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On McShane integrability of Banach space-valued functions

Abstract

The McShane integral of Banach space-valued functions $f : I \to X$ defined on an m-dimensional interval I is considered in the paper.

We show that a McShane integrable function is integrable over measurable sets contained in I (Theorem 9). A certain type of absolute continuity of the indefinite McShane integral with respect to the Lebesgue measure is derived (Theorem 11) and we show that the indefinite McShane integral is countably additive (Theorem 16).

Allowing more general partitions using measurable sets instead of intervals another McShane type integral is defined and we show that it is equivalent to the original McShane integral (Theorem 21).

We consider functions $f : I \to X$ where $I \subset \mathbb{R}^m$ is a compact interval, $m \geq 1$ and X is a Banach space with the norm $\| \cdot \|_X$.

By μ let the Lebesgue measure in $\mathbb{R}^m$ be denoted.

A system (finite collection) of point-interval pairs $\{(t_i, I_i),\ i = 1, \dots, p\}$ is called an *M-system in I* if I_i are non-overlapping (int $I_i \cap$ int $I_j = \emptyset$ for $i \neq j$, int I_i is the interior of I_i), t_i are arbitrary points in I.

An M-system in I is called an *M-partition of I* if $\bigcup_{i=1}^{p} I_i = I$.

By $B(t, \rho)$ the ball in $\mathbb{R}^m$ centered at t with the radius ρ is denoted. For simplicity we use $\text{dist}(s, t) = \max_{i=1,\dots,m} |t_i - s_i|$ for the distance of two points $t, s \in \mathbb{R}^m$.

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Given $\Delta : I \to (0, +\infty)$, called a *gauge,* an M-system $\{(t_i, I_i),\ i = 1, \dots, p\}$ in I is called Δ-*fine* if

$$I_i \subset B(t_i, \Delta(t_i)),\ i = 1, \dots, p.$$

The set of Δ-fine partitions of I is nonempty (Cousin's lemma, see e.g. [3]).

Definition 1. $f : I \to X$ is *McShane integrable* and $J \in X$ is its *McShane integral over* I if for every $\varepsilon > 0$ there exists a gauge $\Delta : I \to (0, +\infty)$ such that for every Δ-fine M-partition $(t_i, I_i), i = 1, \dots, p$ of I the inequality

$$\| \sum_{i=1}^{p} f(t_i)\mu(I_i) - J\|_X \le \varepsilon$$

holds. Denote $J = \int_I f$.

Given a set $E \subset I$ we denote by χ_E its characteristic function ($\chi_E(t) = 1$ for $t \in E$, $\chi_E(t) = 0$ otherwise).

A function $f : I \to X$ is called *McShane integrable over the set* $E \subset I$ if the function $f \cdot \chi_E : I \to X$ is McShane integrable.

In this case we write $\int_I f \cdot \chi_E = \int_E f$.

By a *figure* we mean a finite union of compact nondegenerate intervals in $\mathbb{R}^m$.

We mention the fact that that if for the notion of an M-system $\{(t_i, I_i),\ i = 1, \dots, p\}$ the intervals I_i are replaced by figures, we can develop the same theory and M-systems and M-partitions of this kind can be used everywhere in our forthcomming considerations because if in the M-system $(t_i, I_i), i = 1, \dots, p$ some of the t_i are the same, then the intervals corresponding I_i to this common point form a figure and vice versa if we have (t_i, F_i) where F_i is a figure then this point-figure pair can be divided into point-interval pairs where the intervals are those which give the figure F_j.

Theorem 2. *The function $f : I \to X$ is McShane integrable if and only if for every $\varepsilon > 0$ there exists a gauge $\Delta : I \to (0, +\infty)$ such that*

$$\| \sum_{i=1}^{p} f(t_i)\mu(I_i) - \sum_{j=1}^{r} f(s_j)\mu(K_j)\|_X < \varepsilon \tag{1}$$

for any Δ-fine M-partitions $\{(t_i, I_i), i = 1, \dots, p\}$ and $\{(s_j, K_j), j = 1, \dots, r\}$ of I.

Proof. If f is McShane integrable then (1) clearly holds for the gauge δ which corresponds to $\frac{\varepsilon}{2} > 0$ in the definition of McShane integrability.

Given $\varepsilon > 0$ assume that (1) holds for any δ-fine M-partitions $\{(t_i, I_i), i = 1, \dots, p\}$ and $\{(s_j, K_j), i = 1, \dots, r\}$ of I.

Denote

$$S(\varepsilon) = \{S(f, D) = \sum_{i=1}^{k} f(t_i)\mu(J_i); \quad D = \{(t_i, J_i),\, i = 1, \dots, k\} \subset X$$

where D is an arbitrary δ-fine M-partition of I. The set $S(\varepsilon) \subset X$ is nonempty because by Cousin's lemma there exists a δ-fine M-partition $\{(t_i, J_i),\, i = 1, \dots, k\}$ of I. Since by (1) we have

$$\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j) \|_X < \varepsilon$$

for all δ-fine M-partitions $\{(t_i, J_i),\, i = 1, \dots, k\}$ and $\{(s_j, L_j),\, j = 1, \dots, l\}$ of I, we have also

$$\operatorname{diam}\, S(\varepsilon) \leq \varepsilon$$

(by diam $S(\varepsilon)$ the diameter of the set $S(\varepsilon)$ is denoted). Further evidently

$$S(\varepsilon_1) \subset S(\varepsilon_2),$$

provided $\varepsilon_1 < \varepsilon_2$. Hence the set

$$\bigcap_{\varepsilon > 0} \operatorname{cl}\, S(\varepsilon) = S_f \in X$$

consists of a single point because the space X is complete (by cl $S(\varepsilon)$ the closure of the set $S(\varepsilon)$ in X is denoted).

For the integral sum $S(f, D)$ we get

$$\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - S_f \|_X \leq \varepsilon,$$

whenever $D = \{(t_i, J_i),\, i = 1, \dots, k\}$ is an arbitrary δ-fine M-partition of I and this means that f is McShane integrable with $\int_I f = S_f$. □

Theorem 3. *Assume that $f : I \to X$ is McShane integrable and let $J \subset I$ be a compact interval. Then f is McShane integrable on the interval J.*

Proof. By Theorem 2 for any given $\varepsilon > 0$ there exists a gauge $\delta : I \to (0, +\infty)$ such that for every δ-fine M-partitions $\{(t_i, I_i), i = 1, \dots, p\}$ and $\{(s_j, J_j), i = 1, \dots, r\}$ of I the inequality (1) is satisfied.

Let $\{(\tau_i, K_i), i = 1, \dots, q\}$ and $\{(\sigma_j, L_j), i = 1, \dots, s\}$ be arbitrary δ-fine M-partitions of the interval J.

The complement $I \setminus J$ can be expressed as a finite union of intervals contained in I. Taking an arbitrary δ-fine M-partition of every of those intervals we obtain a finite collection $\{(\rho_l, M_l), l = 1, \dots, t\}$ of tagged intervals which together with $\{(\tau_i, K_i), i = 1, \dots, q\}$ or $\{(\sigma_j, L_j), i = 1, \dots, s\}$ form two δ-fine M-partitions of the interval I.

Taking the difference of the integral sums corresponding to this two δ-fine M-partitions of I we can see that its value is

$$\sum_{i=1}^{q} f(\tau_i)\mu(K_i) - \sum_{j=1}^{s} f(\sigma_j)\mu(L_j)$$

because the remaining $\sum_{l=1}^{t} f(\rho_l)\mu(M_l)$ is the same for both of them. Therefore by (1) we have

$$\|\sum_{i=1}^{q} f(\tau_i)\mu(K_i) - \sum_{j=1}^{s} f(\sigma_j)\mu(L_j)\|_X < \varepsilon$$

and this inequality shows by Theorem 2 the McShane integrability of f on J. □

Theorem 4. *Let $f : I \to X$. If $f = 0$ almost everywhere on I then f is McShane integrable on I and $\int_I f = 0$.*

Proof. Assume that $\varepsilon > 0$ is given.

Let $N = \{t \in I;\ f(t) \neq 0\}$ and for each $n \in \mathbb{N}$, let

$$N_n = \{t \in N;\ n - 1 \le \|f(t)\|_X < n\}.$$

Since $\mu(N) = 0$, we have also $\mu(N_n) = 0$ for $n \in \mathbb{N}$ and therefore there are open sets $G_n \subset I$ such that $N_n \subset G_n$ and $\mu(G_n) < \dfrac{\varepsilon}{n2^n}$.

Define a gauge $\delta : I \to (0, +\infty)$ in such a way that $\delta(t) = 1$ for $t \in I \setminus N$ and $B(t, \delta(t)) \subset G_n$ if $t \in N_n$.

Suppose that $\{(t_i, I_i), i = 1, \dots, p\}$ is a δ-fine M-partition of I. Then

$$\|\sum_{i=1}^{p} f(t_i)\mu(I_i)\|_X \le \|\sum_{n=1}^{\infty} \sum_{i=1,\ t_i \in N_n}^{p} f(t_i)\mu(I_i)\|_X \le$$

$$\leq \sum_{n=1}^{\infty} \| \sum_{i=1,\, t_i \in N_n}^{p} f(t_i)\mu(I_i)\|_X < \sum_{n=1}^{\infty} n \sum_{i=1,\, t_i \in N_n}^{p} \mu(I_i) <$$

$$< \sum_{n=1}^{\infty} n\mu(G_n) < \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^n} = \varepsilon.$$

Hence $f : I \to X$ is McShane integrable and $\int_I f = 0$. □

Lemma 5. (Saks-Henstock) *Assume that $f : I \to X$ is McShane integrable. Given $\varepsilon > 0$ assume that the gauge Δ on I is such that*

$$\| \sum_{i=1}^{p} f(t_i)\mu(I_i) - \int_I f\|_X \leq \varepsilon$$

for every Δ-fine M-partition $\{(t_i, I_i), i = 1, \dots, p\}$ of I.

Then if $\{(r_j, K_j), j = 1, \dots, q\}$ is an arbitrary Δ-fine M-system we have

$$\| \sum_{j=1}^{q} [f(r_j)\mu(K_j) - \int_{K_j} f]\|_X \leq \varepsilon.$$

Proof. Since $\{(r_j, K_j), j = 1, \dots, q\}$ is a Δ-fine M-system, the complement $I \setminus \bigcup_{j=1}^{q} \operatorname{int} K_j$ can be expressed as a finite system M_l, $l = 1, \dots, r$ of non-overlapping intervals in I. The function f is McShane integrable and therefore the integrals $\int_{M_l} f d\mu$ exist and by definition for any $\eta > 0$ there is a gauge δ_l on M_l with $\delta_l(t) < \delta(t)$ for $t \in M_l$ such that for every $l = 1, \dots, r$ we have

$$\| \sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - \int_{M_l} f\|_X < \frac{\eta}{r+1}$$

provided $\{(s_i^l, J_i^l), i = 1, \dots, k_l\}$ is a δ_l-fine M-partition of the interval M_l.

The sum

$$\sum_{j=1}^{q} f(r_j)\mu(K_j) + \sum_{l=1}^{r} \sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l)$$

represents an integral sum which corresponds to a certain δ-fine M-partition of I, namely $\{(r_j, K_j), (s_i^l, J_i^l);\ j = 1, \dots, q,\ l = 1, \dots, r,\ i = 1, \dots, k_l\}$, and consequently by the assumption we have

$$\| \sum_{j=1}^{q} f(r_j)\mu(K_j) + \sum_{l=1}^{r} \sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - \int_I f\|_X < \varepsilon.$$

Hence

$$\|\sum_{j=1}^{q} f(r_j)\mu(K_j) - \int_{K_j} f\|_X =$$

$$= \|\sum_{j=1}^{q} f(r_j)\mu(K_j) + \sum_{l=1}^{r}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - \int_I f +$$

$$- \sum_{l=1}^{r}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) + \int_{M_l} f\|_X$$

$$\leq \|\sum_{j=1}^{q} f(r_j)\mu(K_j) + \sum_{l=1}^{r}\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - \int_I f\|_X +$$

$$+ \sum_{l=1}^{r} \|\sum_{i=1}^{k_l} f(s_i^l)\mu(J_i^l) - \int_{M_l} f\|_X < \varepsilon + r\frac{\eta}{r+1} < \varepsilon + \eta$$

Since this inequality holds for any $\eta > 0$ we obtain the statement of the lemma. □

Our main goal is to show that if $f : I \to X$ is McShane integrable then f is McShane integrable over every measurable set $E \subset I$.

Theorem 3 shows that if $f : I \to X$ is McShane integrable then f is McShane integrable over every subinterval $J \subset I$.

It is clear that if $E \subset I$ is a finite union of non-overlapping intervals contained in I then a McShane integrable $f : I \to X$ is integrable over E.

Lemma 6. *If $f : I \to X$ is McShane integrable on I, then for every $\varepsilon > 0$ there is an $\eta > 0$ such that for any finite collection $\{J_j :\ j = 1, \dots, p\}$ of non-overlapping intervals in I with $\sum_{j=1}^{p} \mu(J_j) < \eta$ we have*

$$\|\sum_{j=1}^{p} \int_{J_j} f\|_X < \varepsilon.$$

Proof. Let $\epsilon > 0$ be given. Since f is McShane integrable on I, there exists a gauge δ on I such that $\| \sum_{i=1}^{q} f(t_i)\mu(I_i) - \int_I f\|_X < \epsilon$ whenever $\{(t_i, I_i);\ i = 1, \dots, q\}$ is an arbitrary δ-fine M-partition of I. Fix a δ-fine M-partition of I

$$\{(t_i, I_i);\ i = 1, \dots, q\},$$

put $K = \max\{\|f(t_i)\|_X; 1 \leq i \leq q\}$ and set $\eta = \dfrac{\varepsilon}{K+1}$.

Suppose that $\{J_j : j = 1, \dots, p\}$ is a finite collection of non-overlapping intervals in I such that $\sum_{j=1}^{p} \mu(J_j) < \eta$. By subdividing these intervals if necessary, we may assume that for each j, $J_j \subseteq I_i$ for some i. For each i, $1 \le i \le q$ let $M_i = \{j; 1 \le j \le p \text{ with } J_j \subseteq I_i\}$ and let

$$D = \{(J_j, t_i) : j \in M_i, i = 1, \dots, q\}.$$

Note that D is a δ-fine M-system in I.

Using the Saks-Henstock Lemma 5 we get

$$\|\sum_{j=1}^{p} \int_{J_j} f\|_X \le \|\sum_{j=1}^{p} \int_{J_j} f - f(t_i)\mu(J_j)\|_X + \sum_{j=1}^{p} \|f(t_i)\|_X \mu(J_j) \le$$

$$\le \varepsilon + M \sum_{j=1}^{p} \mu(J_j) < \varepsilon + K\eta < 2\varepsilon$$

and this proves the lemma.

□

Lemma 7. *If $f : I \to X$ is McShane integrable then*

(a) *for any sequence $\{I_i : i = 1, 2, \cdots\}$ of non-overlapping intervals $I_i \subset I$, $i \in \mathbb{N}$ the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{I_i} f = \sum_{i=1}^{\infty} \int_{I_i} f \in X$$

exists,

(b) *for every $\varepsilon > 0$ there is an $\eta > 0$ such that if for the sequence $\{I_i : i = 1, 2, \cdots\}$ of non-overlapping intervals $I_i \subset I$ we have $\sum_{i=1}^{\infty} \mu(I_i) < \eta$ then*

$$\|\sum_{i=1}^{\infty} \int_{I_i} f\|_X \le \varepsilon.$$

Proof. Assume that $\varepsilon > 0$ is given. Let $\eta > 0$ corresponds to ε by Lemma 6.

Since $\sum_{i=1}^{\infty} \mu(I_i) \le \mu(I) < \infty$, there is an $N \in \mathbb{N}$ such that for $n > N$ we have $\sum_{i=n}^{\infty} \mu(I_i) < \eta$.

Assume that $n, m \in \mathbb{N}$, $N < n < m$. Then by Lemma 6 we have

$$\|\sum_{i=1}^{m} \int_{I_i} f - \sum_{i=1}^{n} \int_{I_i} f\|_X = \|\sum_{i=n+1}^{m} \int_{I_i} f\|_X < \varepsilon$$

because $\sum_{i=n+1}^{m} \mu(I_i) \le \sum_{i=n+1}^{\infty} \mu(I_i) < \eta$.

This implies that $\sum_{i=1}^{n} \int_{I_i} f$, $n \in \mathbb{N}$ is a Cauchy sequence in X and (a) is proved.

If $\sum_{i=1}^{\infty}\mu(I_i)<\eta$ then $\sum_{i=1}^{n}\mu(I_i)<\eta$ for every $n\in\mathbb{N}$ and therefore

$$\|\sum_{i=1}^{n}\int_{I_i} f\|_X<\varepsilon$$

by Lemma 6. Since by (a) the series $\sum_{i=1}^{\infty}\int_{I_i} f$ converges in X, we obtain

$$\|\sum_{i=1}^{\infty}\int_{I_i} f\|_X=\|\lim_{n\to\infty}\sum_{i=1}^{n}\int_{I_i} f\|_X\le\varepsilon$$

and (b) is proved. □

Notation. For simplifying writing we will from now use the notation $\{(u_l,U_l)\}$ for M-systems instead of $\{(u_l,U_l);\ l=1,\dots,r\}$ which specifies the number r of elements of the M-system. For a function $f:I\to X$ and an M-system $\{(u_l,U_l)\}$ we write $\sum_l f(u_l)\mu(U_l)$ instead of $\sum_{l=1}^{r} f(u_l)\mu(U_l)$, etc.

Lemma 8. *Assume that $f:I\to X$ is McShane integrable.*

Then for every $\varepsilon>0$ there exists an $\eta>0$ such that

(a) *if F is closed, G open, $F\subset G\subset I$, $\mu(G\setminus F)<\eta$ then there is a gauge $\xi:I\to(0,\infty)$ such that*

$$B(t,\xi(t))\subset G\ \text{ for }\ t\in G,$$

$$B(t,\xi(t))\subset I\setminus F\ \text{ for }\ t\in I\setminus F$$

and

(b) *for ξ-fine M-systems $\{(u_l,U_l)\}$, $\{(v_m,V_m)\}$ satisfying*

$$u_l,v_m\in G,\ F\subset\ \operatorname{int}\bigcup_{u_l\in F}U_l\ F\subset\ \operatorname{int}\bigcup_{v_m\in F}V_m$$

we have

$$\|\sum_l f(u_l)\mu(U_l)-\sum_m f(v_m)\mu(V_m)\|_X\le\varepsilon.$$

Proof. Denote $\Phi(J)=\int_J f$ for an interval $J\subset I$ (the indefinite integral or primitive of f) and put $\widehat{\varepsilon}=\dfrac{\varepsilon}{10}$.

Since f is McShane integrable we obtain by the Saks-Henstock lemma 5 that there is a gauge Δ on I such that

$$\|\sum_j[f(r_j)\mu(K_j)-\Phi(K_j)]\|_X\le\widehat{\varepsilon}\tag{2}$$

for every Δ-fine M-system $\{(r_j, K_j)\}$.

Assume that

$$\{(w_p, W_p)\} \text{ is a } \Delta\text{-fine } M\text{-partition of } I. \tag{3}$$

Put

$$\kappa = \max_p \{\|f(w_p)\|_X\}, \tag{4}$$

assume that $\eta > 0$ satisfies

$$\eta \cdot \kappa \leq \widehat{\varepsilon} \tag{5}$$

and take

$$0 < \xi(t) \leq \Delta(t),\ t \in I. \tag{6}$$

Since the sets G and $I \setminus F$ are open it is clear that the gauge ξ can be chosen in such a way thet $B(t, \xi(t)) \subset G$ for $t \in G$ and $B(t, \xi(t)) \subset I \setminus F$ for $t \in I \setminus F$.

This is the introductory part (a) of the lemma and now we will show the part (b).

Since $\{(w_p, W_p)\}$ is a partition of I, we have $\bigcup_p W_p = I$ and therefore

$$\sum_l f(u_l)\mu(U_l) = \sum_p \sum_{l,\ u_l \in F} \sum_{m,\ v_m \in F} f(u_l)\mu(W_p \cap U_l \cap V_m) + \tag{7}$$

$$+ \sum_p \sum_{l,\ u_l \in F} f(u_l)\mu(W_p \cap U_l \setminus \bigcup_{m,\ v_m \in F} V_m) + \sum_p \sum_{l,\ u_l \in I \setminus F} f(u_l)\mu(W_p \cap U_l)$$

and similarly

$$\sum_m f(v_m)\mu(V_m) = \sum_p \sum_{l,\ u_l \in F} \sum_{m,\ v_m \in F} f(u_l)\mu(W_p \cap U_l \cap V_m) + \tag{8}$$

$$+ \sum_p \sum_{m,\ u_m \in F} f(u_l)\mu(W_p \cap V_m \setminus \bigcup_{l,\ u_l \in F} U_l) + \sum_p \sum_{m,\ v_m \in I \setminus F} f(v_m)\mu(W_p \cap V_m).$$

The M-systems

$$\{(u_l, W_p \cap U_l \cap V_m);\ p, u_l \in F, v_m \in F\},$$

$$\{(w_p, W_p \cap U_l \cap V_m);\ p, u_l \in F, v_m \in F\}$$

are Δ-fine and therefore, by (2), we have the inequalities

$$\|\sum_p \sum_{l,\ u_l \in F} \sum_{m,\ v_m \in F} f(u_l)\mu(W_p \cap U_l \cap V_m) - \Phi(W_p \cap U_l \cap V_m)\|_X \leq \widehat{\varepsilon},$$

$$\|\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(w_p)\mu(W_p\cap U_l\cap V_m) - \Phi(W_p\cap U_l\cap V_m)\|_X \le \widehat{\varepsilon}.$$

Hence

$$\|\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(u_l)\mu(W_p\cap U_l\cap V_m) -$$

$$-\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(w_p)\mu(W_p\cap U_l\cap V_m)\|_X \le 2\widehat{\varepsilon}$$

and similarly also

$$\|\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(v_m)\mu(W_p\cap U_l\cap V_m) -$$

$$-\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(w_p)\mu(W_p\cap U_l\cap V_m)\|_X \le 2\widehat{\varepsilon}.$$

Therefore

$$\|\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(u_l)\mu(W_p\cap U_l\cap V_m) -$$

$$-\sum_p \sum_{l,\ u_l\in F} \sum_{m,\ v_m\in F} f(v_m)\mu(W_p\cap U_l\cap V_m)\|_X \le 4\widehat{\varepsilon}. \tag{9}$$

Since $\{(u_l, U_l)\}$ is a ξ-fine M-system with $u_l \in G$, we obtain by the properties of the gauge ξ given in (a) and from the assumption $F \subset \ \text{int} \bigcup_{u_l\in F} U_l$, $F \subset \ \text{int} \bigcup_{v_m\in F} V_m$ that

$$\Big(\bigcup_{p,u_l\in F} W_p\cap U_l \setminus \bigcup_{v_m\in F} V_m\Big) \cup \bigcup_{p,u_l\in I\setminus F} W_p\cap U_l \subset G\setminus F. \tag{10}$$

Further the M-systems

$$\{(u_l, W_p\cap U_l\setminus \bigcup_{v_m\in F} V_m);\ p, u_l\in F\} \cup \{(u_l, W_p\cap U_l);\ p, u_l\in I\setminus F\},$$

$$\{(w_p, W_p\cap U_l\setminus \bigcup_{v_m\in F} V_m);\ p, u_l\in F\} \cup \{(w_p, W_p\cap U_l);\ p, u_l\in I\setminus F\}$$

are Δ-fine (note that here we have figures instead of intervals). Therefore by (2) we have

$$\|\sum_{p,u_l\in F} [f(u_l)\mu(W_p\cap U_l\setminus \bigcup_{v_m\in F} V_m) - \Phi(W_p\cap U_l\setminus \bigcup_{v_m\in F} V_m)] +$$

$$+\sum_{p,u_l\in I\setminus F}[f(u_l)\mu(W_p\cap U_l)-\Phi(W_p\cap U_l)]\|_X\le\widehat{\varepsilon},$$

$$\|\sum_{p,u_l\in F}[f(w_p)\mu(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)-\Phi(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)]+$$

$$+\sum_{p,u_l\in I\setminus F}[f(w_p)\mu(W_p\cap U_l)-\Phi(W_p\cap U_l)]\|_X\le\widehat{\varepsilon}.$$

This yields

$$\|\sum_{p,u_l\in F}f(u_l)\mu(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)+\sum_{p,u_l\in I\setminus F}f(u_l)\mu(W_p\cap U_l)-$$

$$-\sum_{p,u_l\in F}f(w_p)\mu(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)-\sum_{p,u_l\in I\setminus F}f(w_p)\mu(W_p\cap U_l)\|_X\le 2\widehat{\varepsilon}.$$

With respect to (10) and (5) we have

$$\|\sum_{p,u_l\in F}f(w_p)\mu(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)-\sum_{p,u_l\in I\setminus F}f(w_p)\mu(W_p\cap U_l)\|_X\le$$

$$\le\kappa\cdot\eta\le\widehat{\varepsilon}$$

and therefore

$$\|\sum_{p,u_l\in F}f(u_l)\mu(W_p\cap U_l\setminus\bigcup_{v_m\in F}V_m)+\sum_{p,u_l\in I\setminus F}f(u_l)\mu(W_p\cap U_l)\|_X\le 3\widehat{\varepsilon}\quad(11)$$

and similarly also

$$\|\sum_{p,v_m\in F}f(v_m)\mu(W_p\cap U_l\setminus\bigcup_{u_l\in F}U_l)+\sum_{p,v_m\in I\setminus F}f(v_m)\mu(W_p\cap V_m)\|_X\le 3\widehat{\varepsilon}.\quad(12)$$

From (7), (8), (9), (11) and (12) we get

$$\|\sum_{l}f(u_l)\mu(U_l)-\sum_{m}f(v_m)\mu(V_m)\|_X\le 10\widehat{\varepsilon}\le\varepsilon$$

and (8) is satisfied. This proves part (b) of the lemma. □

Theorem 9. *If $f:I\to X$ is McShane integrable then $f\cdot\chi_E$ is McShane integrable for every measurable set $E\subset I$ (f is McShane integrable over E).*

Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ corresponds to ε by Lemma 8. Assume that $E \subset I$ is measurable. Then there exist $F \subset I$ closed and $G \subset I$ open such that $F \subset E \subset G$ where $\mu(G \setminus F) < \eta$. Assume that the gauge $\xi : I \to (0, \infty)$ is given as in the Lemma 8 and that $\{(u_l, U_l)\}$, $\{(v_m, V_m)\}$ are ξ-fine M-partitions of I.

We have the following:

$$\text{if } \ u_l \in E \ \text{ then } U_l \subset G, F \subset \operatorname{int} \bigcup_{u_l \in F} U_l$$

and

$$\text{if } \ v_m \in E \ \text{ then } V_m \subset G, F \subset \operatorname{int} \bigcup_{v_m \in F} V_m.$$

Hence by (8) from Lemma 8 we have

$$\| \sum_{l, u_l \in E} f(u_l)\mu(U_l) - \sum_{m, v_m \in E} f(v_m)\mu(V_m) \|_X \leq \varepsilon$$

and therefore also

$$\| \sum_{l} f(u_l)\chi_E(u_l)\mu(U_l) - \sum_{m} f(v_m)\chi_E(v_m)\mu(V_m) \|_X \leq \varepsilon$$

and by Theorem 2 we can see that the McShane integral $\int_I f \cdot \chi_E = \int_E f$ exists. □

Remark 10. Theorem 9 was proved in [1], 2E Theorem by a different approach for the case when $I \subset \mathbb{R}$.

Theorem 11. *If $f : I \to X$ is McShane integrable then for every $\varepsilon > 0$ there is an $\eta > 0$ such that if $E \subset I$ is measurable with $\mu(E) < \eta$ then*

$$\| \int_I f \cdot \chi_E \|_X = \| \int_E f \|_X \leq 2\varepsilon.$$

Proof. Let $\varepsilon > 0$ be given and let $\eta > 0$ corresponds to ε by Lemma 6 and assume that $\mu(E) < \eta$. Then there is an open set $G \subset I$ such that $E \subset G$ and $\mu(G) < \eta$.

The McShane integrability of f over I implies the existence of a gauge $\Delta : I \to (0, +\infty)$ such that for every Δ-fine M-partition $\{(t_i, I_i)\}$ of I the inequality

$$\| \sum_i f(t_i)\mu(I_i) - \int_I f \|_X < \varepsilon$$

holds.

By Theorem 9 the integral $\int_I f \cdot \chi_E$ exists and by the definition of the integral to every $\theta > 0$ there is a gauge $\delta : I \to (0, +\infty)$ which satisfies $B(t, \delta(t)) \subset G$ if $t \in G$, $\delta(t) \le \Delta(t)$, $t \in I$ and

$$\|\sum_m f(v_m) \cdot \chi_E(v_m)\mu(V_m) - \int_I f \cdot \chi_E\|_X \le \theta$$

holds for any δ-fine M-partition $\{(v_m, V_m)\}$ of I.

If $v_m \in E \subset G$ then $V_m \subset G$ and $\sum_{m, v_m \in E} \mu(V_m) \le \eta$.

Since $\{(v_m, V_m);\ v_m \in E\}$ is a Δ-fine M-system, we have by the Saks-Henstock lemma 5 the inequality

$$\|\sum_{m, v_m \in E} [f(v_m)\mu(V_m) - \int_{V_m} f]\|_X \le \varepsilon$$

and by Lemma 6 we get

$$\|\sum_{m, v_m \in E} \int_{V_m} f\|_X \le \varepsilon.$$

Hence

$$\|\int_E f\|_X \le \theta + \|\sum_{m, v_m \in E} f(v_m)\mu(V_m)\|_X \le$$

$$\le \theta + \|\sum_{m, v_m \in E} [f(v_m)\mu(V_m) - \int_{V_m} f]\|_X + \|\sum_{m, v_m \in E} \int_{V_m} f\|_X \le \theta + 2\varepsilon.$$

This proves the statement because $\theta > 0$ can be chosen arbitrarily small. □

Remark 12. Theorem 10 represents an analog of absolute continuity of the indefinite McShane integral which was extended to measurable sets $E \subset I$ by Theorem 9.

Theorem 13. *If $f : I \to X$ is McShane integrable and $E \subset I$ is measurable where $F_i \subset E$, $i \in \mathbb{N}$ are closed sets with $F_i \subset F_{i+1}$ and $\mu(E \setminus \bigcup_i F_i) = 0$, then*

$$\int_I f \cdot \chi_E = \lim_{i \to \infty} \int_I f \cdot \chi_{F_i}.$$

Proof. First note that to a given measurable set E a sequence of closed sets F_i with the properties given in the theorem always exists.

Let an arbitrary $\varepsilon > 0$ be given and let $\eta > 0$ corresponds to it by Lemma 6 and Lemma 8, as well. Let $G \subset I$ be open such that $E \subset G$ and $\mu(G \setminus E) < \dfrac{\eta}{2}$.

Further there is a $k_0 \in \mathbb{N}$ such that $\mu(E \setminus F_{k_0}) < \dfrac{\eta}{2}$ and therefore $\mu(G \setminus F_{k_0}) < \eta$ and of course also $\mu(G \setminus F_k) < \eta$ for all $k \geq k_0$.

Let the gauge $\xi : I \to (0,\infty)$ be given by the Lemma 8 for the set F_{k_0} in the role of F, in particular we have

$$B(t,\xi(t)) \subset I \setminus F_{k_0} \text{ for } t \in I \setminus F_{k_0}.$$

Assume that $\{(u_l, U_l)\}$ is an arbitrary ξ-fine M-partition of I.

Fix an arbitrary $k \geq k_0$.

By Theorem 9 the gauge ξ can be chosen in such a way that in addition we have

$$\| \int_I f - \sum_l f(u_l)\mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_E - \sum_l f(u_l) \cdot \chi_E(u_l)\mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_k} - \sum_l f(u_l) \cdot \chi_{F_k}(u_l)\mu(U_l) \|_X \leq \varepsilon.$$

The last two inequalities can be written in the form

$$\| \int_I f \cdot \chi_E - \sum_{u_l \in E} f(u_l)\mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_k} - \sum_{u_l \in F_k} f(u_l)\mu(U_l) \|_X \leq \varepsilon.$$

This gives

$$\| \int_I f \cdot \chi_E - \int_I f \cdot \chi_{F_k} \|_X \leq 2\varepsilon + \| \sum_{u_l \in E \setminus F_k} f(u_l)\mu(U_l) \|_X.$$

Further by the Saks-Henstock lemma 5 we have

$$\| \sum_{u_l \in E \setminus F_k} f(u_l)\mu(U_l) \|_X \leq \| \sum_{u_l \in E \setminus F_k} [f(u_l)\mu(U_l) - \int_{U_l} f] \|_X +$$

$$+ \| \sum_{u_l \in E \setminus F_k} \int_{U_l} f \|_X \leq \varepsilon + \| \sum_{u_l \in E \setminus F_k} \int_{U_l} f \|_X$$

while

$$\| \sum_{u_l \in E \setminus F_k} \int_{U_l} f \|_X \leq \varepsilon$$

by Lemma 6 because we have $\sum_{u_l \in E \setminus F_k} \mu(U_l) \leq \mu(G \setminus F_{k_0}) < \eta$. Hence

$$\| \int_I f \cdot \chi_E - \int_I f \cdot \chi_{F_k} \|_X \leq 4\varepsilon$$

for $k \geq k_0$ and this proves the theorem. $\square$

Theorem 14. *If $f : I \to X$ is McShane integrable and $F_1, F_2 \subset I$ are closed sets with $F_1 \cap F_2 = \emptyset$ then*

$$\int_I f \cdot \chi_{F_1 \cup F_2} = \int_I f \cdot \chi_{F_1} + \int_I f \cdot \chi_{F_2}.$$

Proof. Assume that $\varepsilon > 0$ is given and that $\eta > 0$ corresponds to ε by Lemma 8.

Since F_1 and F_2 are disjoint closed sets, we have dist $(F_1, F_2) > 0$ and therefore there exist open sets G_1 and G_2 such that $F_1 \subset G_1$, $F_2 \subset G_2$, $G_1 \cap G_2 = \emptyset$, $\mu(G_1 \setminus F_1) < \dfrac{\eta}{2}$, $\mu(G_2 \setminus F_2) < \dfrac{\eta}{2}$.

Hence

$$\mu(G_1 \cup G_2 \setminus (F_1 \cup F_2)) < \eta.$$

For the open set $G = G_1 \cup G_2$ and the closed set $F = F_1 \cup F_2$ let the gauge $\xi : I \to (0, +\infty)$ be given by Lemma 8.

For a given ξ-fine M-partition $\{(u_l, U_l)\}$ of I we have

$$\| \int_I f \cdot \chi_{F_1} - \sum_l f(u_l) \cdot \chi_{F_1}(u_l) \mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_2} - \sum_l f(u_l) \cdot \chi_{F_2}(u_l) \mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_1 \cup F_2} - \sum_l f(u_l) \cdot \chi_{F_1 \cup F_2}(u_l) \mu(U_l) \|_X \leq \varepsilon$$

and this means in other words

$$\| \int_I f \cdot \chi_{F_1} - \sum_{u_l \in F_1} f(u_l)(u_l) \mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_2} - \sum_{u_l \in F_2} f(u_l)(u_l) \mu(U_l) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{F_1 \cup F_2} - \sum_{u_l \in F_1 \cup F_2} f(u_l)(u_l) \mu(U_l) \|_X \leq \varepsilon.$$

This yields

$$\| \int_I f \cdot \chi_{F_1} + \int_I f \cdot \chi_{F_2} - \int_I f \cdot \chi_{F_1 \cup F_2} \|_X \le 3\varepsilon$$

and the statement of the theorem is proved because $\varepsilon > 0$ can be taken arbitrarily small. □

Theorem 15. *If $f : I \to X$ is McShane integrable and $E_1, E_2 \subset I$ are measurable sets with $E_1 \cap E_2 = \emptyset$ then*

$$\int_I f \cdot \chi_{E_1 \cup E_2} = \int_I f \cdot \chi_{E_1} + \int_I f \cdot \chi_{E_2}.$$

Proof. By Theorem 14 the statement holds for closed sets, Theorem 13 yields the result by passing to limits for sequences of closed sets contained in E_1, E_2. □

Theorem 16. *If $f : I \to X$ is McShane integrable and $E_i \subset I$, $i \in \mathbb{N}$ are measurable sets with $E_i \cap E_i = \emptyset$ for $i \neq j$ then*

$$\int_I f \cdot \chi_{\bigcup_i E_i} = \sum_i \int_I f \cdot \chi_{E_i}.$$

Proof. By Theorem 9 all the integrals $\int_I f \cdot \chi_{\bigcup_i E_i}$, $\int_I f \cdot \chi_{E_i}$, $i \in \mathbb{N}$ exist.

Let $\varepsilon > 0$ be given; by the definition of the McShane integral there exist gauges $\delta : I \to (0, +\infty)$, $\delta_i : I \to (0, +\infty)$, $i \in \mathbb{N}$ such that

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_j f(t_j) \cdot \chi_{\bigcup_i E_i}(t_j)\mu(I_j) \|_X \le \varepsilon$$

for any δ-fine M-partition $\{(t_j, I_j)\}$ of the interval I and

$$\| \int_I f \cdot \chi_{E_i} - \sum_j f(t_j) \cdot \chi_{E_i}(t_j)\mu(I_j) \|_X \le \frac{\varepsilon}{2^i}$$

for every δ_i-fine M-partition $\{(t_j, I_j)\}$ of the interval I, $i \in \mathbb{N}$.

Assume now that $\eta > 0$ corresponds to the given ε by Lemma 8 and that $k \in \mathbb{N}$ is such that

$$\mu(\bigcup_{i>k} E_i) < \frac{\eta}{2}.$$

Assume further that a closed set $F \subset I$ is contained in the measurable union $\bigcup_{i \le k} E_i$ ($F \subset \bigcup_{i \le k} E_i$) while

$$\mu(\bigcup_{i \le k} E_i \setminus F) < \frac{\eta}{2}.$$

Hence we have

$$\mu(\bigcup_i E_i \setminus F) < \eta$$

and there is an open set $G \subset I$ such that $\bigcup_i E_i \subset G$ and $\mu(G \setminus F) < \eta$.

Let $\xi : I \to (0, +\infty)$ be the gauge given by (a) from Lemma 8.

Take

$$\theta_i = \min(\xi, \delta_i),\ i \in \mathbb{N},\ \theta = \min(\xi, \delta)$$

and let $l \in \mathbb{N}$ be such that $l > k$. Put

$$\omega = \min(\theta, \theta_1, \dots, \theta_l)$$

and take an arbitrary ω-fine M-partition $\{(s_j, K_j)\}$ of I. For such a partition we have

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_j f(s_j) \cdot \chi_{\bigcup_i E_i}(s_j) \mu(K_j) \|_X \leq \varepsilon,$$

$$\| \int_I f \cdot \chi_{E_i} - \sum_j f(s_j) \cdot \chi_{E_i}(s_j) \mu(K_j) \|_X \leq \frac{\varepsilon}{2^i},\ i = 1, \dots, l$$

i.e.

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_{s_j \in \bigcup_i E_i} f(s_j) \mu(K_j) \|_X \leq \varepsilon$$

and

$$\| \int_I f \cdot \chi_{E_i} - \sum_{s_j \in E_i} f(s_j) \mu(K_j) \|_X \leq \frac{\varepsilon}{2^i},\ i = 1, \dots, l.$$

Therefore

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_{i=1}^{l} \int_I f \cdot \chi_{E_i} - \sum_{s_j \in \bigcup_{i>l} E_i} f(s_j) \mu(K_j) \|_X \leq 2\varepsilon$$

and

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_{i=1}^{l} \int_I f \cdot \chi_{E_i} \|_X \leq 2\varepsilon + \| \sum_{s_j \in \bigcup_{i>l} E_i} f(s_j) \mu(K_j) \|_X.$$

Using Lemma 6 we get

$$\| \sum_{s_j \in \bigcup_{i>l} E_i} f(s_j) \mu(K_j) \|_X \leq 2\varepsilon$$

and this yields

$$\| \int_I f \cdot \chi_{\bigcup_i E_i} - \sum_{i=1}^{l} \int_I f \cdot \chi_{E_i} \|_X \leq 4\varepsilon$$

and the statement is proved because this can be done for every $l > k$. □

Remark 17. Theorem 16 extends the statement (a) from Lemma 7 to sequences of measurable sets an it says that the indefinite McShane integral of a given McShane integrable $f : I \to X$ is countably additive.

The notion of the McShane integral of a function given in Definition 1 is based on the concept of M-partitions of the interval I.

Let us define the following:

A system (finite collection) of pairs $\{(t_i, E_i),\ i = 1, \dots, p\}$ with $E_i \subset I$ measurable, $E_i \cap E_j = \emptyset$ for $i \neq j$ is called an *M^*-system in I.*

An M^*-system in I is called an *M^*-partition of I* if $\bigcup_{i=1}^{p} E_i = I$.

Given a gauge $\Delta : I \to (0, +\infty)$, an M^*-system $\{(t_i, E_i),\ i = 1, \dots, p\}$ in I is called *Δ-fine* if

$$E_i \subset B(t_i, \Delta(t_i)),\ i = 1, \dots, p.$$

Definition 18. A function $f : I \to X$ is *McShane* integrable* and $J \in X$ is its *McShane* integral over I* if for every $\varepsilon > 0$ there exists a gauge $\Delta : I \to (0, +\infty)$ such that for every Δ-fine M^*-partition $(s_i, E_i), i = 1, \dots, p$ of I the inequality

$$\| \sum_{i=1}^{p} f(s_i)\mu(E_i) - J \|_X \leq \varepsilon$$

holds. Denote $J = \int_I^* f$.

It is clear that *if $f : I \to X$ is McShane* integrable then f is McShane integrable* in the sense of Definition 1.

We will show that the concept of the McShane* integral from Definition 18 is not less general than that of the McShane integral from Definition 1. To this aim let us prove the next lemma.

Lemma 19. *Assume that $A \subset I$ is a figure and that $\varepsilon > 0$ is given. Let $\delta : I \to (0, +\infty)$ be a gauge and let $\{(t_i, E_i),\ i = 1, \dots, k\}$ is such $t_i \in I$, $E_i \subset A$ are measurable sets with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E_i \subset B(t_i, \frac{1}{2}\delta(t_i))$.*

Then for $i = 1, \dots, k$ there exist figures $C_i \subset A$ such that $\mu(C_i \cap C_j) = 0$ for $i \neq j$ and

$$\mu(E_i \bigtriangleup C_i) < \varepsilon \ \text{ for } \ i = 1, \dots, k,$$

($E_i \triangle C_i = E_i \setminus C_i \cup C_i \setminus E_i$ is the symmetric difference of the sets E_i and C_i)

$$C_i \subset B(t_i, \delta(t_i)).$$

Proof. We prove the statement by induction.

Assume that $k = 1$, i.e. we have $t_1 \in I$ and $E_1 \subset A$.

Let $\lambda > 0$ be arbitrary. Then there is a set $G \subset A$ which is open in A such that $G \subset B(t_1, \delta(t_1))$, $E_1 \subset G$, $\mu(G \setminus E_1) < \lambda$ and a figure C_1 such that $C_1 \subset G$ and $\mu(G \setminus C_1) < \lambda$.

We have

$$\mu(E_1 \triangle C_1) = \mu(E_1 \setminus C_1) + \mu(C_1 \setminus E_1) \leq$$
$$\leq \mu(G \setminus C_1) + \mu(G \setminus E_1) \leq 2\lambda$$

and the statement holds in this case if we put $\lambda = \dfrac{\varepsilon}{2}$ because $C_1 \subset G \subset B(t_1, \delta(t_1))$.

Coming to the induction step, assume that the statement of the lemma holds for some $k \in \mathbb{N}$ and let $(t_0, E_0), (t_1, E_1), \ldots, (t_k, E_k)$ are $k + 1$ point-set pairs satisfying the assumption. Let $\lambda > 0$ be arbitrary.

Using the first part of the proof there is a figure C_0 contained in A, such that

$$\mu(E_0 \triangle C_0) \leq 2\lambda \tag{13}$$

and

$$C_0 \subset B(t_0, \delta(t_0))$$

holds.

Put $A^* = \text{cl}\,(A \setminus C_0)$, ($\text{cl}\,(A \setminus C_0)$ is the closure of the set $A \setminus C_0$) and let $E_i^* = E_i \cap A^*$ for $i = 1, \ldots, k$. A^* is evidently a figure while $\{(t_i, E_i^*),\ i = 1, \ldots, k\}$ satisfy the assumptions of the lemma.

By the induction assumption there exist figures $C_1, \ldots, C_k$ contained in A^*, such that $\mu(C_i \cap C_j) = 0$ for $i \neq j$ and

$$\mu(E_i^* \triangle C_i) \leq \lambda, \tag{14}$$

$$C_i \subset B(t_i, \delta(t_i)),\ i = 1, \ldots, k.$$

We also have

$$\mu(C_0 \cap C_i) = 0 \ \text{ for } \ i = 1, \ldots, k$$

because $C_i \subset A^* = \text{cl}\,(A \setminus C_0)$.

For $i = 1, \ldots, k$ we have $E_0 \cap E_i = \emptyset$ and therefore

$$E_i \cap C_0 = E_i \cap (C_0 \setminus E_0),$$

$$\mu(E_i \cap C_0) \leq \mu(C_0 \setminus E_0) < \lambda. \tag{15}$$

Since $E_i \subset E_i^* \cup E_i \cap C_0$ we get

$$\mu(C_i \setminus E_i) = \mu(C_i \setminus E_i^*) < \lambda. \tag{16}$$

On the other hand we have, by (14) and (15),

$$\mu(E_i \setminus C_i) \le \mu(E_i^* \setminus C_i) + \mu((E_i \cap C_0) \setminus C_i) \le 2\lambda$$

and this together with (16) shows that for $i = 1, \dots, k$ we have

$$\mu(E_i \bigtriangleup C_i) \le 3\lambda.$$

Taking into account (13) we obtain the result because $\lambda > 0$ can be taken arbitrarily small. □

Theorem 20. *If $f : I \to X$ is McShane integrable then f is McShane* integrable and*

$$\int_I^* f = \int_I f.$$

Proof. Let $\varepsilon > 0$ be given. By the Saks-Henstock lemma 5 there exists a gauge $\delta : I \to (0, +\infty)$ such that for every δ-fine M-system $(r_j, K_j), j = 1, \dots, q$ of I the inequality

$$\| \sum_j [f(r_j)\mu(K_j) - \int_{K_j} f] \|_X \le \varepsilon$$

holds. This implies that if $\{(r_j, C_j),\ j = 1, \dots, q\}$, $r_j \in I$, C_j are non-overlapping figures contained in I with $C_j \subset B(r_j, \delta(r_j))$ then

$$\| \sum_j [f(r_j)\mu(C_j) - \int_{C_j} f] \|_X \le \varepsilon. \tag{17}$$

Assume that $\{(s_i, E_i), i = 1, \dots, p\}$ is an arbitrary $\frac{1}{2}\delta$-fine M^*-partition of I.

By Theorem 11 there is an $\eta > 0$ such that if $E \subset I$ is measurable and $\mu(E) < \eta$ then

$$\| \int_E f \|_X \le \frac{\varepsilon}{p}. \tag{18}$$

By Lemma 19 there exist figures $C_i \subset I$ such that $\mu(C_i \cap C_j) = 0$ for $i \ne j$ with

$$\mu(E_i \bigtriangleup C_i) < \min\{\eta, \frac{\varepsilon}{p \cdot [\max_j (\|f(s_j\|_X) + 1]}\} \tag{19}$$

and

$$C_i \subset B(t_i, \delta(t_i))$$

for $i = 1, \dots, p$.

We have

$$\| \sum_i f(s_i)\mu(E_i) - \int_I f \|_X = \| \sum_i [f(s_i)\mu(E_i) - \int_{E_i} f] \|_X = \tag{20}$$

$$= \| \sum_i [f(s_i)\mu(C_i) + f(s_i)[\mu(E_i) - \mu(C_i)] -$$

$$-[\int_{C_i} f + \int_{E_i \setminus C_i} f - \int_{C_i \setminus E_i} f]] \|_X \le$$

$$\le \| \sum_i [f(s_i)\mu(C_i) - \int_{C_i} f] \|_X + \sum_i \|f(s_i)\|_X |\mu(E_i) - \mu(C_i)| +$$

$$+ \sum_i \| \int_{E_i \setminus C_i} f \|_X + \sum_i \| \int_{C_i \setminus E_i} f \|_X.$$

Since

$$|\mu(E_i) - \mu(C_i)| \le \mu(E_i \triangle C_i) < \frac{\varepsilon}{p \cdot [\max_j(\|f(s_j\|_X) + 1]}$$

we have

$$\sum_i \|f(s_i)\|_X |\mu(E_i) - \mu(C_i)| \le \sum_i \|f(s_i)\|_X \frac{\varepsilon}{p \cdot [\max_j(\|f(s_j\|_X) + 1]} < \varepsilon \tag{21}$$

and because

$$\mu(E_i \setminus C_i) \le \mu(E_i \triangle C_i) < \eta, \ \mu(C_i \setminus E_i) \le \mu(E_i \triangle C_i) < \eta$$

we obtain by (18)

$$\sum_i \| \int_{E_i \setminus C_i} f \|_X < \sum_i \frac{\varepsilon}{p} = \varepsilon, \qquad \sum_i \| \int_{E_i \setminus C_i} f \|_X < \varepsilon. \tag{22}$$

Using the figure version (17) of the Saks-Henstock lemma we obtain finally from (20), (21) and (22)

$$\| \sum_i f(s_i)\mu(E_i) - \int_I f \|_X < 4\varepsilon$$

and this shows that f is McShane* integrable and that $\int_I^* f = \int_I f$ holds. □

Hence we arrive at the following result.

Theorem 21. *A function $f : I \to X$ is McShane integrable if and only if f is is McShane* integrable and both the integrals $\int_I^* f$, $\int_I f$ coincide.*

Remark 22. The concept of McShane* integrability was considered in a more general setting in Fremlin's paper [2], 1A Definitions. See also [1], 2H Lemma.

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