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LINEAR STIELTJES INTEGRAL EQUATIONS IN BANACH SPACES

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Abstract. Fundamental results concerning Stieltjes integrals for functions with values in Banach spaces have been presented in [5]. The background of the theory is the Kurzweil approach to integration, based on Riemann type integral sums (see e.g. [3]). It is known that the Kurzweil theory leads to the (non-absolutely convergent) Perron-Stieltjes integral in the finite dimensional case. Here basic results concerning equations of the form

$$x(t) = x(a) + \int_a^t \mathrm{d}[A(s)]x(s) + f(t) - f(a)$$

are presented on the basis of the Kurzweil type Stieltjes integration. We are looking for generally discontinuous solutions which belong to the space of Banach space-valued regulated functions in the case that A is a suitable operator-valued function and f is regulated.

Keywords: linear Stieltjes integral equations, generalized linear differential equation, equation in Banach space

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1. Preliminaries on functions and Stieltjes integrals

In this section some basic concepts and results concerning Stieltjes type integration are collected for the readers convenience. The presentation is based on the results given in the paper [5].

Assume that X is a Banach space and that $L(X)$ is the Banach space of all bounded linear operators $A\colon X \to X$ with the uniform operator topology. Defining the bilinear form $B\colon L(X) \times X \to X$ by $B(A, x) = Ax \in X$ for $A \in L(X)$ and

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$x \in X$, we obtain in a natural way the bilinear triple $\mathcal{B} = (L(X), X, X)$ because using the usual operator norm we have

$$\|B(A,x)\|_X \leqslant \|A\|_{L(X)}\|x\|_X.$$

Similarly, if we define the bilinear form $B^*\colon\ L(X)\times L(X)\to L(X)$ by the relation $B^*(A,C) = AC \in L(X)$ for $A, C \in L(X)$ where AC is the composition of the linear operators A and C we get the bilinear triple $\mathcal{B}^* = (L(X), L(X), L(X))$ because we have

$$\|B^*(A,C)\|_{L(X)} \leqslant \|AC\|_{L(X)} \leqslant \|A\|_{L(X)}\|C\|_{L(X)}.$$

Assume that $[a,b] \subset \mathbb{R}$ is a bounded interval.

Given $A\colon\ [a,b]\to L(X)$, the function A is of *bounded variation on* $[a,b]$ if

$$\operatorname*{var}_{[a,b]}(A) = \sup\left\{\sum_{j=1}^{k}\|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}\right\} < \infty,$$

where the supremum is taken over all finite partitions

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$. The set of all functions $A\colon\ [a,b]\to L(X)$ with $\operatorname{var}_{[a,b]}(A) < \infty$ will be denoted by $\mathrm{BV}([a,b];L(X))$.

It is easy to show that if $A \in \mathrm{BV}([a,b];L(X))$ then the function $t \in [a,b] \to \operatorname{var}_{[a,t]}(A) \in \mathbb{R}$ is nondecreasing and bounded for $t \in [a,b]$ and it is additive, i.e.

$$\operatorname*{var}_{[a,c]}(A) + \operatorname*{var}_{[c,b]}(A) = \operatorname*{var}_{[a,b]}(A)$$

for any $c \in [a,b]$.

For $A\colon\ [a,b]\to L(X)$ and a partition D of the interval $[a,b]$ define

$$V_a^b(A,D) = \sup\left\{\left\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]y_j\right\|_X\right\}$$

where the supremum is taken over all possible choices of $y_j \in X,\ j = 1,\ldots,k$ with $\|y_j\|_X \leqslant 1$, and similarly

$$\overset{*}{V}{}_a^b(A,D) = \sup\left\{\left\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]C_j\right\|_{L(X)}\right\}$$

where the supremum is taken over all possible choices of $C_j \in L(X), j = 1, \ldots, k$ with $\|C_j\|_{L(X)} \leqslant 1$.

Let us set

$$(\mathcal{B}) \underset{[a,b]}{\operatorname{var}}(A) = \sup V_a^b(A, D)$$

and

$$(\mathcal{B}^*) \underset{[a,b]}{\operatorname{var}}(A) = \sup \overset{*}{V}{}_a^b(A, D)$$

where the supremum is taken over all finite partitions

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a, b]$.

An operator valued function $A\colon [a, b] \to L(X)$ with $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$ is called a *function with bounded $\mathcal{B}$-variation on* $[a, b]$ (sometimes also a *function of bounded semi-variation* cf. [2]) and similarly, if $(\mathcal{B}^*) \operatorname{var}_{[a,b]}(A) < \infty$ then A is of *bounded $\mathcal{B}^*$-variation on* $[a, b]$.

We denote by $(\mathcal{B}) \operatorname{BV}([a, b]; L(X))$ the set of all functions $A\colon [a, b] \to L(X)$ with $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$ and by $(\mathcal{B}^*) \operatorname{BV}([a, b]; L(X))$ the set of all functions $A\colon [a, b] \to L(X)$ with $(\mathcal{B}^*) \operatorname{var}_{[a,b]}(A) < \infty$

Concerning these concepts the following proposition holds.

1.1. Proposition.

$$(\mathcal{B}) \operatorname{BV}([a, b]; L(X)) = (\mathcal{B}^*) \operatorname{BV}([a, b]; L(X)),$$

and if $A \in (\mathcal{B}) \operatorname{BV}([a, b]; L(X))$ *then*

$$(\mathcal{B}) \underset{[a,b]}{\operatorname{var}}(A) = (\mathcal{B}^*) \underset{[a,b]}{\operatorname{var}}(A).$$

P r o o f. It is sufficient to show that for a given $A\colon [a, b] \to L(X)$ and every finite partition $D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$ of $[a, b]$ we have

$$V_a^b(A, D) = \overset{*}{V}{}_a^b(A, D).$$

Assume that the partition D is arbitrary and that $C_j \in L(X)$, $\|C_j\|_{L(X)} \leqslant 1$ for $j = 1, \ldots, k$. Then

$$\begin{aligned}
\Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} &= \sup_{\|y\|_X \leqslant 1} \Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] C_j y \Big\|_X \\
&\leqslant \sup_{\|y_j\|_X \leqslant 1, y_j \in X} \Big\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] y_j \Big\|_X \\
&= V_a^b(A, D)
\end{aligned}$$

because $\|C_j y\|_X \leqslant 1$ for $j = 1,\ldots,k$. Hence passing to the supremum over all $C_j \in L(X)$, $\|C_j\|_{L(X)} \leqslant 1$, $j = 1,\ldots,k$ we obtain the inequality

$$\overset{*}{V}{}_a^b(A,D) \leqslant V_a^b(A,D).$$

For the opposite inequality assume that $y_j \in X,\ j = 1,\ldots,k$ with $\|y_j\|_X \leqslant 1$ are given. Let us take $w \in X$ such that $\|w\|_X = 1$. By the Hahn-Banach Theorem there is a bounded linear functional $f \in X^*$ such that $\|f\|_{X^*} = 1$ and $f(w) = 1$. Define $C_j \in L(X)$ for $j = 1,\ldots,k$ in such a way that

$$C_j x = y_j f(x) \quad \text{for } x \in X, j = 1,\ldots,k.$$

Evidently $C_j w = y_j f(w) = y_j$,

$$\|C_j\|_{L(X)} = \sup_{\|x\|_X \leqslant 1} \|C_j x\|_X = \sup_{\|x\|_X \leqslant 1} \|y_j f(x)\|_X \leqslant \|y_j\|_X \cdot \|f\|_{X^*} = \|y_j\|_X.$$

Then

$$\begin{aligned}
\Big\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})] y_j \Big\|_X &= \Big\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})] C_j w \Big\|_X \\
&\leqslant \Big\| \sum_{j=1}^k [A(\alpha_j) - A(\alpha_{j-1})] C_j \Big\|_{L(X)} \|w\|_X \\
&\leqslant \overset{*}{V}{}_a^b(A,D)
\end{aligned}$$

and therefore also

$$V_a^b(A,D) \leqslant \overset{*}{V}{}_a^b(A,D).$$

Hence $V_a^b(A,D) = \overset{*}{V}{}_a^b(A,D)$ and the statement is proved. □

The following statement holds.

1.2 (see [5, Proposition 1]). *We have*

$$\mathrm{BV}([a,b];L(X)) \subset (\mathcal{B})\,\mathrm{BV}([a,b];L(X)),$$

and if $A \in \mathrm{BV}([a,b];L(X))$*, then*

$$(\mathcal{B}) \operatorname*{var}_{[a,b]}(A) \leqslant \operatorname*{var}_{[a,b]}(A).$$

R e m a r k. It is not difficult to see that if $A\colon [a,b] \to L(X)$ and the space X is finite dimensional then $A \in (\mathcal{B})\,\mathrm{BV}([a,b];L(X))$ if and only if $A \in \mathrm{BV}([a,b];L(X))$.

Therefore the concept of $\mathcal{B}$-variation of a function $A\colon [a,b] \to L(X)$ is relevant for infinite-dimensional Banach spaces X only.

Given $x\colon [a,b] \to X$, the function x is called *regulated on* $[a,b]$ if it has one-sided limits at every point of $[a,b]$, i.e. if for every $s \in [a,b)$ there is a value $x(s+) \in X$ such that

$$\lim_{t \to s+} \|x(t) - x(s+)\|_X = 0,$$

and for every $s \in (a,b]$ there is a value $x(s-) \in X$ such that

$$\lim_{t \to s-} \|x(t) - x(s-)\|_X = 0.$$

The set of all regulated functions $x\colon [a,b] \to X$ will be denoted by $G([a,b];X)$.

Assume now that $\mathcal{B} = (L(X), X, X)$ is the bilinear triple of Banach spaces mentioned above.

A function $A\colon [a,b] \to L(X)$ is called *$\mathcal{B}$-regulated on* $[a,b]$ if for every $y \in X$, $\|y\|_X \leqslant 1$, the function $Ay\colon [a,b] \to X$ given by $t \in [a,b] \mapsto A(t)y \in X$ for $t \in [a,b]$ is regulated, i.e. $Ay \in G([a,b];X)$ for every $y \in X$, $\|y\|_X \leqslant 1$.

We denote by $(\mathcal{B})G([a,b];L(X))$ the set of all $\mathcal{B}$-regulated functions $A\colon [a,b] \to L(X)$.

A function $x\colon [a,b] \to X$ is called a *(finite) step function on* $[a,b]$ if there exists a finite partition

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[a,b]$ such that x has a constant value in X on (α_{j-1}, α_j) for every $j = 1, \ldots, k$, and similarly for operator valued functions.

The following result is well known for regulated functions.

1.3 (see e.g. [2, Theorem 3.1, p. 16]). *A function $x\colon [a,b] \to X$ is regulated ($x \in G([a,b];X)$) if and only if x is the uniform limit of step functions.*

By Proposition 3 in [5] we can state the following.

1.4. *If $A \in G([a,b];L(X))$ then $A \in (\mathcal{B})G([a,b];L(X))$, i.e.*

$$G([a,b];L(X)) \subset (\mathcal{B})G([a,b];L(X)).$$

In addition to this we also have

1.5. *If $A \in \mathrm{BV}([a,b];L(X))$ then $A \in G([a,b];L(X))$, i.e.*

$$\mathrm{BV}([a,b];L(X)) \subset G([a,b];L(X)) \subset (\mathcal{B})G([a,b];L(X)).$$

P r o o f. For $s,t\in[a,b]$, $s\leqslant t$ we have

$$\|A(t)-A(s)\|_{L(X)}\leqslant \operatorname*{var}_{[s,t]}(A)=\operatorname*{var}_{[a,t]}(A)-\operatorname*{var}_{[a,s]}(A)$$

and this implies (e.g. by the Bolzano-Cauchy condition for the existence of onesided limits of the nondecreasing bounded real function $\operatorname{var}_{[a,t]}(A)$) that the onesided limits of the function $A\colon [a,b]\to L(X)$ exist at any point of $[a,b]$, i.e. that $A\in G([a,b];L(X))$. □

R e m a r k. Again, if the Banach space X is finite dimensional, then it is easy to check that a function $A\colon [a,b]\to L(X)$ is $\mathcal{B}$-regulated if and only if it is regulated.

Let us now give the definition of the Stieltjes integral.

A finite system of points

$$\{\alpha_0,\tau_1,\alpha_1,\tau_2,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$$

such that

$$a=\alpha_0<\alpha_1<\dots<\alpha_{k-1}<\alpha_k=b$$

and

$$\tau_j\in[\alpha_{j-1},\alpha_j]\qquad\text{for}\quad j=1,\dots,k$$

is called a *P-partition* of the interval $[a,b]$.

Any positive function $\delta\colon [a,b]\to(0,\infty)$ is called a *gauge on* $[a,b]$.

For a given gauge δ on $[a,b]$, a P-partition $\{\alpha_0,\tau_1,\alpha_1,\tau_2,\dots,\alpha_{k-1},\tau_k,\alpha_k\}$ of $[a,b]$ is called *δ-fine* if

$$[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))\quad\text{for } j=1,\dots,k.$$

The following statement called the Cousin lemma is important for defining the Stieltjes integral.

1.6 (see e.g. [3]). *Given an arbitrary gauge δ on $[a,b]$ there is a δ-fine P-partition of $[a,b]$.*

Definition. Assume that functions $A\colon [a,b]\to L(X)$ and $x\colon [a,b]\to X$ are given.

We say that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists if there is an element $J\in X$ such that for every $\varepsilon>0$ there is a gauge δ on $[a,b]$ such that for

$$S(\mathrm{d}A,x,D)=\sum_{j=1}^{k}\big[A(\alpha_j)-A(\alpha_{j-1})\big]x(\tau_j)$$

we have

$$\|S(\,\mathrm{d}A, x, D) - J\|_X < \varepsilon$$

provided D is a δ-fine P-partition of $[a,b]$. We denote $J = \int_a^b \mathrm{d}[A(s)]x(s)$. For the case $a = b$ it is convenient to set $\int_a^b \mathrm{d}[A(s)]x(s) = 0$ and if $b < a$, then $\int_a^b \mathrm{d}[A(s)]x(s) = -\int_b^a \mathrm{d}[A(s)]x(s)$.

The Stieltjes integral introduced in this way is determined uniquely. Let us recall some elementary properties of the Stieltjes integral from [5].

1.7 (see [5, Proposition 6]). *Assume that functions $A\colon [a,b] \to L(X)$ and $x_i\colon [a,b] \to X$ are such that the Stieltjes integrals $\int_a^b \mathrm{d}[A(s)]x_i(s)$, $i = 1,2$ exist.*

Then for every $c_1, c_2 \in \mathbb{R}$ the integral $\int_a^b \mathrm{d}[A(s)](c_1x_1(s) + c_2x_2(s))$ exists and

$$\int_a^b \mathrm{d}[A(s)](c_1x_1(s) + c_2x_2(s)) = c_1 \int_a^b \mathrm{d}[A(s)]x_1(s) + c_2 \int_a^b \mathrm{d}[A(s)]x_2(s).$$

If functions $A_i\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integrals $\int_a^b \mathrm{d}[A_i(s)]x(s)$, $i = 1,2$ exist then for every $c_1, c_2 \in \mathbb{R}$ the integral $\int_a^b \mathrm{d}[c_1A_1(s) + c_2A_2(s)]x(s)$ exists and

$$\int_a^b \mathrm{d}[c_1A_1(s) + c_2A_2(s)]x(s) = c_1 \int_a^b \mathrm{d}[A_1(s)]x(s) + c_2 \int_a^b \mathrm{d}[A_2(s)]x(s).$$

Also the following Bolzano-Cauchy condition holds.

1.8 (see [5, Proposition 7]). *For $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that*

$$\|S(\,\mathrm{d}A, x, D_1) - S(\,\mathrm{d}A, x, D_2)\|_Z < \varepsilon$$

provided D_1, D_2 are δ-fine P-partitions of $[a,b]$.

1.9 (see [5, Proposition 8]). *If for $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists, then for every interval $[c,d] \subset [a,b]$ also the integral $\int_c^d \mathrm{d}[A(s)]x(s)$ exists.*

1.10 (see [5, Proposition 9]). *Assume that functions $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that for $c \in [a,b]$ the Stieltjes integrals $\int_a^c \mathrm{d}[A(s)]x(s)$ and $\int_c^b \mathrm{d}[A(s)]x(s)$ exist.*

Then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ *exists and*

$$\int_a^b \mathrm{d}[A(s)]x(s) = \int_a^c \mathrm{d}[A(s)]x(s) + \int_c^b \mathrm{d}[A(s)]x(s).$$

In the opposite direction we evidently have:

If $c \in [a,b]$ *and the integral* $\int_a^b \mathrm{d}[A(s)]x(s)$ *exists, then the Stieltjes integrals* $\int_a^c \mathrm{d}[A(s)]x(s)$ *and* $\int_c^b \mathrm{d}[A(s)]x(s)$ *exist and*

$$\int_a^b \mathrm{d}[A(s)]x(s) = \int_a^c \mathrm{d}[A(s)]x(s) + \int_c^b \mathrm{d}[A(s)]x(s).$$

1.11 (see [5, Proposition 10]). *If functions* $A\colon [a,b] \to L(X)$ *and* $x\colon [a,b] \to X$ *are such that the Stieltjes integral* $\int_a^b \mathrm{d}[A(s)]x(s)$ *exists then*

$$\left\| \int_a^b \mathrm{d}[A(s)]x(s) \right\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[a,b]}(A). \sup_{s\in[a,b]} \|x(s)\|_X.$$

The uniform convergence theorem holds for Stieltjes integrals in the following form.

1.12 (see [5, Theorem 11]). *Assume that functions* $A\colon [a,b] \to L(X)$ *and* $x, x_n\colon [a,b] \to X$, $n = 1,2,\ldots$ *are given. If* $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$, *the Stieltjes integrals* $\int_a^b \mathrm{d}[A(s)]x_n(s)$ *exist and the sequence* x_n *converges on* $[a,b]$ *uniformly to* x, *i.e.*

$$\lim_{n\to\infty} \|x_n(s) - x(s)\|_X = 0 \quad \textit{uniformly on } [a,b],$$

then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ *exists and*

$$\int_a^b \mathrm{d}[A(s)]x(s) = \lim_{n\to\infty} \int_a^b \mathrm{d}[A(s)]x_n(s).$$

The facts given in 1.12 together with 1.3 yield the following existence result.

1.13 (see [5, Proposition 15]). *Assume that* $A\colon [a,b] \to L(X)$ *is* $\mathcal{B}$*-regulated on* $[a,b]$ $(A \in (\mathcal{B})G([a,b], L(X)))$ *and* $(\mathcal{B}) \operatorname{var}_{[a,b]}(A) < \infty$. *Let* $x\colon [a,b] \to X$ *be a regulated function.*

Then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ *exists.*

A Hake type theorem holds for our Stieltjes integral, too.

1.14 (see [5, Theorem 17]). *Assume that functions $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integral $\int_a^c \mathrm{d}[A(s)]x(s)$ exists for every $c \in [a,b)$, and let the limit*

$$\lim_{c \to b-} \left[\int_a^c \mathrm{d}[A(s)]x(s) + [A(b) - A(c)]x(b) \right] = J \in X$$

exist. Then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists and

$$\int_a^b \mathrm{d}[A(s)]x(s) = J.$$

The "left endpoint" analog of this statement has the following form.

If functions $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integral $\int_c^b \mathrm{d}[A(s)]x(s)$ exists for every $c \in (a,b]$ and if the limit

$$\lim_{c \to a+} \left[\int_c^b \mathrm{d}[A(s)]x(s) + [A(c) - A(a)]x(a) \right] = J \in X$$

exists, then the integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists and

$$\int_a^b \mathrm{d}[A(s)]x(s) = J.$$

1.15 (see [5, Theorem 19]). *If functions $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists and $c \in [a,b]$, then*

$$\lim_{\substack{r \to c \\ r \in [a,b]}} \left[\int_a^r \mathrm{d}[A(s)]x(s) + [A(c) - A(r)]x(c) \right] = \int_a^c \mathrm{d}[A(s)]x(s).$$

The last statement shows that the function given by

$$r \in [a,b] \mapsto \int_a^r \mathrm{d}[A(s)]x(s) \in X,$$

i.e. the *indefinite Stieltjes integral* is not continuous in general. The indefinite integral is continuous at a point $c \in [a,b]$ if and only if $\lim\limits_{r \to c} [A(c) - A(r)]x(c) = 0$.

1.16 (see [5, Corollary 21]). *If $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists, $c \in [a,b]$ and $A \in (\mathcal{B})G([a,b];L(X))$, then*

$$\lim_{r\to c^+_-} \int_a^r \mathrm{d}[A(s)]x(s) = \lim_{r\to c^+_-} [A(r) - A(c)]x(c) + \int_a^c \mathrm{d}[A(s)]x(s)$$
$$= \lim_{r\to c^+_-} A(r)x(c) - A(c)x(c) + \int_a^c \mathrm{d}[A(s)]x(s).$$

By the Banach-Steinhaus theorem (see e.g. [4]) the following can be deduced.

1.17 (see [5, Proposition 22]). *If $A\colon [a,b] \to L(X)$ is $\mathcal{B}$-regulated (i.e. $A \in (\mathcal{B})G([a,b];L(X))$) then for every $c \in [a,b)$ there exists $A(c+) \in L(X)$ such that*

$$\lim_{t\to c+} A(t)x = A(c+)x$$

for every $x \in X$, and for every $c \in (a,b]$ there exists $A(c-) \in L(X)$ such that

$$\lim_{t\to c-} A(t)x = A(c-)x$$

for every $x \in X$.

1.18 (see [5, Corollary 24]). *Suppose that functions $A\colon [a,b] \to L(X)$ and $x\colon [a,b] \to X$ are such that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists and let $c \in [a,b]$. If $A \in (\mathcal{B})G([a,b],L(X))$ then*

$$\lim_{r\to c^+_-} \int_a^r \mathrm{d}[A(s)]x(s) = [A(c^+_-) - A(c)]x(c) + \int_a^c \mathrm{d}[A(s)]x(s)$$

where $A(c^+_-) \in L(X)$ is given by the relation

$$\lim_{r\to c^+_-} A(r)x = A(c^+_-)x,\ x \in X.$$

R e m a r k. In the situation of 1.18, i.e. if $A \in (\mathcal{B})G([a,b];L(X))$ and $x\colon [a,b] \to X$ is such a function that the Stieltjes integral $\int_a^b \mathrm{d}[A(s)]x(s)$ exists, the indefinite integral given by

$$F(r) = \int_a^r \mathrm{d}[A(s)]x(s) \quad \text{for } r \in [a,b]$$

is a function $F\colon [a,b] \to X$ which is regulated, i.e. $F \in G([a,b];X)$.

Let us assume that $[a,b]=[0,1]$. All the forthcoming consideration can be done for the case of a general compact interval $[a,b]\subset\mathbb{R}$.

First of all let us recall that the space $G(X)=G([0,1];X)$ of all regulated functions $x\colon [0,1]\to X$ is a Banach space with the norm

$$\|x\|_{G(X)}=\sup_{s\in[0,1]}\|x(s)\|_X$$

(see e.g. [2]).

Let us denote

$$(\mathcal{B})\,\mathrm{BV}(L(X))=(\mathcal{B})\,\mathrm{BV}([0,1];L(X)),\qquad (\mathcal{B})G(L(X))=(\mathcal{B})G([0,1];L(X))$$

and

$$\mathrm{BV}(X)=\mathrm{BV}([0,1];X),\qquad G(X)=G([0,1];X)$$

and assume that $A\colon [0,1]\to L(X)$ is given where

$$A\in(\mathcal{B})\,\mathrm{BV}(L(X))\cap(\mathcal{B})G(L(X)). \tag{2.1}$$

Then by 1.13 for every $x\in G(X)$ the Stieltjes integral $\int_0^1 \mathrm{d}[A(s)]x(s)$ exists. Therefore by 1.9 also the integral $\int_0^t \mathrm{d}[A(s)]x(s)$ exists for every $t\in[0,1]$ and $x\in G(X)$. Hence the relation

$$t\in[0,1]\to\int_0^t \mathrm{d}[A(s)]x(s)\in X \tag{2.2}$$

defines an X-valued function.

Using 1.18 we conclude that this function is regulated.

Define an operator $T\colon G(X)\to G(X)$ by the relation

$$(Tx)(t)=\int_0^t \mathrm{d}[A(s)]x(s),\ x\in G(X),\ t\in[0,1]. \tag{2.3}$$

Remark. Let us mention that if for some $A\colon [0,1]\to L(X)$ by (2.3) a regulated function is given for any choice of $x\in G(X)$ then this holds also for every constant function, i.e. for every $x^*\in X$ the onesided limits

$$\lim_{t\to r+}\int_0^t \mathrm{d}[A(s)]x^*=\lim_{t\to r+}[A(t)-A(0)]x^*=\lim_{t\to r+}A(t)x^*-A(0)x^*,\quad r\in[0,1)$$

and

$$\lim_{t\to r-}\int_0^t \mathrm{d}[A(s)]x^* = \lim_{t\to r-} A(t)x^* - A(0)x^*, \quad r\in(0,1]$$

exist and this means that $A\in(\mathcal{B})G(L(X))$.

Hence the assumption $A\in(\mathcal{B})G(L(X))$ is necessary for the operator Tx given by (2.3) to be a mapping of $G(X)$ into $G(X)$.

2.1. Proposition. *If $A\colon\ [0,1]\to L(X)$ satisfies* (2.1), *then $T\colon\ G(X)\to G(X)$ given by* (2.3) *is a bounded linear operator on $G(X)$, i.e. $T\in L(G(X))$.*

P r o o f. The linearity of the operator follows immediately from the linearity of the Stieltjes integral given by 1.7.

By 1.11 for every $t\in[0,1]$ we have

$$\|(Tx)(t)\|_X \leqslant (\mathcal{B})\operatorname*{var}_{[0,t]}(A)\cdot\sup_{s\in[0,t]}\|x(s)\|_X \leqslant (\mathcal{B})\operatorname*{var}_{[0,1]}(A)\cdot\|x\|_{G(X)}.$$

Hence

$$\|Tx\|_{G(X)} = \sup_{t\in[0,1]}\|(Tx)(t)\|_X \leqslant (\mathcal{B})\operatorname*{var}_{[0,1]}(A).\|x\|_{G(X)}$$

and this yields the boundedness of the operator T.

Clearly we have

$$\|T\|_{L(G(X))} \leqslant (\mathcal{B})\operatorname*{var}_{[0,1]}(A) \tag{2.4}$$

for the strong operator norm $\|T\|_{L(G(X))}$ of $T\in L(G(X))$. □

Assume now that $f\in G(X)$ and let us consider the equation

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d), \quad t\in[0,1] \tag{2.5}$$

where $\widetilde{x}\in X$ and $d\in[0,1]$.

If $d\in[\alpha,\beta]\subset[0,1]$ then $x\colon\ [\alpha,\beta]\to X$ is called a solution of (2.5) if x satisfies

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d)$$

for every $t\in[\alpha,\beta]$. Clearly $x(d)=\widetilde{x}$ for any solution x of (2.5).

Equations of the form (2.5) are called *generalized linear ordinary differential equations.* In the special case of $X=\mathbb{R}^n$, $n\in\mathbb{N}$ the equation (2.5) represents a generalization of a linear system of ordinary differential equations (see [6], [7] for more details).

Since by 1.18 the function $t \to \int_d^t \mathrm{d}[A(s)]x(s)$ is regulated provided the integral $\int_d^t \mathrm{d}[A(s)]x(s)$ exists, we can easily conclude that the following holds.

2.2. Proposition. *If $A\colon [0,1] \to L(X)$ satisfies (2.1), $f \in G(X)$, $d \in [\alpha,\beta]$ and $x\colon [\alpha,\beta] \to X$ is a solution of (2.5) on $[\alpha,\beta]$, then x is regulated on $[\alpha,\beta]$, i.e. $x \in G([\alpha,\beta];X)$.*

2.3. Proposition. *If $A\colon [0,1] \to L(X)$ satisfies (2.1), $f \in G(X)$, $d \in [\alpha,\beta]$ and $x\colon [\alpha,\beta] \to X$ is a solution of (2.5) on $[\alpha,\beta]$, then*

$$x(t+) = \lim_{r\to t+} x(r) = [I + \Delta^+ A(t)]x(t) + \Delta^+ f(t), \quad t \in [\alpha,\beta) \tag{2.6}$$

and

$$x(t-) = \lim_{r\to t-} x(r) = [I - \Delta^- A(t)]x(t) - \Delta^- f(t), \quad t \in (\alpha,\beta] \tag{2.7}$$

where I is the identity operator on X, $\Delta^+ A(t) = A(t+) - A(t)$, $\Delta^- A(t) = A(t) - A(t-)$ and $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$.

P r o o f. For $t \in [\alpha,\beta)$ we obtain by 1.15

$$\begin{aligned} x(t+) &= \lim_{r\to t+} x(r) \\ &= \widetilde{x} + \lim_{r\to t+} \int_d^r \mathrm{d}[A(s)]x(s) + \lim_{r\to t+} f(r) - f(d) \\ &= \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + \lim_{r\to t+} \int_t^r \mathrm{d}[A(s)]x(s) \\ &\quad + \lim_{r\to t+} f(r) - f(t) + f(t) - f(d) \\ &= x(t) + \lim_{r\to t+} \int_t^r \mathrm{d}[A(s)]x(s) + \lim_{r\to t+} f(r) - f(t) \\ &= x(t) + (A(t+) - A(t))x(t) + f(t+) - f(t) \\ &= [I + \Delta^+ A(t)]x(t) + \Delta^+ f(t) \end{aligned}$$

and (2.6) is fulfilled. The relation (2.7) can be proved similarly. □

For $[c,d] \subset [0,1]$, $c < d$ define

$$(\mathcal{B}) \operatorname*{var}_{(c,d]}(A) = \lim_{\delta\to 0+} (\mathcal{B}) \operatorname*{var}_{[c+\delta,d]}(A)$$

and

$$(\mathcal{B}) \operatorname*{var}_{[c,d)}(A) = \lim_{\delta\to 0+} (\mathcal{B}) \operatorname*{var}_{[c,d-\delta]}(A).$$

Since $A \in (\mathcal{B})G(L(X))$ and the functions $t \in [0,d] \to (\mathcal{B})\operatorname{var}_{[t,d]}(A)$, $t \in [c,1] \to (\mathcal{B})\operatorname{var}_{[c,t]}(A)$ are bounded and monotone, these limits exist.

2.4. Proposition. *Assume that $A\colon [0,1] \to L(X)$ satisfies (2.1) and assume that the following condition (E) is satisfied: for every $d \in [0,1]$ there are $\varrho = \varrho(d) < 1$ and $\Delta = \Delta(d) > 0$, $0 < \varrho < 1$, such that*

$$\text{(E+)} \qquad (\mathcal{B}) \operatorname*{var}_{(d,d+\Delta]\cap[0,1]}(A) < \varrho$$

and

$$\text{(E−)} \qquad (\mathcal{B}) \operatorname*{var}_{[d-\Delta,d)\cap[0,1]}(A) < \varrho.$$

Then for every $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1];X)$ there is a unique function $x \in G(J_d;X)$ defined on the interval $J_d = [d-\Delta, d+\Delta] \cap [0,1]$ such that

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d), \quad t \in J_d.$$

Proof. Let $\Delta = \Delta(d) > 0$ be the value given by the assumption (E). Define the operator

$$(Tx)(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d), \quad t \in J_d.$$

By Proposition 2.1 T is a bounded linear operator on $G(J_d;X)$. If $x, y \in G(J_d;X)$ with $x(d) = y(d) = \widetilde{x}$, then

$$(Tx)(t) - (Ty)(t) = \int_d^t \mathrm{d}[A(s)](x(s) - y(s))$$

and for $t \in J_d$, $t < d$ we have

$$\begin{aligned}(Tx)(t) - (Ty)(t) &= \int_t^d \mathrm{d}[A(s)](y(s) - x(s)) \\ &= \lim_{\delta\to 0+} \Big[\int_t^{d-\delta} \mathrm{d}[A(s)](y(s) - x(s)) \\ &\qquad + (A(d) - A(d-\delta))(y(d) - x(d)) \Big] \\ &= \lim_{\delta\to 0+} \int_t^{d-\delta} \mathrm{d}[A(s)](y(s) - x(s)).\end{aligned}$$

By 1.11 we obtain

$$\Big\| \int_t^{d-\delta} \mathrm{d}[A(s)](y(s)-x(s)) \Big\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[t,d-\delta]}(A). \sup_{[t,d-\delta]} \|y(s)-x(s)\|_X$$
$$\leqslant (\mathcal{B}) \operatorname*{var}_{[t,d-\delta]}(A). \|x-y\|_{G(J_d;X)}.$$

Hence by (E) we get for $t \in J_d$, $t < d$ the inequality

$$\begin{aligned}\|(Tx)(t)-(Ty)(t)\|_X &\leqslant \lim_{\delta\to 0+} \| \int_t^{d-\delta} \mathrm{d}[A(s)](y(s)-x(s))\|_X \\ &\leqslant \lim_{\delta\to 0+} (\mathcal{B}) \operatorname*{var}_{[t,d-\delta]}(A) \cdot \|x-y\|_{G(J_d;X)} \\ &\leqslant (\mathcal{B}) \operatorname*{var}_{[d-\Delta,d)\cap[0,1]}(A) \cdot \|x-y\|_{G(J_d;X)} \\ &< \varrho \|x-y\|_{G(J_d;X)}.\end{aligned}$$

For $t \in J_d$, $d < t$ we can show similarly that

$$\|(Tx)(t)-(Ty)(t)\|_X < \varrho \|x-y\|_{G(J_d;X)}.$$

Passing to the supremum over $t \in J_d$ we get

$$\|Tx-Ty\|_{G(J_d;X)} = \sup_{t\in J_d} \|Tx(t)-Ty(t)\|_X \leqslant \varrho \|x-y\|_{G(J_d;X)}.$$

Since $\varrho = \varrho(d) < 1$, the operator T acting on $\{x \in G(J_d;X);\ x(d) = \widetilde{x}\}$ is a contraction and by the Banach Contraction Principle it has a unique fixed point, i.e. there is $x \in G(J_d;X)$ such that

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d), \quad t \in J_d.$$

□

Remark. Proposition 2.4 is in fact a local existence and uniqueness result for the equation

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d), \quad t \in [0,1] \tag{2.5}$$

for a given "initial value" $x(d) = \widetilde{x} \in X$ and $f \in G(X)$.

Our goal is to show a global existence and uniqueness result for the solution of (2.5).

To this end the assumptions on A have to be strengthened.

Instead of (2.1) we assume that

$$A \in (\mathcal{B})\,\mathrm{BV}(L(X)) \cap G(L(X)), \tag{2.8}$$

i.e. we require that the operator valued function $A\colon [0,1] \to L(X)$ is regulated. It should be mentioned that 1.4 implies that for A satisfying (2.8) also (2.1) is fulfilled.

Remark. For $A\colon [0,1] \to L(X)$ satisfying (2.8) the onesided limits

$$A(t+) = \lim_{r\to t+} A(r) \in L(X), \quad t \in [0,1)$$

and

$$A(t-) = \lim_{r\to t-} A(r) \in L(X), \quad t \in (0,1]$$

exist because $A \in G(L(X))$ and for every $\varepsilon > 0$ the sets

$$\{t \in [0,1); \|A(t+) - A(t)\|_{L(X)} \geqslant \varepsilon\}, \quad \{t \in (0,1]; \|A(t) - A(t-)\|_{L(X)} \geqslant \varepsilon\}$$

are finite (see [2]). Therefore the set of discontinuity points of A is at most countable and there is a finite set $\{t_1, t_2, \dots, t_m\} \subset [0,1]$ such that for $t \in [0,1]$, $t \neq t_i$, $i = 1, \dots, m$, the operators

$$I + \Delta^+ A(t), \qquad I - \Delta^- A(t) \in L(X)$$

are invertible, i.e. the inverse operators

$$[I + \Delta^+ A(t)]^{-1}, \qquad [I - \Delta^- A(t)]^{-1} \in L(X)$$

exist.

Indeed, as was stated above, the sets

$$\{t \in [0,1); \|\Delta^+ A(t)\|_{L(X)} \geqslant 1\} \quad \text{and} \quad \{t \in (0,1]; \|\Delta^- A(t)\|_{L(X)} \geqslant 1\}$$

are finite and if we set

$$\{t_1, t_2, \dots, t_m\} = \{t \in [0,1); \|\Delta^+ A(t)\|_{L(X)} \geqslant 1\} \cup \{t \in (0,1]; \|\Delta^- A(t)\|_{L(X)} \geqslant 1\}$$

then for $t \in [0,1] \setminus \{t_1, t_2, \dots, t_m\}$ we have

$$\|\Delta^+ A(t)\|_{L(X)} < 1 \quad \text{and} \quad \|\Delta^- A(t)\|_{L(X)} < 1.$$

Hence for these values of t the operators $I + \Delta^+ A(t), I - \Delta^- A(t) \in L(X)$ possess inverses $[I + \Delta^+ A(t)]^{-1}, [I - \Delta^- A(t)]^{-1} \in L(X)$ (see e.g. [1, Lemma VII.6.1]).

Assume that $0 \leqslant d < 1$ and define

$$L(d) = 0, \quad L(t) = \Delta^- A(t) \quad \text{for } t \in (d,1]. \tag{2.9}$$

By the definition of $L\colon [d,1] \to L(X)$ we evidently have $L(t) \neq 0$ for an at most countable set of points in $[d,1]$. Since for every $k \in \mathbb{N}$ the set

$$\bigl\{t \in [d,1]; \|\Delta^- A(t)\|_{L(X)} = \|L(t)\|_{L(X)} \geqslant \tfrac{1}{k}\bigr\}$$

is finite, the function $L\colon [d,1] \to L(X)$ is the uniform limit of simple step functions and therefore $L \in G([d,1]; L(X))$.

2.5. Lemma. *For every finite system of points $d \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_k \leqslant 1$ and every choice of $y_j \in X$, $\|y_j\|_X \leqslant 1$, $j = 1, \ldots, k$ we have*

$$\Bigl\| \sum_{j=1}^{k} L(\alpha_j) y_j \Bigr\|_X \leqslant (\mathcal{B}) \operatorname*{var}_{[0,1]}(A) \tag{2.10}$$

for the function $L\colon [d,1] \to L(X)$ given by (2.9).

P r o o f. Since $L(d) = 0$, it can be assumed that $\alpha_1 > d$. Put $\alpha_0 = d$.

Given $d < \alpha_1 < \alpha_2 < \ldots < \alpha_k \leqslant 1$, for every $\varepsilon > 0$ there is a $\beta_j \in (\alpha_{j-1}, \alpha_j)$, $j = 1, \ldots, k$ such that

$$\|A(\beta_j) - A(\alpha_j -)\|_{L(X)} < \frac{\varepsilon}{2^j}, \quad j = 1, \ldots, k.$$

Then

$$\begin{aligned}
\Bigl\| \sum_{j=1}^{k} L(\alpha_j) y_j \Bigr\|_X &= \Bigl\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_j -)] y_j \Bigr\|_X \\
&= \Bigl\| \sum_{j=1}^{k} [A(\alpha_j) - A(\beta_j)] y_j + [A(\beta_j) - A(\alpha_j -)] y_j \Bigr\|_X \\
&\leqslant \Bigl\| \sum_{j=1}^{k} [A(\alpha_j) - A(\beta_j)] y_j \Bigr\|_X + \Bigl\| \sum_{j=1}^{k} [A(\beta_j) - A(\alpha_j -)] y_j \Bigr\|_X \\
&\leqslant (\mathcal{B}) \operatorname*{var}_{[0,1]}(A) + \sum_{j=1}^{k} \bigl\| [A(\beta_j) - A(\alpha_j -)] \bigr\|_{L(X)} \\
&\leqslant (\mathcal{B}) \operatorname*{var}_{[0,1]}(A) + \varepsilon.
\end{aligned}$$

Hence (2.10) holds. □

2.6. Corollary. *For L given by* (2.9) *we have* $L \in (\mathcal{B})\operatorname{BV}([d,1]; L(X))$.

Proof. Assume that $d = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ is an arbitrary finite partition of $[d,1]$. Then for $y_j \in X$, $\|y_j\|_X \leqslant 1$, $j = 1, \ldots, k$ we have by Lemma 2.5

$$\Big\| \sum_{j=1}^{k} [L(\alpha_j) - L(\alpha_{j-1})] y_j \Big\|_X \leqslant \Big\| \sum_{j=1}^{k} L(\alpha_j) y_j \Big\|_X + \Big\| \sum_{j=1}^{k} L(\alpha_{j-1}) y_j \Big\|_X$$
$$\leqslant 2(\mathcal{B}) \operatorname*{var}_{[0,1]}(A).$$

Passing to the corresponding suprema we obtain by definition the inequality

$$(\mathcal{B}) \operatorname*{var}_{[0,1]}(L) \leqslant 2(\mathcal{B}) \operatorname*{var}_{[0,1]}(A)$$

and therefore $L \in (\mathcal{B})\operatorname{BV}([d,1]; L(X))$. $\square$

2.7. Proposition. *If L is given by* (2.9) *then for every $x \in G([d,1]; X)$ the integral $\int_d^1 \mathrm{d}[L(s)]x(s)$ exists and*

$$\int_d^t \mathrm{d}[L(s)]x(s) = L(t)x(t) = \Delta^- A(t)x(t) \tag{2.11}$$

for $t \in (d,1]$.

Proof. Since $L \in (\mathcal{B})\operatorname{BV}([d,1]; L(X)) \cap G([d,1]; L(X))$ and by 1.4,

$$G([d,1]; L(X)) \subset (\mathcal{B})G([d,1]; L(X))$$

the existence of $\int_d^1 \mathrm{d}[L(s)]x(s)$ follows immediately from 1.13.

Hence by 1.10 for every $t \in [d,1]$ the integral $\int_d^t \mathrm{d}[L(s)]x(s)$ exists and the only fact we have to show is the formula (2.11).

Assume that $t \in (d,1]$ is such that $L(t) = 0$. Let $\delta\colon [d,t] \to (0,\infty)$ be an arbitrary gauge on $[d,t]$. From the system of intervals $(\tau - \delta(\tau), \tau + \delta(\tau))$, $\tau \in [d,t]$ we choose a finite system $J_j = (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$, $j = 1, \ldots, k$ such that $\tau_j < \tau_{j+1}$, $j = 1, \ldots, k-1$, $[d,t] \subset \bigcup_{j=1}^{k} J_j$ and $[d,t] \setminus \bigcup_{j=1, j\neq r}^{k} J_j \neq \emptyset$ for any $r = 1, \ldots, k$. Hence $J_i \cap J_{i+1} \neq \emptyset$ is an interval for all $i = 1, \ldots, k-1$ because the intervals J_i are open. Since the set of points at which $L(s) = 0$ is dense in $[d,1]$, there is an $\alpha_j \in J_j \cap J_{j+1} \cap [d,1]$ such that $L(\alpha_j) = 0$ for $j = 1, \ldots, k-1$. Let us set $\alpha_0 = d$ and $\alpha_k = t$. The system of points $d = \alpha_0 < \alpha_1 < \ldots < \alpha_k = t$ with $\tau_j \in [\alpha_{j-1}, \alpha_j]$, $j = 1, \ldots, k$ evidently forms a δ-fine P-partition D of $[d,t]$.

Assume that $\varepsilon > 0$ is given arbitrarily. Since the integral $\int_d^t \mathrm{d}[L(s)]x(s)$ exists, there is a gauge $\widetilde{\delta}\colon [d,t] \to (0,+\infty)$ such that

$$\left\| S(\mathrm{d}L, x, \widetilde{D}) - \int_d^t \mathrm{d}[L(s)]x(s) \right\| < \varepsilon$$

for every $\widetilde{\delta}$-fine P-partition $\widetilde{D}$ of $[d,t]$.

Let us construct a $\widetilde{\delta}$-fine P-partition D for this gauge $\widetilde{\delta}$ as was described above for an arbitrary gauge δ. Then for the corresponding integral sum we have

$$S(\mathrm{d}L, x, D) = \sum_{j=1}^{k} [L(\alpha_j) - L(\alpha_{j-1})]x(\tau_j) = 0$$

because $L(\alpha_j) = 0$, $j = 0, \dots, k$. Hence

$$\left\| \int_d^t \mathrm{d}[L(s)]x(s) \right\| = \left\| S(\mathrm{d}L, x, D) - \int_d^t \mathrm{d}[L(s)]x(s) \right\| < \varepsilon$$

and since $\varepsilon > 0$ can be arbitrarily small, we get $\int_d^t \mathrm{d}[L(s)]x(s) = 0$ in this case.

If $t \in (d,1]$ and $L(t) \neq 0$, then define $L^0(s) = L(s)$ for $s \in [d,t)$, $L^0(t) = 0$ and $L^1(s) = 0$ for $s \in [d,t)$, $L^1(t) = L(t)$. Then evidently

$$\int_d^t \mathrm{d}[L(s)]x(s) = \int_d^t \mathrm{d}[L^0(s)]x(s) + \int_d^t \mathrm{d}[L^1(s)]x(s), \tag{2.12}$$

where $\int_d^t \mathrm{d}[L^0(s)]x(s) = 0$ by the result given above. Using the definition of L^1 we have $\int_d^r \mathrm{d}[L^1(s)]x(s) = 0$ for every $r < d$ and

$$\lim_{r \to t-} \left[\int_d^r \mathrm{d}[L^1(s)]x(s) + (L^1(t) - L^1(r))x(t) \right] = L^1(t)x(t) = L(t)x(t).$$

According to 1.14 we get

$$\int_d^t \mathrm{d}[L^1(s)]x(s) = L(t)x(t).$$

Using (2.12) and (2.9) we finally obtain (2.11). □

Since $A, L \in (\mathcal{B})\,\mathrm{BV}([d,1]; L(X)) \cap G([d,1]; L(X))$ and this set has a linear structure, we deduce easily that the function $B\colon [d,1] \to L(X)$ given by

$$B(d) = A(d), \quad B(s) = A(s-) \quad \text{for } s \in (d,1] \tag{2.13}$$

also belongs to $(\mathcal{B})\,\mathrm{BV}([d,1];L(X))\cap G([d,1];L(X))$ because

$$B(s)=A(s)-L(s)\quad\text{for every } s\in[d,1].$$

Therefore for every $t\in[d,1]$ the integral $\int_d^t \mathrm{d}[B(s)]x(s)$ exists and by (2.11) we have from Proposition 2.7

$$\begin{aligned}\int_d^t \mathrm{d}[A(s)]x(s)&=\int_d^t \mathrm{d}[B(s)]x(s)+\int_d^t \mathrm{d}[L(s)]x(s)\\&=\int_d^t \mathrm{d}[B(s)]x(s)+\Delta^-A(t)x(t)\end{aligned}\tag{2.14}$$

for $t\in(d,1]$.

Since for every $s\in(d,1]$ the function $B\colon\ [d,1]\to L(X)$ given by (2.13) is continuous from the left at s, we get by 1.18 the equality

$$\lim_{r\to t-}\int_d^r \mathrm{d}[B(s)]x(s)=\int_d^t \mathrm{d}[B(s)]x(s)$$

for every $x\in G([d,1];X)$ and this shows that the integral $\int_d^t \mathrm{d}[B(s)]x(s)$ does not depend on the value $x(t)$.

2.8. Proposition. *Assume that* $d\in[0,1)$, $\widetilde{x}\in X$, $f\in G([0,1];X)$ *and that* $A\colon\ [0,1]\to L(X)$ *satisfies* (2.8), (E) *and*

$$[I-\Delta^-A(t)]^{-1}\in L(X)\quad\text{exists for every } t\in(d,1].\tag{2.15}$$

Then there exists $x\in G([d,1];X)$ *such that*

$$x(t)=\widetilde{x}+\int_d^t \mathrm{d}[A(s)]x(s)+f(t)-f(d)\tag{2.16}$$

for every $t\in[d,1]$.

P r o o f. By the local existence result given in Proposition 2.4 there is a $\Delta>0$ such that there is a function $x\in G([d,d+\Delta];X)$ satisfying (2.16) for $t\in[d,d+\Delta]$.

Define

$$t^*=\sup\bigl\{T\in(d,1];\ \text{there is } x\in G([d,T];X)\text{ such that (2.16) holds for } t\in[d,T]\bigr\}$$

and assume that $t^*<1$. Evidently $t^*>d$ and there exists a function $x\colon\ [d,t^*)\to X$ such that (2.16) holds for every $t\in[d,t^*)$. Let us define

$$x(t^*)=[I-\Delta^-A(t^*)]^{-1}\biggl\{\widetilde{x}+\int_d^{t^*}\mathrm{d}[B(s)]x(s)+f(t^*)-f(d)\biggr\}\tag{2.17}$$

where B is given by (2.13).

Since $\int_d^{t^*} \mathrm{d}[B(s)]x(s)$ depends only on the values of $x(s)$ at $s \in [d, t^*)$ and the inverse $[I - \Delta^- A(t^*)]^{-1}$ exists by the assumption (2.15), the value $x(t^*) \in X$ is well defined and we have by its definition

$$[I - \Delta^- A(t^*)]x(t^*) = \widetilde{x} + \int_d^{t^*} \mathrm{d}[B(s)]x(s) + f(t^*) - f(d).$$

Using (2.14) we get

$$\begin{aligned} x(t^*) &= \widetilde{x} + \int_d^{t^*} \mathrm{d}[B(s)]x(s) + \Delta^- A(t^*)x(t^*) + f(t^*) - f(d) \\ &= \widetilde{x} + \int_d^{t^*} \mathrm{d}[A(s)]x(s) + f(t^*) - f(d). \end{aligned}$$

Hence the function x completed at t^* by (2.17) is a solution of (2.16) on $[d, t^*]$.

Now for t^* and the well defined initial value $x(t^*)$ the local existence result given by Proposition 2.4 can be used again to show that there is a $\Delta(t^*) > 0$ such that a solution $y \in G([t^*, t^* + \Delta(t^*)]; X)$ of

$$y(t) = x(t^*) + \int_{t^*}^{t} \mathrm{d}[A(s)]x(s) + f(t) - f(t^*)$$

exists on $[t^*, t^* + \Delta(t^*)] \cap [0, 1]$.

Putting $z(t) = x(t)$ for $t \in [d, t^*]$, $z(t) = y(t)$ for $t \in [t^*, t^* + \Delta(t^*)]$ we obtain a solution of (2.16) on $[d, t^* + \Delta(t^*)]$. But this contradicts the properties of the supremum and therefore $t^* = 1$. $\square$

Proposition 2.8 gives conditions such that given $d \in [0, 1)$, $\widetilde{x} \in X$ and $f \in G([0, 1]; X)$ there is a global forward solution of the equation (2.16).

In a completely analogous way the following statement can be proved.

2.9. Proposition. *Assume that* $d \in [0, 1)$, $\widetilde{x} \in X$, $f \in G([0, 1]; X)$ *and that* $A\colon [0, 1] \to L(X)$ *satisfies* (2.8), (E) *and*

(2.18) $$[I + \Delta^+ A(t)]^{-1} \in L(X) \quad \textit{exists for every } t \in [0, d).$$

Then there exists $x \in G([0, d]; X)$ *such that*

(2.16) $$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d)$$

for every $t \in [0, d]$.

Putting together the results given by Proposition 2.8 and Proposition 2.9 we obtain the following

2.10. Theorem. *Assume that* $A\colon [0,1] \to L(X)$ *satisfies* (2.8), (E) *and assume that the following condition* (U) *is satisfied:*

$$\text{(U+)} \qquad [I + \Delta^+ A(t)]^{-1} \in L(X) \quad \textit{exists for every } t \in [0,1)$$

and

$$\text{(U−)} \qquad [I - \Delta^- A(t)]^{-1} \in L(X) \quad \textit{exists for every } t \in (0,1].$$

Then for every choice of $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1];X)$ *there exists* $x \in G([0,1];X)$ *such that*

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d) \tag{2.16}$$

for every $t \in [0,1]$.

This solution of (2.16) *is determined uniquely.*

Proof. Given $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1];X)$, Proposition 2.8 can be used for proving the existence of a forward solution $y \in G([d,1];X)$ and Proposition 2.9 for proving the existence of a backward solution $z \in G([0,d];X)$ of the equation (2.16). Taking

$$x(t) = y(t), \quad t \in [d,1] \quad \text{and} \quad x(t) = z(t), \quad t \in [0,d]$$

we get $x \in G([0,1];X)$, which satisfies (2.16) for all $t \in [0,1]$.

To prove the uniqueness of the solution $x \in G([0,1];X)$ assume that there are two solutions $x_1, x_2 \in G([0,1];X)$ which satisfy (2.16) for $t \in [0,1]$. Then for $t \in [0,1]$ we have

$$x_2(t) - x_1(t) = \int_d^t \mathrm{d}[A(s)](x_2(s) - x_2(s)),$$

i.e. the difference $z(t) = x_2(t) - x_1(t)$ satisfies

$$z(t) = \int_d^t \mathrm{d}[A(s)]z(s), \quad t \in [0,1]. \tag{2.19}$$

Taking assumption (E) into account we obtain by Proposition 2.4 that the equation (2.19) has a unique solution $z(t) = 0$ on the interval $J_d = [d - \Delta, d + \Delta] \cap [0,1]$ where $\Delta > 0$.

If $d < 1$, then in a standard manner we take

$$T^* = \sup \bigl\{ \tau \in [d,1]; z(t) = 0 \quad \text{for } t \in [d,\tau] \bigr\}.$$

Then evidently $z(t) = 0$ for all $t \in [d, T^*)$. Hence $z(T^*-) = \lim_{r \to T^*-} z(r) = 0$. Since z is a solution of (2.19) we have by Proposition 2.3

$$z(T^*-) = [I - \Delta^- A(T^*)] z(T^*) = 0$$

and therefore by the assumption of the existence of the inverse $[I - \Delta^- A(T^*)]^{-1}$ we get $z(T^*) = 0$.

If we had $T^* < 1$, then for the equation

$$z(t) = z(T^*) + \int_{T^*}^t \mathrm{d}[A(s)] z(s) = \int_{T^*}^t \mathrm{d}[A(s)] z(s)$$

we could show that there is a $\Delta(T^*) > 0$ such that $z(t) = 0$ for $t \in [T^*, T^* + \Delta(T^*)]$ and this would contradict the definition of the supremum T^*. Hence $T^* = 1$ and $z(t) = 0$ for $t \in [d,1]$. Analogously it can be shown that also $z(t) = 0$ for $t \in [0,d]$ and therefore $x_1(t) = x_2(t)$ for $t \in [0,1]$. □

By Theorem 2.10 conditions are given which are sufficient for every choice of $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1];X)$.

Let us show that if (2.8) and (E) are satisfied then the condition (U) is also necessary for the existence and uniqueness of a solution x of the equation (2.16) on the whole interval $[0,1]$ in this sense.

Assume that (2.8) and (E) hold where (U) is not valid. As was shown above there is a finite set of points $\{t_1, t_2, \ldots, t_m\} \subset [0,1]$ at which the condition (U) can be violated. Assume e.g. that $d \in [0,1)$ is given and that there is a point $t^* \in (d,1]$ such that the operator $I - \Delta^- A(t^*) \in L(X)$ has not an inverse while $[I - \Delta^- A(t)]^{-1}$ exists for every $t \in (d, t^*)$. Then there exists $y \in X$ such that the linear equation

$$[I - \Delta^- A(t^*)] z = y$$

has no solution z in X.

Define $g(t) = 0$ for $t \in [0,1]$, $t \neq t^*$ and $g(t^*) = y$. Evidently $g \in G(X)$. Suppose that x is a solution of

$$x(t) = \int_d^t \mathrm{d}[A(s)] x(s) + g(t) - g(d) \tag{2.20}$$

for every $t \in [0,1]$. Then $x(t) = 0$ for $t \in [d, t^*)$ and by Proposition 2.3 we have

$$0 = x(t^*-) = [I - \Delta^- A(t^*)]x(t^*) - \Delta^- g(t^*) = [I - \Delta^- A(t^*)]x(t^*) - y$$

and this means that for the value $x(t^*)$ we have

$$[I - \Delta^- A(t^*)]x(t^*) = y.$$

But by the assumption such a value $x(t^*) \in X$ cannot exist and consequently neither can the equation (2.20) have a solution in $G([0,1]; X)$ for the given choice of $g \in G([0,1]; X)$.

Assuming that the operator $I + \Delta^+ A(t^*) \in L(X)$ has no inverse for some $t^* \in [0,1)$ we can proceed analogously.

Together with Theorem 2.10 we arrive at the following result.

2.11. Theorem. *If* $A\colon [0,1] \to L(X)$ *satisfies* (2.8) *and* (E), *then the equation*

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d) \tag{2.16}$$

has a unique solution $x\colon [0,1] \to X$ *for any choice of* $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1]; X)$ *if and only if the condition* (U) *is satisfied.*

Remark. For the finite-dimensional case $X = \mathbb{R}^n$, equations of the form (2.16) have been studied thoroughly in [7] for the case when $A\colon [0,1] \to L(\mathbb{R}^n)$ is a function of bounded variation $A \in \mathrm{BV}([0,1]; L(\mathbb{R}^n))$ and $f \in \mathrm{BV}([0,1]; \mathbb{R}^n)$. It has to be mentioned that in this case we have $x \in \mathrm{BV}([0,1]; \mathbb{R}^n)$ for a solution $x\colon [0,1] \to \mathbb{R}^n$ of (2.16).

It was shown (see [7, Theorem III.1.4]) that the condition (U) is necessary and sufficient for having a unique global solution $x\colon [0,1] \to \mathbb{R}^n$ of (2.16) for every $f \in \mathrm{BV}([0,1]; \mathbb{R}^n)$ and $\widetilde{x} \in \mathbb{R}^n$. This result corresponds in some sense to Theorem 2.11. The techniques used in [7] are different from the approach used there.

Let us mention that if we assume that $A \in \mathrm{BV}([0,1]; L(X))$ then by 1.2 and 1.5 we have $A \in (\mathcal{B})\,\mathrm{BV}(L(X)) \cap G(L(X))$, i.e. (2.8) is satisfied. Moreover, for $[c,d] \subset [0,1]$ we have by 1.2

$$(\mathcal{B}) \operatorname*{var}_{[c,d]}(A) \leqslant \operatorname*{var}_{[c,d]}(A) = \operatorname*{var}_{[0,d]}(A) - \operatorname*{var}_{[0,c]}(A)$$

because the variation is known to be linear. The function given by

$$t \in [0,1] \mapsto \operatorname*{var}_{[0,t]}(A) \in \mathbb{R}$$

is nondecreasing and has therefore onesided limits at every point of $[0,1]$. Hence for every $d \in [0,1]$ there are $0 < \varrho(d) < 1$ and $\Delta(d) > 0$ such that

$$\operatorname*{var}_{[0,d+\Delta(d)]}(A) - \lim_{r\to d-} \operatorname*{var}_{[0,r]}(A) < \varrho,$$

and this yields $\operatorname{var}_{(d,d+\Delta(d)]\cap[0,1]}(A) < \varrho$ and similarly $\operatorname{var}_{[d-\Delta(d),d)\cap[0,1]}(A) < \varrho$.

Hence the condition (E) is also satisfied if $A \in \mathrm{BV}([0,1];L(X))$. Using Theorem 2.11 we obtain the following statement.

If $A\colon [0,1] \to L(X)$ *satisfies* $A \in \mathrm{BV}([0,1];L(X))$ *then the equation*

$$x(t) = \widetilde{x} + \int_d^t \mathrm{d}[A(s)]x(s) + f(t) - f(d) \tag{2.16}$$

has a unique solution $x\colon [0,1] \to X$ *for any choice of* $d \in [0,1]$, $\widetilde{x} \in X$, $f \in G([0,1];X)$ *if and only if the condition* (U) *is satisfied.*

This statement is in fact the above mentioned result from [7] for the case of a general Banach space X.

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