

# MCSHANE EQUI-INTEGRABILITY AND VITALI'S CONVERGENCE THEOREM

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ABSTRACT. The McShane integral of functions  $f : I \rightarrow \mathbb{R}$  defined on an  $m$ -dimensional interval  $I$  is considered in the paper. This integral is known to be equivalent to the Lebesgue integral for which the Vitali convergence theorem holds.

For McShane integrable sequences of functions a convergence theorem based on the concept of equi-integrability is proved and it is shown that this theorem is equivalent to the Vitali convergence theorem.

We consider functions  $f : I \rightarrow \mathbb{R}$  where  $I \subset \mathbb{R}^m$  is a compact interval,  $m \geq 1$ .

A system (finite collection) of point-interval pairs  $\{(t_i, I_i), i = 1, \dots, p\}$  is called an  $M$ -system in  $I$  if  $I_i$  are non-overlapping ( $\text{int } I_i \cap \text{int } I_j = \emptyset$  for  $i \neq j$ ,  $\text{int } I_i$  is the interior of  $I_i$ ),  $t_i$  are arbitrary points in  $I$ .

Denote by  $\mu$  the Lebesgue measure in  $\mathbb{R}^m$ .

An  $M$ -system in  $I$  is called an  $M$ -partition of  $I$  if  $\bigcup_{i=1}^p I_i = I$ .

Given  $\Delta : I \rightarrow (0, +\infty)$ , called a gauge, an  $M$ -system  $\{(t_i, I_i), i = 1, \dots, p\}$  in  $I$  is called  $\Delta$ -fine if

$$I_i \subset B(t_i, \Delta(t_i)), \quad i = 1, \dots, p.$$

The set of  $\Delta$ -fine partitions of  $I$  is nonempty (Cousin's lemma, see e.g. [1]).

**1. Definition.**  $f : I \rightarrow \mathbb{R}$  is McShane integrable and  $J \in \mathbb{R}$  is its McShane integral if for every  $\varepsilon > 0$  there exists a gauge  $\Delta : I \rightarrow (0, +\infty)$  such that for every  $\Delta$ -fine  $M$ -partition  $\{(t_i, I_i), i = 1, \dots, p\}$  of  $I$  the inequality

$$\left| \sum_{i=1}^p f(t_i) \mu(I_i) - J \right| < \varepsilon$$

holds. Denote  $J = \int_I f$ .

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**Notation.** For simplifying writing we will from now use the notation  $\{(u_l, U_l)\}$  for  $M$ -systems instead of  $\{(u_l, U_l); l = 1, \dots, r\}$  which specifies the number  $r$  of elements of the  $M$ -system. For a function  $f : I \rightarrow \mathbb{R}$  and an  $M$ -system  $\{(u_l, U_l)\}$  we write  $\sum_l f(u_l)\mu(U_l)$  instead of  $\sum_{l=1}^r f(u_l)\mu(U_l)$ , etc.

**2. Theorem.**  $f : I \rightarrow \mathbb{R}$  is McShane integrable if and only if  $f$  is Lebesgue integrable.

See [2] or [4].

**3. Definition.** A collection  $\mathcal{M}$  of functions  $f : I \rightarrow \mathbb{R}$  is called equi-integrable if every  $f \in \mathcal{M}$  is McShane integrable and for every  $\varepsilon > 0$  there is a gauge  $\Delta$  such that for any  $f \in \mathcal{M}$  the inequality

$$\left| \sum_i f(t_i)\mu(I_i) - \int_I f \right| < \varepsilon$$

holds provided  $\{(t_i, I_i)\}$  is a  $\Delta$ -fine  $M$ -partition of  $I$ .

**4. Theorem.** A collection  $\mathcal{M}$  of functions  $f : I \rightarrow X$  is equi-integrable if and only if for every  $\varepsilon > 0$  there exists a gauge  $\Delta : I \rightarrow (0, +\infty)$  such that

$$\left\| \sum_i f(t_i)\mu(I_i) - \sum_j f(s_j)\mu(K_j) \right\|_X < \varepsilon$$

for every  $\Delta$ -fine  $M$ -partitions  $\{(t_i, I_i)\}$  and  $\{(s_j, K_j)\}$  of  $I$  and any  $f \in \mathcal{M}$ .

*Proof.* If  $\mathcal{M}$  is equi-integrable then the condition clearly holds for the gauge  $\delta$  which corresponds to  $\frac{\varepsilon}{2} > 0$  in the definition of equi-integrability.

If the condition of the theorem is fulfilled, then every individual function  $f \in \mathcal{M}$  is McShane integrable (see e.g. [5]) with the same gauge  $\delta$  for a given  $\varepsilon > 0$  independently on the choice of  $f \in \mathcal{M}$  and this proves the theorem.

**5. Theorem.** Assume that  $\mathcal{M} = \{f_k : I \rightarrow \mathbb{R}; k \in \mathbb{N}\}$  is an equi-integrable sequence such that

$$\lim_{k \rightarrow \infty} f_k(t) = f(t), \quad t \in I.$$

Then the function  $f : I \rightarrow \mathbb{R}$  is McShane integrable and

$$\lim_{k \rightarrow \infty} \int_I f_k = \int_I f$$

holds.

*Proof.* If  $\Delta$  is the gauge from the definition of equi-integrability of the sequence  $f_k$  corresponding to the value  $\varepsilon > 0$  then for any  $k \in \mathbb{N}$

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < \varepsilon \quad (1)$$

for every  $\Delta$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$ .

If the partition  $\{(t_i, I_i)\}$  is fixed then the pointwise convergence  $f_k \rightarrow f$  yields

$$\lim_{k \rightarrow \infty} \sum_i f_k(t_i) \mu(I_i) = \sum_i f(t_i) \mu(I_i).$$

Choose  $k_0 \in \mathbb{N}$  such that for  $k > k_0$  the inequality

$$\left| \sum_i f_k(t_i) \mu(I_i) - \sum_i f(t_i) \mu(I_i) \right| < \varepsilon$$

holds. Then we have

$$\begin{aligned} & \left| \sum_i f(t_i) \mu(I_i) - \int_I f_k \right| \leq \\ & \leq \left| \sum_i f(t_i) \mu(I_i) - f_k(t_i) \mu(I_i) \right| + \left| \sum_i f_k(t_i) \mu(I_i) - \int_I f_k \right| < 2\varepsilon \end{aligned}$$

for  $k > k_0$ .

This gives for  $k, l > k_0$  the inequality

$$\left| \int_I f_k - \int_I f_l \right| < 4\varepsilon$$

which shows that the sequence of real numbers  $\int_I f_k$ ,  $k \in \mathbb{N}$  is Cauchy and therefore

$$\lim_{k \rightarrow \infty} \int_I f_k = J \in \mathbb{R} \text{ exists.} \quad (2)$$

Let  $\varepsilon > 0$ . By hypothesis there is a gauge  $\Delta$  such that (1) holds for all  $k$  whenever  $\{(t_i, I_i)\}$  is a  $\Delta$ -fine  $M$ -partition of  $I$ . By (2) choose a  $N \in \mathbb{N}$  such that  $|\int_I f_k - J| < \varepsilon$  for all  $k \geq N$ . Suppose that  $\{(t_i, I_i)\}$  is a  $\Delta$ -fine  $M$ -partition of  $I$ . Since  $f_k$  converges to  $f$  pointwise there is a  $k_1 \geq N$  such that

$$\left| \sum_i f_{k_1}(t_i) \mu(I_i) - \sum_i f(t_i) \mu(I_i) \right| < \varepsilon.$$

Therefore

$$\begin{aligned} \left| \sum_i f(t_i) \mu(I_i) - J \right| & \leq \left| \sum_i f(t_i) \mu(I_i) - \sum_i f_{k_1}(t_i) \mu(I_i) \right| + \\ & + \left| \sum_i f_{k_1}(t_i) \mu(I_i) - \int_I f_{k_1} \right| + \left| \int_I f_{k_1} - J \right| < 3\varepsilon \end{aligned}$$

and it follows that  $f$  is McShane integrable and  $\lim_{k \rightarrow \infty} \int_I f_k = J = \int_I f$ .

**6. Remark.** By a *figure* we mean a finite union of compact nondegenerate intervals in  $\mathbb{R}^m$ .

We mention the fact that if for the notion of an  $M$ -system  $\{(t_i, I_i), i = 1, \dots, p\}$  the intervals  $I_i$  are replaced by figures, we can develop the same theory and  $M$ -systems and  $M$ -partitions of this kind can be used everywhere in our considerations.

**7. Definition.** Let  $\mathcal{M}$  be a collection of Lebesgue integrable functions  $f : I \rightarrow \mathbb{R}$ .

If for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that for  $E \subset I$  measurable with  $\mu(E) < \delta$  we have  $|\int_E f| < \varepsilon$  for every  $f \in \mathcal{M}$  then the collection  $\mathcal{M}$  is called *uniformly absolutely continuous*.

**8. Theorem.** Assume that a sequence of Lebesgue integrable functions  $f_k : I \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$  is given such that  $f_k$  converges to  $f$  in measure.

If the set  $\{f_k; k \in \mathbb{N}\}$  is uniformly absolutely continuous then the function  $f$  is Lebesgue integrable and

$$\lim_{k \rightarrow \infty} \int_I f_k = \int_I f.$$

See [3, p. 168] or [1, p. 203, Theorem 13.3].

We will consider Theorem 8 in a less general form

**9. Theorem.** Assume that a sequence of Lebesgue integrable functions  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  is given such that  $f_k$  converges to  $f$  pointwise in  $I$ .

If the set  $\{f_k; k \in \mathbb{N}\}$  is uniformly absolutely continuous then the function  $f$  is Lebesgue integrable and

$$\lim_{k \rightarrow \infty} \int_I f_k = \int_I f.$$

**10. Remark.** It is possible to assume in Theorem 9 that  $f_k$  converges to  $f$  almost everywhere in  $I$ , but changing the values of  $f_k$  and  $f$  to 0 on a set  $N$  of zero Lebesgue measure ( $\mu(N) = 0$ ) it can be seen easily that such a change has no effect on Lebesgue integrability and on the corresponding indefinite Lebesgue integrals.

Our goal is to show that the relaxed Vitali convergence Theorem 9 is a consequence of our convergence Theorem 4 for the McShane integral.

**11. Lemma (Saks-Henstock).** Assume that a collection  $\mathcal{M}$  of functions  $f : I \rightarrow \mathbb{R}$  is equi-integrable. Given  $\varepsilon > 0$  assume that the gauge  $\Delta$  on  $I$  is such that

$$|\sum_i f(t_i)\mu(I_i) - \int_I f| < \varepsilon$$

for every  $\Delta$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$  and  $f \in \mathcal{M}$ .

Then if  $\{(r_j, K_j)\}$  is an arbitrary  $\Delta$ -fine  $M$ -system we have

$$|\sum_j [f(r_j)\mu(K_j) - \int_{K_j} f]| \leq \varepsilon$$

for every  $f \in \mathcal{M}$ .

*Proof.* Since  $\{(r_j, K_j)\}$  is a  $\Delta$ -fine  $M$ -system the complement  $I \setminus \text{int}(\bigcup_j K_j)$  can be expressed as a finite system  $M_l$ ,  $l = 1, \dots, r$  of non-overlapping intervals in  $I$ . The functions  $f \in \mathcal{M}$  are equi-integrable and therefore they are equi-integrable over all

$M_l$  and by definition for any  $\eta > 0$  there is a gauge  $\delta_l$  on  $M_l$  with  $\delta_l(t) < \delta(t)$  for  $t \in M_l$  such that for every  $l = 1, \dots, r$  we have

$$\left\| \sum_i f(s_i^l) \mu(J_i^l) - \int_{M_l} f d\mu \right\|_Y < \frac{\eta}{r+1}$$

provided  $\{(s_i^l, J_i^l)\}$  is a  $\delta_l$ -fine  $M$ -partition of the interval  $M_l$  and  $f \in \mathcal{M}$ .

The sum

$$\sum_j f(r_j) \mu(K_j) + \sum_l \sum_i f(s_i^l) \mu(J_i^l)$$

represents an integral sum which corresponds to a certain  $\delta$ -fine  $M$ -partition of  $I$ , namely  $\{(r_j, K_j), (s_i^l, J_i^l)\}$ , and consequently by the assumption we have

$$\left| \sum_j f(r_j) \mu(K_j) + \sum_l \sum_i f(s_i^l) \mu(J_i^l) - \int_I f d\mu \right| < \varepsilon.$$

Hence

$$\begin{aligned} & \left| \sum_j f(r_j) \mu(K_j) - \int_{K_j} f d\mu \right| \leq \\ & \leq \left| \sum_j f(r_j) \mu(K_j) + \sum_l \sum_i f(s_i^l) \mu(J_i^l) - \int_I f d\mu \right| + \\ & + \sum_l \left| \sum_i f(s_i^l) \mu(J_i^l) - \int_{M_l} f d\mu \right| < \varepsilon + r \frac{\eta}{r+1} < \varepsilon + \eta \end{aligned}$$

Since this inequality holds for every  $\eta > 0$  and  $f \in \mathcal{M}$  we obtain immediately the statement of the lemma.

Looking at Lemma 11 it can be seen immediately that if the equi-integrable collection  $\mathcal{M}$  consists of a single McShane integrable function  $f$ , then the following standard Saks-Henstock Lemma holds.

*Assume that  $f : I \rightarrow \mathbb{R}$  is McShane integrable. Given  $\varepsilon > 0$  assume that the gauge  $\Delta$  on  $I$  is such that*

$$\left| \sum_i f(t_i) \mu(I_i) - \int_I f \right| < \varepsilon$$

*for every  $\Delta$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$ .*

*Then if  $\{(r_j, K_j)\}$  is an arbitrary  $\Delta$ -fine  $M$ -system we have*

$$\left| \sum_j [f(r_j) \mu(K_j) - \int_{K_j} f] \right| \leq \varepsilon.$$

**12. Proposition.** *Assume that  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  are McShane (=Lebesgue) integrable functions such that*

- 1)  $f_k(t) \rightarrow f(t)$  for  $t \in I$ ,
- 2) the set  $\{f_k; k \in \mathbb{N}\}$  is uniformly absolutely continuous.

*Then the set  $\{f_k; k \in \mathbb{N}\}$  is equi-integrable.*

*Proof.* Assuming 1) we will use Egoroff's Theorem (see [3] or [1, Th. 2.13, p. 22]) in the following form:

*For every  $j \in \mathbb{N}$  there is a measurable set  $E_j \subset I$  such that  $\mu(I \setminus E_j) < \frac{1}{j}$ ,  $E_j \subset E_{j+1}$  and  $f_k(t) \rightarrow f(t)$  uniformly for  $t \in E_j$ , i.e. for every  $\varepsilon > 0$  there is a  $K_j \in \mathbb{N}$  such that for  $k > K_j$  we have*

$$|f_k(t) - f(t)| < \varepsilon \text{ for } t \in E_j. \quad (3)$$

Let us mention that for  $N = I \setminus \bigcup_{j=1}^{\infty} E_j$  we have  $\mu(N) = 0$  because  $\mu(N) \leq \mu(I \setminus E_j) < \frac{1}{j}$  for every  $j \in \mathbb{N}$ .

Using Remark 10 we may assume without any loss of generality that  $f_k(t) = f(t) = 0$  for  $k \in \mathbb{N}$  and  $t \in N$ .

Assume now that  $\varepsilon > 0$  is given. By the assumption 2) there is a  $j \in \mathbb{N}$  such that

$$\int_{I \setminus E_j} |f_k| < \varepsilon \text{ for all } k \in \mathbb{N}. \quad (4)$$

Then (by (3) and (4))

$$\begin{aligned} \int_I |f_k - f_l| &= \int_{E_j} |f_k - f_l| + \int_{I \setminus E_j} |f_k - f_l| \leq \\ &\leq \int_{E_j} |f_k - f| + \int_{E_j} |f - f_l| + \int_{I \setminus E_j} |f_k| + \int_{I \setminus E_j} |f_l| < \\ &< 2\varepsilon\mu(E_j) + 2\varepsilon \leq 2\varepsilon(\mu(I) + 1) \end{aligned}$$

for all  $k, l > K_j$ . This shows that the sequence  $f_k$ ,  $k \in \mathbb{N}$  is Cauchy in the Banach space  $L$  of Lebesgue integrable functions on  $I$  and implies that the function  $f : I \rightarrow \mathbb{R}$  also belongs to  $L$  and

$$\lim_{k \rightarrow \infty} \int_I |f_k - f| = 0, \quad (5)$$

i.e. there is a  $K \in \mathbb{N}$  such that

$$\int_I |f_k - f| < \varepsilon \text{ for all } k > K. \quad (6)$$

By Theorem 2 we know at this point that all the functions  $f$ ,  $f_k$ ,  $k \in \mathbb{N}$  are also McShane integrable and the values of the integrals are the same.

Using Definition 1 there exists a gauge  $\Delta_1 : I \rightarrow (0, +\infty)$  such that

$$\left| \sum_i f(t_i)\mu(I_i) - \int_I f \right| < \varepsilon \quad (7)$$

for every  $\Delta_1$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$ .

Further there exists a gauge  $\Delta_2 : I \rightarrow (0, +\infty)$  such that

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < \varepsilon \quad (8)$$

for every  $\Delta_2$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$  for all  $k \leq K$ ,  $K$  given by (6). (A finite set of integrable functions is evidently equi-integrable.)

Similarly for any  $j \in \mathbb{N}$  we have a gauge  $\delta_j : I \rightarrow (0, +\infty)$  such that

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < \frac{\varepsilon}{2^j} \quad (9)$$

for every  $\delta_j$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$  and all  $k \leq K_j$ .

Since  $\mu(N) = 0$ , for every  $\delta > 0$  there is an open set  $U \subset \mathbb{R}^m$  such that  $N \subset U$  and  $\mu(U) < \delta$ . Using 2) the value of  $\delta$  can be chosen in such a way that

$$\left| \int_{U \cap I} f_k \right| < \varepsilon \quad \text{for all } k \in \mathbb{N}, \quad (10)$$

cf. Definition 7.

For  $t \in E_1 \setminus N$  define  $\Delta_3(t) = \delta_1(t)$ , for  $t \in (E_2 \setminus E_1) \setminus N$  define  $\Delta_3(t) = \delta_2(t)$ ,  $\dots$ , for  $t \in (E_j \setminus E_{j-1}) \setminus N$  define  $\Delta_3(t) = \delta_j(t)$ , etc.

If  $t \in N$  then we define  $\Delta_3(t) > 0$  such that for the ball  $B(t, \Delta_3(t))$  (centered at  $t$  with the radius  $\Delta_3(t)$ ) we have  $B(t, \Delta_3(t)) \subset U$ .

In this way the positive function  $\Delta_3$  defined on  $I$  represents a gauge.

Let us put  $\Delta(t) = \min(\Delta_1(t), \Delta_2(t), \Delta_3(t))$  for  $t \in I$ . The function  $\Delta$  is evidently a gauge on  $I$ .

Assume that  $\{(t_i, I_i)\}$  is an arbitrary  $\Delta$ -fine  $M$ -partition of  $I$ .

If  $k \leq K$  then

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < \varepsilon$$

by (8).

If  $k > K$  then

$$\begin{aligned} & \left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| = \left| \sum_i [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| \leq \\ & \leq \left| \sum_{j=1}^{\infty} \sum_{t_i \in (E_j \setminus E_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| + \left| \sum_{t_i \in N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right|. \quad (11) \end{aligned}$$

For the second term on the right hand side of (11) we know that if  $t_i \in N$  then  $f_k(t_i) = 0$  and  $\bigcup_{t_i \in N} A_i \subset U$  and therefore by (10) we have

$$\left| \sum_{t_i \in N} \int_{I_i} f_k \right| \leq \left| \int_{\bigcup_i I_i} f_k \right| \leq \left| \int_{U \cap I} f_k \right| < \varepsilon. \quad (12)$$

Concerning the first term of the right hand side of (11) we have

$$\begin{aligned} & \left| \sum_{j=1}^{\infty} \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| \leq \\ & \leq \sum_{j=1}^{\infty} \left| \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right|. \end{aligned} \quad (13)$$

If  $k \leq K_j$  the the Saks-Henstock lemma 11 yields by (9) the inequality

$$\left| \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| < \frac{\varepsilon}{2^j}. \quad (14)$$

If  $k > K_j$  then

$$\begin{aligned} & \left| \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| \leq \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} |f_k(t_i)\mu(I_i) - \int_{I_i} f_k| \leq \\ & \leq \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} |f_k(t_i) - f(t_i)|\mu(I_i) + \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} |f(t_i)\mu(I_i) - \int_{I_i} f| \\ & \quad + \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} \int_{I_i} |f - f_k| < \\ & < \varepsilon \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} \mu(I_i) + \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} |f(t_i)\mu(I_i) - \int_{I_i} f| + \\ & \quad + \int_{\bigcup_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} I_i} |f - f_k|. \end{aligned}$$

This together with (14) gives the estimate

$$\left| \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| < \frac{\varepsilon}{2^j} + \varepsilon \sum_{t_i \in (E_j \setminus \dot{E}_{j-1}) \setminus N} \mu(I_i) +$$



$$+ \sum_{t_i \in (E_j \setminus E_{j-1}) \setminus N} |f(t_i)\mu(I_i) - \int_{I_i} f| + \int_{t_i \in (\cup_i I_i) \setminus N} |f - f_k|.$$

Summing over  $j$  we obtain finally

$$\sum_j \left| \sum_{t_i \in (E_j \setminus E_{j-1}) \setminus N} [f_k(t_i)\mu(I_i) - \int_{I_i} f_k] \right| < \varepsilon + \varepsilon\mu(I) + \varepsilon + \varepsilon$$

and by (11) and (12) we obtain

$$\left| \sum_i f_k(t_i)\mu(I_i) - \int_I f_k \right| < (4 + \mu(I))\varepsilon \text{ for all } k \in \mathbb{N}.$$

This inequality proves that the sequence  $f_k, k \in \mathbb{N}$  is equi-integrable.

**13. Lemma.** *Assume that  $f_k : I \rightarrow \mathbb{R}, k \in \mathbb{N}$  are McShane (Lebesgue) integrable functions such that*

- 1)  $f_k(t) \rightarrow f(t)$  for  $t \in I$ ,
- 2) the set  $\{f_k; k \in \mathbb{N}\}$  is equi-integrable.

*Then for every  $\varepsilon > 0$  there is an  $\eta > 0$  such that for any finite collection  $\{J_j : j = 1, \dots, p\}$  of non-overlapping intervals in  $I$  with  $\sum_j \mu(J_j) < \eta$  we have*

$$\left| \sum_j \int_{J_j} f_k \right| < \varepsilon, k \in \mathbb{N}.$$

*Proof.* Let  $\varepsilon > 0$  be given. Since  $f_k$  are equi-integrable on  $I$ , there exists a gauge  $\delta$  on  $I$  such that  $|\sum_i f_k(t_i)\mu(I_i) - \int_I f_k| < \varepsilon$  for  $k \in \mathbb{N}$  whenever  $\{(t_i, I_i)\}$  is an arbitrary  $\delta$ -fine  $M$ -partition of  $I$ . Fixing a  $\delta$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$  let  $k_0 \in \mathbb{N}$  be such that

$$|f_k(t_i) - f(t_i)| < \varepsilon \text{ for } 1 \leq i \leq q, k > k_0$$

put  $M = \max\{|f(t_i)|, |f_k(t_i)|; i, k \leq k_0\}$  and set  $\eta = \frac{\varepsilon}{M+1}$ .

Suppose that  $\{J_j : j = 1, \dots, p\}$  is a finite collection of non-overlapping intervals in  $I$  such that  $\sum_j \mu(J_j) < \eta$ . By subdividing these intervals if necessary, we may assume that for each  $j, J_j \subseteq I_i$  for some  $i$ . For each  $i$  let  $M_i = \{j; J_j \subseteq I_i\}$  and let

$$D = \{(J_j, t_i) : j \in M_i, i\}.$$

Note that  $D$  is a  $\delta$ -fine  $M$ -system in  $I$ .

Using the Saks-Henstock Lemma 11 we get

$$\begin{aligned} \left| \sum_j \int_{J_j} f_k \right| &\leq \left| \sum_j \int_{J_j} f_k - f_k(t_i)\mu(J_j) \right| + \sum_j |f_k(t_i)|\mu(J_j) \leq \\ &\leq \varepsilon + (M + \varepsilon) \sum_j \mu(J_j) < \varepsilon + (M + \varepsilon)\eta < \varepsilon(2 + \frac{\varepsilon}{M+1}) \end{aligned}$$

and this proves the lemma.

**14. Lemma.** *Assume that  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  are McShane (Lebesgue) integrable functions such that*

- 1)  $f_k(t) \rightarrow f(t)$  for  $t \in I$ ,
- 2) the set  $\{f_k; k \in \mathbb{N}\}$  forms an equi-integrable sequence.

*Then for every  $\varepsilon > 0$  there exists an  $\eta > 0$  such that*

(a) *if  $F$  is closed,  $G$  open,  $F \subset G \subset I$ ,  $\mu(G \setminus F) < \eta$  then there is a gauge  $\xi : I \rightarrow (0, \infty)$  such that*

$$B(t, \xi(t)) \subset G \text{ for } t \in G,$$

$$B(t, \xi(t)) \subset I \setminus F \text{ for } t \in I \setminus F$$

*and*

(b) *for  $\xi$ -fine  $M$ -systems  $\{(u_l, U_l)\}$ ,  $\{(v_m, V_m)\}$  satisfying*

$$u_l, v_m \in G, F \subset \text{int} \bigcup_{u_l \in F} U_l \subset \text{int} \bigcup_{v_m \in F} V_m$$

*we have*

$$\left| \sum_l f_k(u_l) \mu(U_l) - \sum_m f_k(v_m) \mu(V_m) \right| \leq \varepsilon \quad (15)$$

*for every  $k \in \mathbb{N}$ .*

*Proof.* Denote  $\Phi_k(J) = \int_J f_k$  for an interval  $J \subset I$  (the indefinite integral or primitive of  $f_k$ ) and put  $\widehat{\varepsilon} = \frac{\varepsilon}{10}$ .

Since  $f_k$  is equi-integrable we obtain by the Saks-Henstock lemma 11 that there is a gauge  $\Delta$  on  $I$  such that

$$\left| \sum_j [f_k(r_j) \mu(K_j) - \Phi_k(K_j)] \right| \leq \widehat{\varepsilon} \quad (16)$$

for every  $\Delta$ -fine  $M$ -system  $\{(r_j, K_j)\}$  and  $k \in \mathbb{N}$ .

Assume that

$$\{(w_p, W_p)\} \text{ is a fixed } \Delta\text{-fine } M\text{-partition of } I. \quad (17)$$

Let  $k_0 \in \mathbb{N}$  be such that

$$|f_k(w_p) - f(w_p)| < 1$$

for  $k > k_0$  and all  $p$ . Put  $\kappa = \max_{p, k \leq k_0} \{1 + |f(w_p)|, |f_k(w_p)|\}$ . Then

$$|f_k(w_p)| < \kappa \text{ for all } k \in \mathbb{N} \text{ and } p. \quad (18)$$

Assume that  $\eta > 0$  satisfies

$$\eta \cdot \kappa \leq \widehat{\varepsilon} \quad (19)$$

and take

$$0 < \xi(t) \leq \Delta(t), t \in I. \quad (20)$$

Since the sets  $G$  and  $I \setminus F$  are open, the gauge  $\xi$  can be chosen such that  $B(t, \xi(t)) \subset G$  for  $t \in G$  and  $B(t, \xi(t)) \subset I \setminus F$  for  $t \in I \setminus F$ .

This shows part (a) of the lemma.

Since  $\{(w_p, W_p)\}$  is a partition of  $I$ , we have  $\bigcup_p W_p = I$  and therefore

$$\begin{aligned} \sum_l f_k(u_l)\mu(U_l) &= \sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) + \\ &+ \sum_p \sum_{u_l \in F} f_k(u_l)\mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) + \sum_p \sum_{u_l \in I \setminus F} f_k(u_l)\mu(W_p \cap U_l) \end{aligned} \quad (21)$$

and similarly

$$\begin{aligned} \sum_m f_k(v_m)\mu(V_m) &= \sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) + \\ &+ \sum_p \sum_{u_l \in F} f_k(u_l)\mu(W_p \cap V_m \setminus \bigcup_{u_l \in F} U_l) + \sum_p \sum_{v_m \in I \setminus F} f_k(v_m)\mu(W_p \cap V_m). \end{aligned} \quad (22)$$

The  $M$ -systems

$$\begin{aligned} &\{(u_l, W_p \cap U_l \cap V_m); p, u_l \in F, v_m \in F\}, \\ &\{(w_p, W_p \cap U_l \cap V_m); p, u_l \in F, v_m \in F\} \end{aligned}$$

are  $\Delta$ -fine and therefore, by (16), we have the inequalities

$$\begin{aligned} &|\sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) - \Phi_k(W_p \cap U_l \cap V_m)| \leq \hat{\varepsilon}, \\ &|\sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m) - \Phi_k(W_p \cap U_l \cap V_m)| \leq \hat{\varepsilon}. \end{aligned}$$

Hence

$$|\sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) - \sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m)| \leq 2\hat{\varepsilon}$$

and similarly also

$$|\sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(v_m)\mu(W_p \cap U_l \cap V_m) - \sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(w_p)\mu(W_p \cap U_l \cap V_m)| \leq 2\hat{\varepsilon}.$$

Therefore

$$|\sum_p \sum_{u_l \in F} \sum_{v_m \in F} f_k(u_l)\mu(W_p \cap U_l \cap V_m) -$$

$$-\sum_p \sum_{\substack{l \\ u_l \in F}} \sum_{\substack{m \\ v_m \in F}} f_k(v_m) \mu(W_p \cap U_l \cap V_m) \leq 4\hat{\varepsilon}. \quad (23)$$

Since  $\{(u_l, U_l)\}$  is a  $\xi$ -fine  $M$ -system with  $u_l \in G$ , we obtain by the properties of the gauge  $\xi$  given in (a) and from the assumption  $F \subset \text{int } \bigcup_{u_l \in F} U_l$ ,  $F \subset \text{int } \bigcup_{v_m \in F} V_m$  that

$$\left( \bigcup_{p, u_l \in F} W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m \right) \cup \bigcup_{p, u_l \in I \setminus F} W_p \cap U_l \subset G \setminus F. \quad (24)$$

Further the  $M$ -systems

$$\{(u_l, W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m); p, u_l \in F\} \cup \{(u_l, W_p \cap U_l); p, u_l \in I \setminus F\},$$

$$\{(w_p, W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m); p, u_l \in F\} \cup \{(w_p, W_p \cap U_l); p, u_l \in I \setminus F\}$$

are  $\Delta$ -fine. Therefore by (16) we have

$$\begin{aligned} & \left| \sum_{p, u_l \in F} [f_k(u_l) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) - \Phi_k(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m)] + \right. \\ & \quad \left. + \sum_{p, u_l \in I \setminus F} [f_k(u_l) \mu(W_p \cap U_l) - \Phi_k(W_p \cap U_l)] \right| \leq \hat{\varepsilon}, \\ & \left| \sum_{p, u_l \in F} [f_k(w_p) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) - \Phi_k(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m)] + \right. \\ & \quad \left. + \sum_{p, u_l \in I \setminus F} [f_k(w_p) \mu(W_p \cap U_l) - \Phi_k(W_p \cap U_l)] \right| \leq \hat{\varepsilon}. \end{aligned}$$

This yields

$$\begin{aligned} & \left| \sum_{p, u_l \in F} f_k(u_l) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) + \sum_{p, u_l \in I \setminus F} f_k(u_l) \mu(W_p \cap U_l) - \right. \\ & \quad \left. - \sum_{p, u_l \in F} f_k(w_p) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) - \sum_{p, u_l \in I \setminus F} f_k(w_p) \mu(W_p \cap U_l) \right| \leq 2\hat{\varepsilon}. \end{aligned}$$

With respect to (24) and (19) we have

$$\begin{aligned} & \left| \sum_{p, u_l \in F} f_k(w_p) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) - \sum_{p, u_l \in I \setminus F} f_k(w_p) \mu(W_p \cap U_l) \right| \leq \\ & \leq \kappa \cdot \eta \leq \hat{\varepsilon} \end{aligned}$$

and therefore

$$\left| \sum_{p, u_l \in F} f_k(u_l) \mu(W_p \cap U_l \setminus \bigcup_{v_m \in F} V_m) + \sum_{p, u_l \in I \setminus F} f_k(u_l) \mu(W_p \cap U_l) \right| \leq 3\widehat{\varepsilon} \quad (25)$$

and similarly also

$$\left| \sum_{p, v_m \in F} f_k(v_m) \mu(W_p \cap U_l \setminus \bigcup_{u_l \in F} U_l) + \sum_{p, v_m \in I \setminus F} f_k(v_m) \mu(W_p \cap V_m) \right| \leq 3\widehat{\varepsilon}. \quad (26)$$

From (21), (22), (23), (25) and (26) we get

$$\left| \sum_l f_k(u_l) \mu(U_l) - \sum_m f_k(v_m) \mu(V_m) \right| \leq 10\widehat{\varepsilon} \leq \varepsilon$$

and (15) is satisfied. This proves part (b) of the lemma.

**15. Theorem.** *Assume that  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  are McShane integrable functions such that*

- 1)  $f_k(t) \rightarrow f(t)$  for  $t \in I$ ,
- 2) the set  $\{f_k; k \in \mathbb{N}\}$  is equi-integrable.

*Then  $f_k \cdot \chi_E$ ,  $k \in \mathbb{N}$  is an equi-integrable sequence for every measurable set  $E \subset I$ .*

*Proof.* Let  $\varepsilon > 0$  be given and let  $\eta > 0$  corresponds to  $\varepsilon$  by Lemma 14. Assume that  $E \subset I$  is measurable. Then there exist  $F \subset I$  closed and  $G \subset I$  open such that  $F \subset E \subset G$  where  $\mu(G \setminus F) < \eta$ . Assume that the gauge  $\xi : I \rightarrow (0, \infty)$  is given as in the Lemma 14 and that  $\{(u_l, U_l)\}$ ,  $\{(v_m, V_m)\}$  are  $\xi$ -fine  $M$ -partitions of  $I$ .

We have

$$\text{if } u_l \in E \text{ then } U_l \subset G, F \subset \text{int } \bigcup_{u_l \in F} U_l$$

and

$$\text{if } v_m \in E \text{ then } V_m \subset G, F \subset \text{int } \bigcup_{v_m \in F} V_m.$$

Hence by (b) from Lemma 14 we have

$$\left| \sum_{l, u_l \in E} f_k(u_l) \mu(U_l) - \sum_{m, v_m \in E} f_k(v_m) \mu(V_m) \right| \leq \varepsilon$$

and therefore also

$$\left| \sum_l f_k(u_l) \chi_E(u_l) \mu(U_l) - \sum_m f_k(v_m) \chi_E(v_m) \mu(V_m) \right| \leq \varepsilon.$$

This is the Bolzano-Cauchy condition from Theorem 4 for equi-integrability of the sequence  $f_k \cdot \chi_E$ ,  $k \in \mathbb{N}$  and the proof is complete.

**16. Proposition.** *Assume that  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  are McShane integrable functions such that*

- 1)  $f_k(t) \rightarrow f(t)$  for  $t \in I$ ,
- 2) the set  $\{f_k; k \in \mathbb{N}\}$  is equi-integrable.

*Then for every  $\varepsilon > 0$  there is an  $\eta > 0$  such that if  $E \subset I$  is measurable with  $\mu(E) < \eta$  then*

$$\left| \int_I f_k \cdot \chi_E \right| = \left| \int_E f_k \right| \leq 2\varepsilon$$

for every  $k \in \mathbb{N}$ .

*Proof.* Let  $\varepsilon > 0$  be given and let  $\eta > 0$  corresponds to  $\varepsilon$  by Lemma 13 and assume that  $\mu(E) < \eta$ . Then there is an open set  $G \subset I$  such that  $E \subset G$  and  $\mu(G) < \eta$ .

The equi-integrability of  $f_k$  implies the existence of a gauge  $\Delta : I \rightarrow (0, +\infty)$  such that for every  $\Delta$ -fine  $M$ -partition  $\{(t_i, I_i)\}$  of  $I$  the inequality

$$\left| \sum_i f_k(t_i) \mu(I_i) - \int_I f_k \right| < \varepsilon$$

holds.

By Theorem 15 the integrals  $\int_I f_k \cdot \chi_E$ ,  $k \in \mathbb{N}$  exist and to every  $\theta > 0$  there is a gauge  $\delta : I \rightarrow (0, +\infty)$  which satisfies  $B(t, \delta(t)) \subset G$  if  $t \in G$ ,  $\delta(t) \leq \Delta(t)$ ,  $t \in I$  and

$$\left| \sum_m f_k(v_m) \cdot \chi_E(v_m) \mu(V_m) - \int_I f_k \cdot \chi_E \right| \leq \theta$$

holds for any  $\delta$ -fine  $M$ -partition  $\{(v_m, V_m)\}$  of  $I$ .

If  $v_m \in E \subset G$  then  $V_m \subset G$  and  $\sum_{m, v_m \in E} \mu(V_m) \leq \eta$ .

Since  $\{(v_m, V_m); v_m \in E\}$  is a  $\Delta$ -fine  $M$ -system, we have by the Saks-Henstock lemma 11 the inequality

$$\left| \sum_{m, v_m \in E} [f_k(v_m) \mu(V_m) - \int_{V_m} f_k] \right| \leq \varepsilon$$

and by Lemma 13 we get

$$\left| \sum_{m, v_m \in E} \int_{V_m} f_k \right| \leq \varepsilon.$$

Hence

$$\begin{aligned} \left| \int_E f_k \right| &\leq \theta + \left| \sum_{m, v_m \in E} f_k(v_m) \mu(V_m) \right| \leq \\ &\leq \theta + \left| \sum_{m, v_m \in E} [f_k(v_m) \mu(V_m) - \int_{V_m} f_k] \right| + \left| \sum_{m, v_m \in E} \int_{V_m} f_k \right| \leq \theta + 2\varepsilon. \end{aligned}$$

This proves the statement because  $\theta > 0$  can be chosen arbitrarily small.

Using Proposition 12 and 16 and the concept of uniform absolute continuity of a sequence of functions given in Definition 7 we obtain the following.

**17. Theorem.** *Assume that  $f_k : I \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$  are McShane integrable functions such that  $f_k(t) \rightarrow f(t)$  for  $t \in I$ .*

*Then the set  $\{f_k; k \in \mathbb{N}\}$  forms an equi-integrable sequence if and only if  $\{f_k; k \in \mathbb{N}\}$  is uniformly absolutely continuous.*

**18. Concluding remarks.** Theorem 17 shows that the relaxed Vitali convergence Theorem 9 is equivalent to our convergence Theorem 4 which uses the concept of equi-integrability.

Therefore Theorem 4 is in the sense of Gordon [1] also a sort of primary theorem because the Lebesgue dominated convergence theorem and the Levi monotone convergence theorem follow from Theorem 4 (see [1, p. 203]).

Note also that if we are looking at the Vitali convergence Theorem 8 where the sequence  $f_k$ ,  $k \in \mathbb{N}$  is assumed to converge to  $f$  in measure then by the Riesz theorem [3] there is a subsequence  $f_{k_i}$  which converges to  $f$  for all  $t \in I \setminus N$  where  $\mu(N) = 0$ . If we set  $f_{k_i}(t) = f(t)$  for  $t \in N$  then Theorem 17 yields that the assumption of the Vitali convergence Theorem implies that the original sequence  $f_k$ ,  $k \in \mathbb{N}$  contains a subsequence which is equi-integrable.

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