On the Kato decomposition of quasi-Fredholm and B-Fredholm operators

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Abstract. We construct a Kato-type decomposition of quasi-Fredholm operators on Banach spaces. This generalizes the corresponding result of Labrousse for Hilbert space operators. The result is then applied to B-Fredholm operators.

Denote by $\mathcal{B}(X)$ the set of all bounded linear operators acting on a Banach space X. For $T \in \mathcal{B}(X)$ denote by $N(T) = \{x \in X : Tx = 0\}$ and $R(T) = TX$ its kernel and range, respectively.

Let $T \in \mathcal{B}(X)$. For $n \geq 0$ set $\alpha_n(T) = \dim N(T^{n+1})/N(T^n)$ and $\beta_n(T) = \dim R(T^n)/R(T^{n+1})$. For $n = 0$ these numbers reduce to the well-known defect numbers $\alpha_0(T) = \dim N(T)$ and $\beta_0(T) = \operatorname{codim} R(T)$.

It is possible to show that $\alpha_n(T) = \dim\bigl(N(T) \cap R(T^n)\bigr)$, and similarly, $\beta_n(T) = \operatorname{codim}\bigl(R(T) + N(T^n)\bigr)$. This implies that the sequences $\alpha_n(T)$ and $\beta_n(T)$ are non-increasing.

Further we define the "difference sequence" $k_n(T)$, see [4], by

$$k_n(T) = \dim\bigl(R(T^n) \cap N(T)\bigr)/\bigl(R(T^{n+1}) \cap N(T)\bigr).$$

Equivalently,

$$k_n(T) = \dim\bigl(R(T) + N(T^{n+1})\bigr)/\bigl(R(T) + N(T^n)\bigr).$$

From this one can see easily that $k_n(T) = \alpha_n(T) - \alpha_{n+1}(T)$ whenever the difference has meaning, i.e., if $\alpha_{n+1}(T) < \infty$. Similarly, $k_n(T) = \beta_n(T) - \beta_{n+1}(T)$ if $\beta_{n+1}(T) < \infty$.

The numbers $\alpha_n(T)$, $\beta_n(T)$ and $k_n(T)$ enable to define many interesting classes of operators that have been studied by many authors. For a survey of such classes see [10].

One of the most important classes is that of semiregular operators. An operator $T \in \mathcal{B}(X)$ is called semiregular if $R(T)$ is closed and $k_i(T) = 0$ for all $i \geq 0$. Semiregular operators have been studied intensely, see e.g. [3], [5], [9], [11], [12].

Let $T \in \mathcal{B}(X)$ be a semiregular operator. It is well-known that $N(T^i) \subset R(T^j)$ for all i, j. Further T^* is semiregular and T^n is semiregular for all n. Conversely, if T^n is semiregular for some $n \geq 1$, then T is semiregular.

In the present paper we concentrate on classes of quasi-Fredholm and B-Fredholm operators.

Definition 1. Let $d \geq 0$. An operator $T \in \mathcal{B}(X)$ is called quasi-Fredholm of degree d if $k_n(T) = 0 \quad (n \geq d)$, and subspaces $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree d.

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Definition 1 is due to Labrousse [8] who introduced and studied quasi-Fredholm operators on Hilbert spaces. The same definition can be used for Banach space operators. The assumption that the subspaces $N(T^d)+R(T)$ and $N(T)\cap R(T^d)$ are closed can be replaced by other equivalent conditions.

First we need the following lemma.

Lemma 2. Let $T\in\mathcal{B}(X)$ be a quasi-Fredholm operator of degree d and let $j\ge 1$. Then $N(T^j)\cap R(T^d)\subset\bigcap_{n=0}^{\infty}R(T^n)$.

Proof. We prove the statement by induction on j.

Since $k_j(T)=0\quad(j\ge d)$, we have $N(T)\cap R(T^d)=N(T)\cap R(T^{n+1})=\cdots$. Hence $N(T)\cap R(T^d)\subset\bigcap_{n=0}^{\infty}R(T^n)$.

Suppose that the statement is true for some $j\ge 1$. Let $x\in N(T^{j+1})\cap R(T^d)$ and let $n\ge d$. Then $Tx\in N(T^j)\cap R(T^d)\subset R(T^{n+1})$, and so $Tx=T^{n+1}y$ for some $y\in X$. Thus $x-T^ny\in N(T)$ and $x=T^ny+u$ for some $u\in N(T)$. Clearly also $u\in R(T^d)$, and so $x\in R(T^n)+\big(N(T)\cap R(T^d)\big)\subset R(T^n)$.

This finishes the proof.

Proposition 3. Let $T\in\mathcal{B}(X)$, $d\ge 0$ and let $k_n(T)=0$ for all $n\ge d$. The following statements are equivalent:

(i) T is quasi-Fredholm, i.e., $R(T)+N(T^d)$ and $N(T)\cap R(T^d)$ are closed;

(ii) $R(T^{d+1})$ is closed;

(iii) $R(T^n)$ is closed for all $n\ge d$;

(iv) $R(T^i)+N(T^j)$ is closed for all i,j with $i+j\ge d$.

Proof. The equivalences (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) were proved in [10].

The implication (iv)$\Rightarrow$(i) is trivial.

(i)$\Rightarrow$(ii): We shall use repeatedly a lemma of Neubauer, see [8], Proposition 2.1.1: if $M,N\subset X$ are paracomplete subspaces (= ranges of bounded operators) such that both $M\cap N$ and $M+N$ are closed, then M and N are closed.

To show that $R(T^{d+1})$ is closed, it is therefore sufficient to prove that $R(T^{d+1})+N(T^d)$ and $R(T^{d+1})\cap N(T^d)$ are closed.

(A) We prove by induction on j that $N(T^j)+R(T^d)$ is closed. This is true for $j=1$. Let $j\ge 1$ and let $N(T^j)\cap R(T^d)=N(T^j)\cap R(T^{d+1})$ be closed. Then the space $T^{-1}\big(N(T^j)\cap R(T^{d+1})\big)=N(T)+\big(N(T^{j+1})\cap R(T^d)\big)$ is closed. Further $N(T)\cap\big(N(T^{j+1})\cap R(T^d)\big)=N(T)\cap R(T^d)$ is closed and the space $N(T^{j+1})\cap R(T^d)$ is paracomplete. By the lemma of Neubauer, $N(T^{j+1})\cap R(T^d)$ is closed.

This proves that $N(T^j)\cap R(T^d)$ is closed for all $j\ge 1$. In particular, $N(T^d)\cap R(T^d)=N(T^d)\cap R(T^{d+1})$ is closed.

(B) We show first that $N(T^{d+1})\subset R(T^j)+N(T^d)$ for each $j\ge 1$. Let $x\in N(T^{d+1})$ and $j\ge 1$. Then $T^dx\in N(T)\cap R(T^d)=N(T)\cap R(T^{d+j})$. Thus $T^dx=T^{d+j}y$ for some $y\in X$ and $x-T^jy\in N(T^d)$. Hence $x\in N(T^d)+R(T^j)$ and $N(T^{d+1})\subset N(T^d)+R(T^j)$.

Consider the operator $\hat{T}:X/N(T^d)\to X/N(T^d)$ induced by T. The previous inclusion gives that $N(\hat{T})\subset\bigcap_{j=1}^{\infty}R(\hat{T}^j)$. Further $R(T)+N(T^d)$ is closed and thus $R(\hat{T})$ is a closed subspace of $X/N(T^d)$. Hence $\hat{T}$ is semiregular and, consequently, $R(\hat{T}^{d+1})$ is closed. Let Q be the canonical projection $Q:X\to X/N(T^d)$. Then the space $R(T^{d+1})+N(T^d)=Q^{-1}R(\hat{T}^{d+1})$ is closed.

This completes the proof.

Lemma 4. Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then $T^* \in \mathcal{B}(X^*)$ is quasi-Fredholm of the same degree d.

Proof. Since $R(T^{d+1})$ is closed, the space $R(T^{*d+1})$ is also closed.

Let $j \geq d$. We have $N(T^{*j}) + R(T^*) \subset \big(R(T^j) \cap N(T)\big)^\perp$. Thus

$$R(T^j) \cap N(T) = {}^\perp\Big(\big(R(T^j) \cap N(T)\big)^\perp\Big) \subset {}^\perp\big(N(T^{*j}) + R(T^*)\big) = R(T^j) \cap N(T).$$

Therefore

$$\begin{aligned} k_j(T^*) &= \dim\big(N(T^{*j+1}) + R(T^*)\big)/\big(N(T^{*j}) + R(T^*)\big) \\ &= \dim\big(R(T^j) \cap N(T)\big)/\big(R(T^{j+1}) \cap N(T)\big) = k_j(T) = 0. \end{aligned}$$

Hence T^* is quasi-Fredholm of degree d.

The main result of Labrousse [8] is that any quasi-Fredholm operator T on a Hilbert space admits a Kato-type decomposition $T = T_1 \oplus T_2$ with T_1 nilpotent and T_2 semiregular. We prove an analogues result for Banach space operators under an additional assumption that the subspaces that appear in the definition of quasi-Fredholm operators are complemented. For Hilbert space operators this condition is satisfied automatically.

Theorem 5. Let $T \in \mathcal{B}(X)$ be a quasi-Fredholm operator of degree d and let the subspaces $R(T) + N(T^d)$ and $N(T) \cap R(T^d)$ be complemented. Then there are closed subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $TX_i \subset X_i \quad (i = 1, 2)$, $T^d|X_1 = 0$ and $T|X_2$ is semiregular.

Proof. Let $T \in \mathcal{B}(X)$ be a quasi-Fredholm operator of degree d. By Lemma 2 and Proposition 3, $R(T^d)$ is closed and $N(T^i) \cap R(T^d) \subset R(T^j)$ for all $i, j \geq 0$.

If $d = 0$ then T is semiregular and the decomposition is trivial. In the following we assume that $d \geq 1$.

By the assumption, there exists a closed subspace L such that $X = \big(R(T^d) \cap N(T)\big) \oplus L$.

We define closed subspaces $N_j \quad (j = 0, \dots, d)$ inductively by $N_0 = \{0\}$ and $N_{j+1} = T^{-1}N_j \cap L \quad (j < d)$.

Clearly $TN_{j+1} \subset N_j \cap R(T)$. Conversely, let $x \in N_j \cap R(T)$. Then $x = Tu$ for some $u \in X$. Express $u = l + v$ with $l \in L$ and $v \in N(T) \cap R(T^d)$. Then $u - v = l \in L$ and $T(u - v) = Tu = x$. Thus $u - v \in N_{j+1}$ and $x \in TN_{j+1}$.

Hence

$$TN_{j+1} = N_j \cap R(T) \quad (j < d).$$

We prove by induction on j that $N_j \subset N_{j+1}$. The statement is clear for $j = 0$. Suppose that $j \geq 0$, $N_j \subset N_{j+1}$, and let $x \in N_{j+1}$. Then $Tx \in N_j \subset N_{j+1}$, and so $x \in T^{-1}N_{j+1}$. Since also $x \in N_{j+1} \subset L$, we conclude that $x \in N_{j+2}$.

Hence

$$N_j \subset N_{j+1} \quad (j = 0, 1, \dots, d - 1).$$

Also one can see easily that $N_j \subset N(T^j)$ for all j.

We prove now by induction on j that

$$N(T^j) \subset N_j + N(T^j) \cap R(T^d). \tag{1}$$

The inclusion is clear for $j = 0$. For $j = 1$ we have $N(T) = N(T) \cap L + N(T) \cap R(T^d) = N_1 + N(T) \cap R(T^d)$. Let $j \geq 1$, $N(T^j) \subset N_j + N(T^j) \cap R(T^d)$ and let $x \in N(T^{j+1})$. Then $Tx \in N(T^j)$, and so $Tx = v_1 + v_2$ for some $v_1 \in N_j$ and $v_2 \in N(T^j) \cap R(T^d) = N(T^j) \cap R(T^{d+1}) = T(N(T^{j+1}) \cap R(T^d))$. Thus $v_1 \in N_j \cap R(T) = TN_{j+1}$ and

$$\begin{aligned} x &\in N_{j+1} + N(T^{j+1}) \cap R(T^d) + N(T) \\ &= N_{j+1} + N(T^{j+1}) \cap R(T^d) + N(T) \cap L + N(T) \cap R(T^d) \\ &= N_{j+1} + N(T^{j+1}) \cap R(T^d). \end{aligned}$$

This proves (1).

Finally, we prove by induction on j that $N_j \cap R(T^d) = \{0\}$. This is clear for $j = 0$. Let $j \geq 0$, $N_j \cap R(T^d) = \{0\}$ and let $x \in N_{j+1} \cap R(T^d)$. Then $Tx \in N_j \cap R(T^d)$ and so, by the induction assumption, $Tx = 0$. Thus $x \in N(T) \cap R(T^d)$ and $x \in N_{j+1} \subset L$. Consequently, $x = 0$. Hence

$$N_j \cap R(T^d) = \{0\} \qquad (j \leq d).$$

Set $N = N_d$. Then $TN \subset N$ and $N \subset N(T^d)$. Further $N(T^d) \subset N + R(T^d)$ and $N \cap R(T^d) = \{0\}$. Note also that the space $N + R(T^d) = N(T^d) + R(T^d)$ is closed.

Since T^* is quasi-Fredholm of degree d, we can use the same construction for T^*. Moreover, since $R(T) + N(T^d)$ is complemented and $N(T^*) \cap R(T^{*d}) = \Big(R(T) + N(T^d)\Big)^\perp$, we can choose a w^*-closed space L' such that $\big(N(T^*) \cap R(T^{*d})\big) \oplus L' = X^*$.

As above, construct subspaces $M'_i \subset X^*$ by $M'_0 = \{0\}$ and $M_{i+1} = T^{*-1} M_i \cap L' \quad (0 \leq i \leq d-1)$. Clearly all speces M'_i are w^*-closed. Set $M' = M'_d$. Thus we have

$$\begin{aligned} &T^* M' \subset M' \subset N(T^{*d}), \\ &M' \cap R(T^{*d}) = \{0\} \quad \text{and} \\ &N(T^{*d}) \subset M' + R(T^{*d}). \end{aligned}$$

Further $M' + R(T^{*d})$ is a closed subspace.

Set $M = {}^\perp M'$. Then $TM \subset M$ and

$$\begin{aligned} &M = {}^\perp M' \supset {}^\perp N(T^{*d}) = R(T^d), \\ &M + N(T^d) = {}^\perp M' + {}^\perp R(T^{*d}) = {}^\perp\big(M' \cap R(T^{*d})\big) = X, \quad \text{and} \\ &R(T^d) = {}^\perp N(T^{*d}) \supset {}^\perp\big(M' + R(T^{*d})\big) = {}^\perp M' \cap {}^\perp R(T^{*d}) = M \cap N(T^d) \end{aligned}$$

(the equality ${}^\perp M' + {}^\perp R(T^{*d}) = {}^\perp\big(M' \cap R(T^{*d})\big)$ follows from the fact that the space $M' + R(T^{*d})$ is closed, see [7], p. 221). Thus

$$M + N \supset M + R(T^d) + N \supset M + N(T^d) = X$$

and

$$M \cap N \subset M \cap N(T^d) \cap N \subset R(T^d) \cap N = \{0\}.$$

Hence $X = N \oplus M$, $TN \subset N$, $TM \subset M$ and $(T|N)^d = 0$.

Let $T_2 = T|M$.

If $x \in N(T_2)$ then $x \in N(T) \cap M \subset N(T^d) \cap M \subset M \cap N(T^d) \cap R(T^d) \subset M \cap \bigcap_{i=0}^{\infty} R(T^i) = \bigcap_{i=0}^{\infty} R(T_2^i)$. Further $R(T_2^d) = T_2^d M = R(T^d)$, and so $R(T_2^d)$ is a closed subspace. Thus T_2^d is semiregular and so is also T_2.

We apply the previous result to B-Fredholm operators.

Definition 6. An operator $T \in \mathcal{B}(X)$ is called B-Fredholm if there exists $d \geq 0$ such that $R(T^d)$ is closed and the restriction $T|R(T^d)$ is Fredholm.

B-Fredholm operators were introduced and studied by Berkani [1], [2]. In [1] it was proved that an operator T is B-Fredholm if and only if $T = T_1 \oplus T_2$ with T_1 nilpotent and T_2 Fredholm. The proof, however, is based on the decomposition of quasi-Fredholm operators of Labrousse [8], which was proved only for Hilbert space operators.

Theorem 7. Let T be an operator on a Banach space X. The following statements are equivalent:

(i) T is B-Fredholm;

(ii) there are closed subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, $TX_i \subset X_i \quad (i = 1, 2)$, $T|X_1$ is nilpotent and $T|X_2$ Fredholm.

Proof. (ii)⇒(i): Let $X = X_1 \oplus X_2$, $TX_i \subset X_i \quad (i = 1, 2)$, $T|X_1$ nilpotent and $T|X_2$ Fredholm. Let $T^n|X_1 = 0$. Then $R(T^n) = R(T^n|X_2)$, which is of finite codimension in X_2. Therefore $R(T^n)$ is closed. It is easy to see that $T|R(T^n)$ is Fredholm.

(i)⇒(ii): Let $n \geq 0$ satisfy that $R(T^n)$ is closed and the restriction $T_0 = T|R(T^n)$ is Fredholm. Then $\alpha_n(T) = \dim N(T) \cap R(T^n) = \dim N(T_0) < \infty$ and $\beta_n(T) = \dim R(T^n)/R(T^{n+1}) = \operatorname{codim} R(T_0) < \infty$. Since the sequences $\alpha_j(T)$ and $\beta_j(T)$ are non-increasing, they are constant for j large enough, i.e., there exists d such that $\alpha_d(T) = \alpha_{d+1}(T) = \cdots < \infty$ and $\beta_d(T) = \beta_{d+1}(T) = \cdots < \infty$. This means that $k_j(T) = \alpha_j(T) - \alpha_{j+1}(T) = 0$ for $j \geq d$. Further $\dim\bigl(N(T) \cap R(T^d)\bigr) = \alpha_d(T) < \infty$ and $\operatorname{codim}\bigl(R(T) + N(T^d)\bigr) = \beta_d(T) < \infty$, and so these two subspaces are complemented.

Thus T is quasi-Fredholm of degree d and, by Theorem 5, $X = X_1 \oplus X_2$ where X_1, X_2 are closed subspaces, $TX_i \subset X_i \quad (i = 1, 2)$, $(T|X_1)^d = 0$ and $T_2 = T|X_2$ is semiregular. Further $\alpha_d(T_2) = \alpha_d(T_1) + \alpha_d(T_2) = \alpha_d(T) < \infty$ and $\beta_d(T_2) = \beta_d(T_1) + \beta_d(T_2) = \beta_d(T) < \infty$. Since $k_j(T_2) = 0$ for all j, we conclude that $\alpha_0(T_2) = \alpha_d(T_2) < \infty$ and $\beta_0(T_2) = \beta_d(T_2) < \infty$, and so T_2 is Fredholm.

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