{A-c,t})P_s\varphi(W_A\eta - W_A\delta_{A-c,t}W_A\eta + \cdots) \wedge \mathrm{d}z.$$

Since $W_A(\delta_{A-c,t}W_A)^k\eta$ has degree $p + k$ in t and $n - k - 1$ in $(s, \mathrm{d}\bar{z})$, the only relevant term in the integral above is $W_A\eta$. Thus

$$\begin{aligned}\int (\bar\partial + \delta_{A-c,t})P_s\varphi\xi_1 \wedge \mathrm{d}z &= \int (\bar\partial + \delta_{A-c,t})P_s\varphi W_A\eta \wedge \mathrm{d}z \\ &= \int \bar\partial P_s\varphi W_A\eta \wedge \mathrm{d}z = -(2\pi i)^n f(A)\eta_0.\end{aligned}$$

Consider now the n-tuple $B = (c_1 I, \dots, c_n I) \in \mathcal{B}(X)^n$. Since f can be approximated by polynomials uniformly on a neighbourhood of c, we note that $f(B) = f(c) \cdot I$.

As above, consider mappings δ_{B-z} and W_B connected with variables s.

Let $\xi_2 = \sum_{k=0}^{n} (-1)^k W_B (\delta_{A-c,t} W_B)^k \eta$. As above, we have $(\bar\partial + \delta_{B-z} + \delta_{A-c,t})\xi_2 = \eta$ and

$$\int (\bar\partial + \delta_{A-c,t}) P_s \varphi \xi_2 \wedge \mathrm{d}z = \int (\bar\partial + \delta_{A-c,t}) P_s \varphi W_B \eta \wedge \mathrm{d}z$$
$$= \int \bar\partial P_s \varphi W_B \eta \wedge \mathrm{d}z = -(2\pi i)^n f(B)\eta_0 = -(2\pi i)^n f(c)\eta_0.$$

To show that $(f(A) - f(c))\eta_0 \in \delta_{A-c,t}\Lambda[t, X]$, consider the linear mapping U acting on $\Lambda[s, t, \mathrm{d}\bar z, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$ defined by

$$U\Big(t_{i_1} \wedge \cdots \wedge t_{i_m} \wedge \psi\Big) = (t_{i_1} - s_{i_1}) \wedge \cdots \wedge (t_{i_m} - s_{i_m}) \wedge \psi$$

for all $i_1, \dots, i_m$ and $\psi \in \Lambda[s, \mathrm{d}\bar z, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$. Clearly $P_s U = P_s$ and, for each $\psi \in \Lambda[s, t, \mathrm{d}\bar z, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$,

$$U(\bar\partial + \delta_{A-z} + \delta_{A-c,t})\psi$$
$$= \bar\partial U\psi + \sum (A_j - z_j) s_j \wedge U\psi + \sum (A_j - c_j)(t_j - s_j) \wedge U\psi$$
$$= (\bar\partial + \delta_{B-z} + \delta_{A-c,t}) U\psi.$$

We have

$$-(2\pi i)^n f(A)\eta_0 = \int (\bar\partial + \delta_{A-c,t}) P_s \varphi \xi_1 \wedge \mathrm{d}z = \int P_s (\bar\partial + \delta_{A-z} + \delta_{A-c,t}) \varphi \xi_1 \wedge \mathrm{d}z$$
$$= \int P_s U (\bar\partial + \delta_{A-z} + \delta_{A-c,t}) \varphi \xi_1 \wedge \mathrm{d}z = \int P_s (\bar\partial + \delta_{B-z} + \delta_{A-c,t}) \varphi U \xi_1 \wedge \mathrm{d}z.$$

Thus

$$-(2\pi i)^n (f(A) - f(c))\eta_0 = \int P_s (\bar\partial + \delta_{B-z} + \delta_{A-c,t}) \varphi (U\xi_1 - \xi_2) \wedge \mathrm{d}z = \int P_s \theta \wedge \mathrm{d}z$$

where $\theta = (\bar\partial + \delta_{B-z} + \delta_{A-c,t})\varphi(U\xi_1 - \xi_2)$. If $\varphi \equiv 1$ then $\theta = (\bar\partial + \delta_{B-z} + \delta_{A-c,t}) U\xi_1 - \eta = U(\bar\partial + \delta_{A-z} + \delta_{A-c,t})\xi_1 - \eta = U\eta - \eta = 0$; so $\operatorname{supp}\theta$ is compact. Furthermore, θ can be written as $\theta = (\bar\partial + \delta_{B-z} + \delta_{A-c,t})\psi$ for some form $\psi \in \Lambda[s, t, \mathrm{d}\bar z, C^\infty(\mathbb{C}^n, X)]$ with compact support. Indeed, by Remark 8, there exists a form $\vartheta \in \Lambda[s, t, \mathrm{d}\bar z, \mathrm{d}\bar w, C^\infty(\mathbb{C}^{2n}, X)]$ with $\operatorname{supp}\vartheta \subset \operatorname{supp}\theta \times \mathbb{C}^n$ such that $(\bar\partial_{z,w} + \delta_{B-z} + \delta_{A-c,t})\vartheta = \theta$.

Set $\psi(z) = \vartheta_0(z, c)$ where ϑ_0 is the part of ϑ containing none of the variables $\mathrm{d}\bar w_j$. Then $\operatorname{supp}\psi \subset \operatorname{supp}\theta$ and $(\bar\partial_z + \delta_{B-z} + \delta_{A-c,t})\psi = \theta$. By the Stokes theorem,

$$\int P_s \theta \wedge \mathrm{d}z = \int P_s (\bar\partial_z + \delta_{B-z} + \delta_{A-c,t})\psi \wedge s \wedge \mathrm{d}z$$
$$= \int \bar\partial_z P_s \psi \wedge dc + \int P_s \delta_{A-c,t}\psi \wedge \mathrm{d}z = \delta_{A-c,t} \int P_s \psi \wedge \mathrm{d}z \in \delta_{A-c,t}\Lambda[t, X].$$

Q.E.D.

Proposition 17. *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of operators on X, $c = (c_1, \dots, c_n) \in \sigma_T(A)$ and let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Then the $(n+1)$-tuple $(A_1 - c_1, \dots, A_n - c_n, f(A))$ is Taylor regular if and only if $f(c) \neq 0$.*

Proof. To the $(n+1)$-tuple $(A - c, f(A))$ we relate exterior variables $s_1, \dots, s_{n+1}$. Write for short $s = (s_1, \dots, s_n)$. Let $\delta_{A-c} : \Lambda[s, X] \to \Lambda[s, X]$ be be defined by $\delta_{A-c}\psi = \sum (A_j - c_j)s_j \wedge \psi \quad (\psi \in \Lambda[s, X])$. Clearly $\Lambda[s, s_{n+1}, X] = \Lambda[s, X] \oplus s_{n+1} \wedge \Lambda[s, X]$. The operator $\delta_{A-c,f(A)}$ corresponding to the $(n+1)$-tuple $(A - c, f(A))$ can be written in this decomposition in the matrix form

$$\delta_{A-c,f(A)} = \begin{pmatrix} \delta_{A-c} & 0 \\ f(A) & -\delta_{A-c} \end{pmatrix}.$$

We distinguish two cases:

(a) $f(c) = 0$.

Since $c \in \sigma_T(A)$, there is a $\psi \in \Lambda[s, X]$ such that $\delta_{A-c}\psi = 0$ and $\psi \notin \delta_{A-c}\Lambda[s, X]$. By the preceding lemma, there is an $\eta \in \Lambda[s, X]$ such that $f(A)\psi = \delta_{A-c}\eta$. Then $\delta_{A-c,f(A)}(\psi + s_{n+1} \wedge \eta) = 0$ and $(\psi + s_{n+1} \wedge \eta) \notin \delta_{A-c,f(A)}\Lambda[s, s_{n+1}, X]$ since $\psi \notin \delta_{A-c}\Lambda[s, X]$. Thus the $(n+1)$-tuple $(A - c, f(A))$ is Taylor singular.

(b) $f(c) \neq 0$. Without loss of generality we can assume that $f(c) = 1$.

Let $\psi, \xi \in \Lambda[s, X]$, $\delta_{A-c,f(A)}(\psi + s_{n+1} \wedge \xi) = 0$. Then $\delta_{A-c}\psi = 0$ and $f(A)\psi - \delta_{A-c}\xi = 0$. By the preceding lemma, $f(A)\psi - \psi \in \delta_{A-c}\Lambda[s, X]$. Since $f(A)\psi \in \delta_{A-c}\Lambda[s, X]$, we have $\psi = \delta_{A-c}\eta$ for some $\eta \in \Lambda[s, X]$.

Further $\delta_{A-c}(f(A)\eta - \xi) = f(A)\psi - \delta_{A-c}\xi = 0$. Thus there is an $\theta \in \Lambda[s, X]$ with $f(A)(f(A)\eta - \xi) - (f(A)\eta - \xi) = \delta_{A-c}\theta$. Set $\eta' = \eta - (f(A)\eta - \xi)$. Then $\delta_{A-c}\eta' = \delta_{A-c}\eta = \psi$ and $f(A)\eta' - \delta_{A-c}\theta = f(A)\eta - f(A)(f(A)\eta - \xi) + \delta_{A-c}\theta = f(A)\eta - (f(A)\eta - \xi) = \xi$. Hence $\delta_{A-c,f(A)}(\eta' - s_{n+1} \wedge \theta) = (\psi + s_{n+1} \wedge \xi)$ and the $(n+1)$-tuple $(A - c, f(A))$ is Taylor regular. Q.E.D.

Theorem 18. (spectral mapping property) *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of operators on X and let $f = (f_1, \dots, f_m)$ be an m-tuple of functions analytic on a neighbourhood of $\sigma_T(A)$. Then $\sigma_T(f(A)) = f\sigma_T(A)$.*

Proof. Consider the commutative Banach algebra $\mathcal{A}$ generated by $A_1, \dots, A_n, I$ and $f_1(A), \dots, f_m(A)$. Since the restriction of σ_T to $\mathcal{A}$ satisfies the projection property, by [23] there is a compact subset K of the maximal ideal space of $\mathcal{A}$ such that $\sigma_T(B) = \{\varphi(B) : \varphi \in K\}$ for each tuple $B = (B_1, \dots, B_k) \subset \mathcal{A}$.

Fix $\varphi \in K$ and i, $1 \le i \le m$. Let $c_j = \varphi(A_j) \quad (j = 1, \dots, n)$ and $c = (c_1, \dots, c_n) \in \sigma_T(A)$. Then the $(n+1)$-tuple $(A_1 - c_1, \dots, A_n - c_n, f_i(A) - \varphi(f_i(A)))$ is Taylor singular. By Proposition 17, $f_i(c) - \varphi(f_i(A)) = 0$, i.e., $\varphi(f_i(A)) = f_i(\varphi(A))$. Then

$$\begin{aligned} \sigma_T(f(A)) &= \{(\varphi(f_1(A), \dots \varphi(f_m(A))) : \varphi \in K\} = \{(f_1(\varphi(A)), \dots, f_m(\varphi(A))) : \varphi \in K\} \\ &= \{f(c) : c \in \sigma_T(A)\} = f\sigma_T(A). \end{aligned}$$

Q.E.D.

Theorem 19. (superposition property [13], [6]) *Let $A = (A_1, \dots, A_m)$ be a commuting n-tuple of operators on X, let $f = (f_1, \dots, f_m)$ be an m-tuple of function analytic on a* *neighbourhood of $\sigma_T(A)$, let $B = f(A)$, let g be a function analytic on a neighbourhood of $\sigma_T(B)$ and let $h(z) = g(f_1(z), \dots, f_m(z))$. Then $h(A) = g(B)$.*

Proof. By Lemma 14, $g(v) - g(w) = \sum_{j=1}^m (v_j - w_j) r_j(v, w)$ for some functions $r_1, \dots, r_m$ analytic on a neighbourhood of the set $\{(v, v) : v \in \sigma_T(B)\}$. Thus

$$g(f(z)) - g(w) = \sum_{j=1}^{m} (f_j(z) - w_j) r'_j(z, w)$$

where $r'_j(z, w) = r_j\big(f(z), w\big)$ and functions r'_j are analytic on a neighbourhood of the set $\{(z, f(z)) : z \in \sigma_T(A)\} = \sigma_T(A, f(A))$. Thus $h(A) - g(B) = \sum_{j=1}^m \big(f_j(A) - B_j\big) r'_j(A, B) = 0$. Hence $h(A) = g(B)$. Q.E.D.

Concluding Remarks

1. There are many variants of formulas (5), (6) defining the Taylor functional calculus that differ from each other in the sign in front of the integral. There are several sources of differences:

(a) Instead of the n-tuple $A - z = (A_1 - z_1, \dots, A_n - z_n)$ it is possible to consider the n-tuple $z - A$ (which appears naturally in the Cauchy formula). In this approach an additional factor $(-1)^n$ in front of the integral (5) would appear.

(b) Instead of (4) it is possible to use convention that the Lebesgue measure in $\mathbb{C}^n$ is $(2i)^{-n} d\bar{z}_1 \wedge dz_1 \wedge \dots \wedge d\bar{z}_n \wedge dz_n$. With this convention the Fubini theorem becomes more natural. In formula (5), however, an additional factor $(-1)^{\binom{n}{2}}$ would appear.

(c) It is also possible to modify the definition of the mappings δ_A^p in the Koszul complex as in [10]: $\delta_A^p x s_{i_1} \wedge \dots \wedge s_{i_p} = \sum_j A_j x s_{i_1} \wedge \dots \wedge s_{i_p} \wedge s_j$. This convention results also in an additional factor $(-1)^{\binom{n}{2}}$ in formula (5).

2. For Hilbert space operators it is possible to choose $V = (\delta_{A-z} + \delta_{A-z}^*)^{-1}$, see [20], [21], [22]. Formula (7) is then quite explicit.

3. The split-spectrum $\sigma_S(A)$ of the n-tuple $A = (A_1, \dots, A_n) \in \mathcal{B}(X)^n$ is defined as the set of all $\lambda \in \mathbb{C}^n$ such that either $\operatorname{Im} \delta_{A-\lambda} \neq \operatorname{Ker} \delta_{A-\lambda}$ or $\operatorname{Im} \delta_{A-\lambda}$ is not complemented in $\Lambda[s, X]$. In general $\sigma_S(A)$ is bigger than $\sigma_T(A)$, see [12] (in Hilbert spaces these two spectra coincide).

On the complement of $\sigma_S(A)$ it is possible to find bounded linear generalized inverses $V(z)$, see [9]. Thus for functions analytic on a neighbourhood of the split-spectrum the proof of basic properties of the Taylor functional calculus becomes simpler. The linearity of $f(A)$ is clear and also the proofs of multiplicativity of the functional calculus and the spectral mapping property are simpler.

4. As in Theorem 18, it is possible to prove the spectral mapping property for functions analytic on a neighbourhood of the Taylor spectrum for each spectral system which is contained in the Taylor spectrum. In particular, this applies to the spectra of Słodkowski and the essential Taylor spectrum, see [14].

5. An interesting problem is to generalize the Taylor spectrum for Banach algebras.

Let $a = (a_1, \dots, a_n)$ be a commuting n-tuple of elements of a Banach algebra. Denote by $L_a = (L_{a_1}, \dots, L_{a_n})$ the n-tuple of left multiplication operators acting on $\mathcal{A}$. A natural idea is to define the Taylor spectrum of a as $\sigma_T(L_a)$. However, if $\mathcal{A} = \mathcal{B}(X)$ is the algebra of operators on a Banach space X and $A \in \mathcal{B}(X)^n$ a commuting n-tuple, then $\sigma_T(L_A) = \sigma_S(A)$. Thus this simple way does not produce the Taylor spectrum in $\mathcal{B}(X)$.

In fact in this situation A can be considered also as a commuting n-tuple of elements of $\mathcal{H}(X)$ where $\mathcal{H}(X)$ satisfies all axioms of Banach algebras except one of the distributive laws; let us call such objects semi-distributive algebras. Define $L'_A = (L'_{A_1}, \dots, L'_{A_n}) \in \mathcal{B}(\mathcal{H}(X))^n$ by $L'_{A_i}\varphi = A_i\varphi \quad (\varphi \in \mathcal{H}(X))$; clearly L'_{A_i} is an extension of L_{A_i}. It is easy to check now that $\sigma_T(L'_A) = \sigma_T(A)$.

It seems that the natural setting for the Taylor spectrum in algebras is to define it for commuting n-tuples $a = (a_1, \dots, a_n)$ of elements of a semi-distributive algebra $\mathcal{A}$ that lie in the "distributive center" of $\mathcal{A}$ (more precisely, $a_i(b+c) = a_ib + a_ic$ for all $b, c \in \mathcal{A}, 1 \le i \le n$. For such an n-tuple, $L_{a_i} : \mathcal{A} \to \mathcal{A}$ defined by $L_{a_i}b = a_ib \quad (b \in \mathcal{A})$ is a linear operator and we can define the Taylor spectrum of a as the Taylor spectrum of $L_a = (L_{a_1}, \dots, L_{a_n}) \in \mathcal{B}(\mathcal{A})^n$.

References

1. D. W. Albrecht, Integral formulae for special cases of Taylor's functional calculus, *Studia Math.* 105 (1993), 51–68.
2. C. Ambrozie, F.-H. Vasilescu, Banach space complexes, Kluwer Academic Publishers, MAIA Series, vol. 334, 1995.
3. M. Andersson, Taylor's functional calculus for commuting operators with Cauchy-Fantappie-Leray formulas, *Int. Math. Res. Not.* 6 (1997), 247–258; correction, ibidem 2 (1998), 123–124.
4. M. Andersson, Taylor's functional calculus and the resolvent mapping, Preprint Goteborg Univ., 2000.
5. M. Andersson, Various constructions of Taylor's functional calculus for commuting operators, Preprint Goteborg Univ., 2000.
6. M. Andersson, S. Sandberg, A constructive proof of the composition rule for Taylor's functional calculus, *Studia Math.* 142 (2000), 65–69.
7. R.C. Gunning, H. Rossi, Analytic functions of several complex variables, Prentice Hall, Englewood Cliffs, N.J. 1965.
8. A.Ya. Helemskii, The homology of Banach and topological algebras (translated from Russian), Mathematics and its applications (Soviet Series) 41, Kluver, Academic Publishers Group, Dordrecht 1989.
9. V. Kordula, V. Müller, Vasilescu-Martinelli formula for operators in Banach spaces, *Studia Math.* 113 (1995), 127–139.
10. R. Levi, Notes on the Taylor joint spectrum of commuting operators, Spectral Theory, Banach Center Publications, Vol. 8, PWN – Polish Scientific Publishers, Warsaw, 1982, 321–332.
11. F. Mantlik, Linear equations depending differentiably on a parameter, *Integral Equations Operator Theory* 13 (1990), 321–250.
12. V. Müller, The splitting spectrum differs from the Taylor spectrum, *Studia Math.* 123 (1997), 291–294.

13. M. Putinar, The superposition property for Taylor's functional calculus, *J. Operator Theory* 7 (1982), 149–155.
14. M. Putinar, Functional calculus and the Gelfand transformation, *Studia Math.* 79 (1984), 83–86.
15. S. Sandberg, On non-holomorphic functional calculus for commuting operators, Preprint Goteborg Univ., 1999.
16. Z. Słodkowski, An infinite family of joint spectra, *Studia Math.* 61 (1977), 239–255.
17. Z. Słodkowski, Operators with closed ranges in spaces of analytic vector-valued functions, *J. Funct. Anal.* 28 (1986), 155–177.
18. J. L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* 6 (1970), 172–191.
19. J. L. Taylor, Analytic-functional calculus for several commuting operators, *Acta Math.* 125 (1970), 1–38.
20. F.- H. Vasilescu, A Martinelli type formula for the analytic functional calculus, *Rev. Roumaine Math. Pures Appl.* 23 (1978), 1587–1605.
21. F.- H. Vasilescu, A multidimensional spectral theory in C^*- algebras, Spectral Theory, Banach Center Publications, Vol. 8, PWN – Polish Scientific Publishers, Warsaw, 1982, 471–491.
22. F.- H. Vasilescu, Analytic Functional Calculus and Spectral Decompositions, Mathematics and its Applications (East European Series), Editura Academiei, Bucuresti and D. Reidel Publishing Company, Dordrecht, 1982.
23. W. Żelazko, An axiomatic approach to joint spectra I, *Studia Math.* 64 (1979), 249–261.