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VARIATIONAL MEASURES AND THE KURZWEIL-HENSTOCK INTEGRAL

1 Introduction

For a given continuous function F on a compact interval E in the set $\mathbb{R}$ of reals the problem is how to describe the "total change" of F on a set $M \subset E$.

Quantities $W_F(M)$ and $V_F(M)$ (see Section 3) are introduced in this work for this aim. They are in fact full variational measures in the sense presented by B.S. Thomson in [10] generated by two slightly different interval functions, namely the oscillation of F over an interval and the value of the additive interval function generated as usual by F. They coincide with the concept of classical total variation if M is an interval and they are zero if on the set M the function F is of negligible variation.

Properties of these variational measures are recalled from [10] and investigated.

The Kurzweil-Henstock integration is shortly described and some of its properties are studied using the variational measure $W_F(M)$ for the indefinite integral F of an integrable function f.

2 Notations, divisions, tags, gauges

Let $-\infty < a < b < \infty$ and let the compact interval $E = [a, b]$ be fixed in the sequel. The topology on E is induced by the usual topology on the set $\mathbb{R}$ of reals.

We denote by $\mathrm{Int}(M)$ the interior of a set $M \subset E$ and $\overline{M}$ denotes the closure of a set $M \subset E$.

In the next I and J always denote closed subintervals of E. The set of all closed subintervals of J will be denoted by $\mathrm{Sub}(J)$. The empty set $\emptyset$ is also assumed to belong to $\mathrm{Sub}(J)$.

If I is nonempty, then by $l(I)$, $r(I)$ we denote the left, right endpoint of I, respectively.

The number $|I| = r(I) - l(I)$ is the length of I.

For the purposes of this paper a mapping T from a set Γ into a set M will be sometimes called a *system of elements* of M.

The notation $T = \{V_j; j \in \Gamma\}$ means that $T(j) = V_j \in M$ for $j \in \Gamma$. A system $\{V_j; j \in \Gamma\}$ of elements of M is called finite if Γ is finite. The usual use of this are mostly the cases $\Gamma = \mathbb{N}$ or $\Gamma = \mathbb{N}_k$ where $\mathbb{N}$ is the set of natural numbers and $N_k = \{j \in \mathbb{N};\ j \leq k\}$.

When we will deal with a system of elements belonging to $\mathrm{Sub}(E)$, we will speak simply about a *system (of intervals).*

The set of all finite unions of closed subintervals of E (i.e. unions of elements of all finite systems) is denoted by $\mathrm{Alg}(E)$.

The set $\mathrm{Alg}(E)$ is closed with respect to finite unions and intersections. Any set $M \in \mathrm{Alg}(E)$ is the union of elements of a finite system $\{I_j; j \in \Gamma\}$, where $I_j \cap I_k = \emptyset$ for $j \neq k$. If $M \in \mathrm{Alg}(E)$, then clearly also $\overline{E \setminus M} \in \mathrm{Alg}(E)$.

A *division* is a finite system $D = \{I_j; j \in \Gamma\}$ of intervals, where $\mathrm{Int}(I_j) \cap I_k = \emptyset$ for $j \neq k$. This means that the elements of a division do not overlap.

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For a given set $M \subset E$ the division D is called a *division in* M if $M \supset \bigcup_{j\in\Gamma} I_j$ D is called a *division of* M if $M = \bigcup_{j\in\Gamma} I_j$; and the division D *covers* M if $M \subset \bigcup_{j\in\Gamma} I_j$.

A division of M exists if and only if $M \in \mathrm{Alg}(E)$.

A map τ from $\mathrm{Sub}(E)$ into E is called a *tag* if $\tau(I) \in I$ for $I \in \mathrm{Dom}(\tau)$. In the sequel only tags of this sort will be used.

A *tagged system* is a pair (D, τ), where $D = \{I_j; j \in \Gamma\}$ is a system and τ is a tag defined on the range of D, i.e. on all I_j, $j \in \Gamma$. In this case we write usually τ_j instead of $\tau(I_j)$.

The tagged system (D, τ) is called M*-tagged* for some set $M \subset E$ if $\tau_j \in M$ for $j \in \Gamma$.

Given a function $f : E \to \mathbb{R}$ and a set $M \subset E$ we denote

$$|f|_M = \sup_{x\in M} |f(x)|.$$

A *gauge* is any function on E with values in the set $\mathbb{R}^+$ of positive reals. The set of all gauges is denoted by $\Delta(E)$.

For $\delta_1, \delta_2 \in \Delta(E)$ we write $\delta_1 \le \delta_2$ if $\delta_1(x) \le \delta_2(x)$ for $x \in E$. In this way a partial ordering in $\Delta(E)$ is defined and any finite set in $\Delta(E)$ has an infimum with respect to this ordering.

If $\delta \in \Delta(E)$, then a tagged system (D, τ), where $D = \{I_j; j \in \Gamma\}$, is called δ*-fine* if $|I_j| < \delta(\tau_j)$ for $j \in \Gamma$.

If $\delta_1, \delta_2 \in \Delta(E)$, $\delta_1 \le \delta_2$, then every δ_1-fine tagged system is also δ_2-fine.

Remark. Let us note that for a given $M \subset E$ and a gauge $\delta \in \Delta(E)$ in some situations it can be helpful to use divisions $D = \{I_j; j \in \Gamma\}$ with the property

$$|I_j| \le |\delta|_{I_j\cap M}, \quad j \in \Gamma$$

instead of δ-fine M-tagged divisions. Let us call divisions of this type δ*-fine and* M*-related.*

If $\{I_j; j \in \Gamma\}$ is δ-fine and M-related and $I_j \cap M = \emptyset$ then $|\delta|_{I_j\cap M} = 0$. Hence $|I_j| = 0$ and the element I_j of the division $D = \{I_j; j \in \Gamma\}$ can be neglected in many of the considerations.

If $(D, \tau) = (\{I_j; j \in \Gamma\}, \tau)$ is an M-tagged δ-fine system then $\tau(I_j) = \tau_j \in M \cap I_j$ and $|I_j| \le \delta(\tau_j) \le |\delta|_{I_j\cap M}$ and $D = \{I_j; j \in \Gamma\}$ is δ-fine and M-related.

If, conversely, $D = \{I_j; j \in \Gamma\}$ is δ-fine and M-related then it need not be possible to find $\tau_j \in M \cap I_j$ for $j \in \Gamma$ such that $|I_j| \le \delta(\tau_j)$.

The following crucial statement is known as Cousin's lemma (see e.g. [5, 3.4 Lemma] or any other relevant text on Kurzweil-Henstock integration).

Proposition 2.1. *To any* $\delta \in \Delta(E)$ *and* $I \in \mathrm{Sub}(E)$ *there exists a* δ*-fine division of* I.

Cousin's lemma can be used in many different ways. We shall use the following statements.

Lemma 2.2. *Let* $I \in \mathrm{Sub}(E)$ *and let* A *be a closed subset of* I*. Then to every* $\delta \in \Delta(E)$ *there is a* δ*-fine* A*-tagged division in* I *which covers* A.

Proof. Denote $\mathrm{dist}(x, A)$ the distance of a point $x \in \mathbb{R}$ from the set A. Let us set

$$\eta(x) = \begin{cases} \min\{\delta(x), \frac{1}{2}\mathrm{dist}(x, A)\} \text{ for } x \in I \setminus A, \\ \delta(x) \text{ for } x \in A \cup (E \setminus I). \end{cases}$$

It is easy to see that $\eta \in \Delta(E)$. Let $(\{I_j; j \in \Phi\}, \tau)$ be an η-fine division of I (it exists by Proposition 2.1) and set $\Gamma = \{j \in \Phi, \tau_j \in A\}$. Then $(\{I_j; j \in \Gamma\}, \tau)$ is a δ-fine A-tagged division which covers A. This follows from the definition of η for $x \notin A$ because for the tag $\tau_j \notin A$ the corresponding interval I_j does not intersect A by the definition of the gauge η. □

Lemma 2.3. *Let* A *be a closed subset of* E*,* $\delta \in \Delta(E)$ *and let* $(\{I_j; j \in \Gamma\}, \tau)$ *be a* δ*-fine* A*-tagged division.*

Then there exists a set $\Phi \supset \Gamma$ *a tag* σ *and a* σ*-fine* A*-tagged division* $(\{I_j; j \in \Phi\}, \sigma)$ *such that* $\sigma_j = \tau_j$ *for* $j \in \Gamma$ *and*

$$A \subset \operatorname{Int}(\bigcup_{j\in\Phi} I_j).$$

Proof. Let $E \setminus \bigcup_{j\in\Gamma} I_j = \bigcup_{k\in\Psi} U_k$ where $\{\overline{U_k}; k \in \Psi\}$ is a pairwise disjoint finite system of closed intervals.

For any $k \in \Psi$ let $(\{I_j; j \in \Gamma_k\}, \tau^{(k)})$ be a δ-fine A-tagged division in $\overline{U_k}$ which covers $A \cap \overline{U_k}$. Now it suffices to set $\Phi = \Gamma \cup (\bigcup_{k\in\Psi} \Gamma_k)$ and $\sigma(I_j) = \tau(I_j)$ for $j \in \Gamma$ and $\sigma(I_j) = \tau^{(k)}(I_j)$ for $j \in \Gamma_k$. □

Remark. Lemma 2.3 means that any δ-fine A-tagged division can be extended to a δ-fine A-tagged division which covers a closed set $A \subset E$.

3 The function W

Assume that $F : E \to \mathbb{R}$ is a real function defined on E. For $I \in \operatorname{Sub}(E)$ define the usual interval function

$$F[I] = F(r(I)) - F(l(I)).$$

Let us denote by $C(E)$ the set of all continuous real-valued functions on E.

The *oscillation* of $F \in C(E)$ on an interval $I \in \operatorname{Sub}(E)$ is defined in the usual way by

$$\omega(F, I) = \sup\{|F(x) - F(y)|; x, y \in I\} = \sup\{|F[J]|; J \in \operatorname{Sub}(I)\}.$$

The following simple properties of the oscillation of a function may be mentioned:

(3.1) $$\omega(F, I) \geq 0,$$

(3.2) $$\omega(F, I) = 0 \text{ if and only if } F \text{ is constant on } I,$$

(3.3) $$\omega(\alpha F, I) = |\alpha|\omega(F, I) \text{ for } \alpha \in \mathbb{R},$$

(3.4) $$\omega(\sum_{j\in\Phi} F_j, I) \leq \sum_{j\in\Phi} \omega(F_j, I) \text{ if } \Phi \text{ is finite },$$

(3.5) $$\omega(F, \bigcup_{j\in\Phi} I_j) \leq \sum_{j\in\Phi} \omega(F, I_j) \text{ if } \Phi \text{ is finite and } \bigcup_{j\in\Phi} I_j \in \operatorname{Sub}(E).$$

Definition 3.1. For $F \in C(E)$ and a division $D = \{I_j; j \in \Gamma\}$ let us set

$$\Omega(F, D) = \sum_{j\in\Gamma} \omega(F, I_j)$$

and

$$A(F, D) = \sum_{j\in\Gamma} |F[I_j]|.$$

If $F \in C(E)$ and $M \subset E$ then for any $\delta \in \Delta(E)$ set

$$W_\delta(F, M) = \sup\{\Omega(F, D);\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}$$

and

$$V_\delta(F,M) = \sup\{A(F,D);\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}$$

and put

$$W_F(M) = \inf\{W_\delta(F,M);\ \delta \in \Delta(E)\}, \tag{3.6}$$

$$V_F(M) = \inf\{V_\delta(F,M);\ \delta \in \Delta(E)\}, \tag{3.7}$$

Let us note that if $\delta_1, \delta_2 \in \Delta(E)$, $\delta_1 \le \delta_2$ then $W_{\delta_1}(F,M) \le W_{\delta_2}(F,M)$ and $V_{\delta_1}(F,M) \le V_{\delta_2}(F,M)$.

Therefore in the definition of $W_F(M)$ and $V_F(M)$ it suffices to take into account gauges which are less than some fixed gauge δ_0 only.

If $D = \{I_j; j \in \Gamma\}$ is a division then

$$|F[I_j]| \le \omega(F, I_j) \quad \text{for} \quad j \in \Gamma.$$

Therefore

$$A(F,D) \le \Omega(F,D)$$

and

$$V_F(M) \le W_F(M) \tag{3.8}$$

Let us recall the notion $V(F,I)$ of *total variation* of a function F over $I \in \mathrm{Sub}(E)$ which is defined by

$$V(F,I) = \sup\{\sum_{j\in\Gamma} |F[I_j]|; \{I_j; j \in \Gamma\} \text{ is a division of } I\}. \tag{3.9}$$

Note that $V(F,I) = 0$ for $I \in \mathrm{Sub}(E)$ if and only if the function F is constant on I and that $V(F,I) = V_F(I)$ for $I \in \mathrm{Sub}(E)$.

First let us show that in the simple situation of an interval $I \in \mathrm{Sub}(E)$ the values $W_F(I)$ and $V_F(I)$ have the classical meaning of the total variation of F over I.

Lemma 3.2. *Let $F \in C(E)$ and $I \in \mathrm{Sub}(E)$. Then*

$$W_F(I) = V_F(I) = V(F,I). \tag{3.10}$$

Proof. Assume that $\varepsilon > 0$ is given.

Since F is uniformly continuous on E there is a $\sigma > 0$ such that $|F[J]| < \frac{1}{2}\varepsilon$ provided $J \subset E$ and $|J| \le \sigma$.

If $\delta(x) = \sigma$ for $x \in E$ then for any δ-fine I-tagged division $\{I_j; j \in \Gamma\}$ we have $\Omega(F,D) = \sum_{j\in\Gamma} \omega(F,I_j) = \sum_{j\in\Gamma} |F[J_j]|$ where $J_j \in \mathrm{Sub}(I_j)$, $j \in \Gamma$ is such that $|F[J_j]| = \omega(F,I_j)$.

Define $\Gamma_1 = \{j \in \Gamma; I_j \subset I\}$ and $\Gamma_2 = \Gamma \setminus \Gamma_1$. Since I is an interval, the set Γ_2 consists of at most two elements. Hence

$$\Omega(F,D) = \sum_{j\in\Gamma} |F[J_j]| = \sum_{j\in\Gamma_1} |F[J_j]| + \sum_{j\in\Gamma_2} |F[J_j]| < V(F,I) + \varepsilon$$

and therefore also

$$W_F(I) \le V(F,I) + \varepsilon$$

and

$$W_F(I) \le V(F,I) \tag{3.11}$$

since $\varepsilon > 0$ can be taken arbitrarily small.

Further let $\{I_j; j \in \mathbb{N}_k\}$ be a division of I, for which

$$V(F,I) < \sum_{j=1}^{k} |F[I_j]| + \frac{\varepsilon}{2}.$$

Let $\delta \in \Delta(E)$ be arbitrary and let $D_j = \{J_i^j; i \in \Phi_j\}$ be a δ-fine division of I_j. Then

$$|F[I_j]| \le \sum_{i \in \Phi_j} |F[J_i^j]|$$

and

$$V(F,I) < \frac{\varepsilon}{2} + \sum_{j=1}^{k} \sum_{i \in \Phi_j} |F[J_i^j]|.$$

Let us set $D = \{J_i^j; j = 1, \dots, k, i \in \Phi_j\}$. Then D is a δ-fine division of I and therefore

$$\sum_{j=1}^{k} \sum_{i \in \Phi_j} |F[J_i^j]| \le V_\delta(F,I).$$

This yields then $V(F,I) < \frac{\varepsilon}{2} + V_\delta(F,I)$ and also $V(F,I) < \varepsilon + V_F(I)$, i.e. we get

$$V(F,I) \le V_F(I).$$

Using (3.8), (3.11) we obtain

$$V_F(I) \le W_F(I) \le V(F,I) \le V_F(I)$$

and this finishes the proof. □

The following simple assertion will be also useful.

Lemma 3.3. *Let $F \in C(E)$, $I \in \mathrm{Sub}(E)$ and $\tau \in I$.*

Then there exists $J \subset I$ such that $\tau \in J$ and

$$\omega(F,I) \le 2|F[J]|.$$

Proof. Since $F \in C(E)$ there is an interval $\widetilde{I} \subset I$ such that $|F[\widetilde{I}]| = \omega(F,I)$.

If $\tau \in \widetilde{I}$, then we may take $J = \widetilde{I}$.

If $\tau \notin \widetilde{I}$, then we have two intervals J_1, where the endpoints of J_1 are τ and $l(\widetilde{I})$ and J_2, where the endpoints of J_2 are τ and $r(\widetilde{I})$ and evidently $\omega(F,I) \le |F[J_1]| + |F[J_2]|$. To get the statement we put $J = J_1$ if $|F[J_1]| \ge |F[J_2]|$ or $J = J_2$ if $|F[J_1]| < |F[J_2]|$. □

Corollary 3.4. *Assume that $F \in C(E)$. If $M \subset E$ then*

$$V_F(M) \le W_F(M) \le 2V_F(M).$$

(This implies e.g. that $V_F(M) = 0$ if and only if $W_F(M) = 0$.)

Given a function $F \in C(E)$ by $W_F(M)$ and $V_F(M)$ two set functions are given. Using the terms presented by B.S.Thomson in [10] we identify $W_F(M)$ and $V_F(M)$ as the *full variational measures* generated by the continuous interval functions given for $I \in \mathrm{Sub}(E)$ by $\omega(F,I)$, $F[I]$, respectively.

By Theorem 3.7 in [10] $W_F(\cdot)$ and $V_F(\cdot)$ are metric outer measures. This means that the following holds.

Proposition 3.5. *Assume that* $F \in C(E)$.

1. *If* $M, M_1, M_2, M_3, \dots$ *is a sequence of sets in* E *for which* $M \subset \bigcup_{i=1}^{\infty} M_i$ *then*

$$W_F(M) \le \sum_{i=1}^{\infty} W_F(M_i)$$

and

$$V_F(M) \le \sum_{i=1}^{\infty} V_F(M_i).$$

2. *If* $M_1, M_2 \subset E$ *are such that there are open sets* G_1, G_2 *with* $M_1 \subset G_1$, $M_2 \subset G_2$ *and* $G_1 \cap G_2 = \emptyset$, *then*

$$W_F(M_1) + W_F(M_2) = W_F(M_1 \cup M_2)$$

and

$$V_F(M_1) + V_F(M_2) = V_F(M_1 \cup M_2).$$

From the second part of this proposition we obtain immediately the following.

Corollary 3.6. *If* $F \in C(E)$ *and* $A_1, A_2 \subset E$ *are closed sets with* $A_1 \cap A_2 = \emptyset$, *then*

$$W_F(A_1 \cup A_2) = W_F(A_1) + W_F(A_2)$$

and

$$V_F(A_1 \cup A_2) = V_F(A_1) + V_F(A_2)$$

Since $\omega(F, I)$ and $F[I]$ are continuous interval functions for the case $F \in C(E)$, by Theorem 3.10 in [10] the outer measures $W_F(\cdot)$ and $V_F(\cdot)$ have the increasing sets property presented in the following statement.

Proposition 3.7. *If* $F \in C(E)$ *and* M_i *is a sequence of sets with* $M_i \subset M_{i+1}$ *then*

$$W_F(\bigcup_{i=1}^{\infty} M_i) = \lim_{n \to +\infty} W_F(M_n)$$

and similarly

$$V_F(\bigcup_{i=1}^{\infty} M_i) = \lim_{n \to +\infty} V_F(M_n).$$

Let us recall another known concept.

Definition 3.8. Let $F \in C(E)$ and $M \subset E$. The function F is called to be of *negligible variation on the set* M if for any $\varepsilon > 0$ there is a $\delta \in \Delta(E)$ such that

$$|\sum_{j \in \Gamma} F[I_j]| < \varepsilon \tag{3.12}$$

for any δ-fine M-tagged division $(\{I_j; j \in \Gamma\}, \tau)$.

Remark. Let us mention that if M is countable then every $F \in C(E)$ is of negligible variation on M.

It is easy to see that the notion of negligible variation on a set M for a function $F \in C(E)$ remains unchanged if (3.12) is replaced by

$$\sum_{j \in \Gamma} |F[I_j]| < \varepsilon$$

in Definition 3.8.

The next statement indicates where the function W_F might be important. It shows that the concept of negligible variation can be characterized by W_F.

Lemma 3.9. *Let $F \in C(E)$ and $M \subset E$. Then F is of negligible variation on M if and only if $W_F(M) = V_F(M) = 0$.*

Proof. Let $\varepsilon > 0$ be given and let $\delta \in \Delta(E)$ be such that (3.12) is satisfied in the case that F is of negligible variation on M.

Assume that $(\{I_j; j \in \Gamma\}, \tau)$ is a δ-fine M-tagged division and let $\Gamma_+ = \{j \in \Gamma; F[I_j] \geq 0\}$ and $\Gamma_- = \Gamma \setminus \Gamma_+$. Then $(\{I_j; j \in \Gamma_+\}, \tau)$ and $(\{I_j; j \in \Gamma_-\}, \tau)$ are again δ-fine M-tagged divisions and this implies that

$$\sum_{j\in\Gamma} |F[I_j]| = \sum_{j\in\Gamma_+} F[I_j] - \sum_{j\in\Gamma_-} F[I_j] < 2\varepsilon$$

holds. By Lemma 3.3 for any $j \in \Gamma$ there is an interval J_j for which $\tau_j \in J_j \subset I_j$ and $\omega(F, I_j) \leq 2|F[J_j]|$ for $j \in \Gamma$. Hence

$$\sum_{j\in\Gamma} \omega(F, I_j) \leq 2 \sum_{j\in\Gamma} |F[J_j]| < 4\varepsilon,$$

because $(\{J_j; j \in \Gamma\}, \tau)$ is also a δ-fine M-tagged division. The last inequality gives $W_\delta(F, M) \leq 4\varepsilon$ and this yields $W_F(M) \leq 4\varepsilon$ for any $\varepsilon > 0$. Hence $W_F(M) = 0$.

If $W_F(M) = 0$ then by definition to every $\varepsilon > 0$ there is a $\delta \in \Delta(E)$ such that $W_\delta(F, M) < \varepsilon$. Hence for every δ-fine M-tagged division $D = (\{I_j; j \in \Gamma\}, \tau)$ we have $\Omega(F, D) < \varepsilon$ and this yields the other implication because $|F[I_j]| \leq \omega(F, I_j)$ for every $j \in \Gamma$.

The quantity $V_F(M)$ appears in the result simply by using Corollary 3.4. □

The basic properties of the function W are summarized in the following statement.

Theorem 3.10. *Let $F, F_j \in C(E)$ and $M, M_j \subset E$, $j \in \mathbb{N}$.*
Then

$$\text{if } M_1 \subset M_2, \text{ then } 0 \leq W_F(M_1) \leq W_F(M_2), \tag{3.13}$$

$$W_F\Big(\bigcup_{j\in\Phi} M_j\Big) \leq \sum_{j\in\Phi} W_F(M_j) \text{ if } \Phi \text{ is at most countable}, \tag{3.14}$$

$$W(\alpha F, I) = |\alpha| W_F(I) \text{ for } \alpha \in \mathbb{R}, \tag{3.15}$$

$$W_{\sum_{j\in\Phi} F_j}(M) \leq \sum_{j\in\Phi} W_{F_j}(M) \text{ if } \Phi \text{ is finite.} \tag{3.16}$$

Proof. The items (3.13), (3.14), (3.16) are easy to prove. (3.14) follows from Proposition 3.5.
□

Remark. The problem under what conditions the equality holds in (3.14), i.e. when

$$W_F\Big(\bigcup_{j\in\Phi} M_j\Big) = \sum_{j\in\Phi} W_F(M_j)$$

if Φ is at most countable, will be important. We give a result of this type in Theorem 3.14 below.

For a given set $M \subset E$ denote by $\mu(M)$ the *Lebesgue measure* of M.

Definition 3.11. By $C^*(E)$ we denote the set of all continuous functions on E which are of negligible variation on sets of Lebesgue measure zero, i.e.

$$C^*(E) = \{F \in C(E);\ W_F(N) = 0 \text{ whenever } \mu(N) = 0\}. \tag{3.17}$$

(See Lemma 3.9.)

It should be mentioned that functions $F \in C^*(E)$ are called in the literature also functions satisfying the *strong Luzin condition* on E (see e.g. [7, Definition 4.1.1]).

If $E = [0,1]$ and $F : E \to \mathbb{R}$ is the well known Cantor function (cf. [3, Theorem 1.21]) then $F \in C(E)$ but $F \notin C^*(E)$.

The following well known assertion will be also needed in the sequel.

Proposition 3.12. *Let M be a (Lebesgue) measurable subset of E. Then there exists a sequence $\{A_j, j \in \mathbb{N}\}$ of closed sets, for which $A_j \subset A_{j+1} \subset M$ for $j \in \mathbb{N}$ and*

$$\mu(M \setminus \bigcup_{j=1}^{\infty} A_j) = 0. \tag{3.18}$$

This statement means that there is an F_σ set F such that $F \subset M$ and $\mu(M \setminus F) = 0$. (See e. g. [3, Theorem 1.12].)

Lemma 3.13. *Let $F \in C^*(E)$, M a measurable subset of E and assume that $\{A_j, j \in \mathbb{N}\}$ is a sequence of closed sets, for which $A_j \subset A_{j+1} \subset M$ for $j \in \mathbb{N}$ and*

$$\mu(M \setminus \bigcup_{j=1}^{\infty} A_j) = 0.$$

Then

$$W_F(M) = \lim_{j\to\infty} W_F(A_j).$$

Proof. Clearly

$$M = (M \setminus \bigcup_{j=1}^{\infty} A_j) \cup \bigcup_{j=1}^{\infty} A_j.$$

Since $F \in C^*(E)$, we have $W_F(M \setminus \bigcup_{j=1}^{\infty} A_j) = 0$. This yields by (3.14) in Theorem 3.10 and by Proposition 3.7

$$W_F(M) \le W_F(M \setminus \bigcup_{j=1}^{\infty} A_j) + W_F(\bigcup_{j=1}^{\infty} A_j) =$$

$$= W_F(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} W_F(A_j).$$

On the other hand, by (3.13) in Theorem 3.10 we have

$$W_F(A_j) \le W_F(A_{j+1}) \le W_F(M)$$

for every $j \in \mathbb{N}$ and therefore

$$\lim_{j\to\infty} W_F(A_j) \le W_F(M).$$

This together with the previous inequality gives the statement of the lemma. □

Theorem 3.14. *Assume that $F \in C^*(E)$ and that $\{M_k; k \in \mathbb{N}\}$ is a sequence of measurable subsets of E.*

If $M_k \cap M_n = \emptyset$ for $k \neq n$, then

$$W_F(\bigcup_{k=1}^{\infty} M_k) = \sum_{k=1}^{\infty} W_F(M_k).$$

Proof. Let $M_k \cap M_n = \emptyset$ for $k, n \in \mathbb{N}$ and $k \neq n$.

First let us show that

$$W_F(M_1 \cup M_2) = W_F(M_1) + W_F(M_2)$$

holds.

If $\{A_j; j \in \mathbb{N}\}$ and $\{B_j; j \in \mathbb{N}\}$ are sequences of closed sets such that $A_j \subset A_{j+1} \subset M_1$, $B_j \subset B_{j+1} \subset M_2$ for $j \in \mathbb{N}$ and

$$\mu(M_1 \setminus \bigcup_{j=1}^{\infty} A_j) = 0, \quad \mu(M_2 \setminus \bigcup_{j=1}^{\infty} B_j) = 0,$$

(cf. Proposition 3.12) then by Lemma 3.13 we have

$$W_F(M_1) = \lim_{j\to\infty} W_F(A_j),\ W_F(M_2) = \lim_{j\to\infty} W_F(B_j).$$

Further clearly

$$\mu((M_1 \cup M_2) \setminus \bigcup_{j=1}^{\infty} (A_j \cup B_j)) = 0$$

and again by Lemma 3.13 we get

$$W_F(M_1 \cup M_2) = \lim_{j\to\infty} W_F(A_j \cup B_j) =$$

$$= \lim_{j\to\infty} W_F(A_j) + \lim_{j\to\infty} W_F(B_j) = W_F(M_1) + W_F(M_2)$$

because

$$W_F(A_j \cup B_j) = W_F(A_j) + W_F(B_j)$$

for every $j \in \mathbb{N}$ by Corollary 3.6.

This easily implies that

$$W_F(\bigcup_{k=1}^{n} M_k) = \sum_{k=1}^{n} W_F(M_k)$$

holds for every $n \in \mathbb{N}$. By (3.13) we have

$$W_F(\bigcup_{k=1}^{n} M_k) \leq W_F(\bigcup_{k=1}^{\infty} M_k)$$

for every $n \in \mathbb{N}$ and therefore

$$\sum_{k=1}^{\infty} W_F(M_k) \leq W_F(\bigcup_{k=1}^{\infty} M_k).$$

From (3.14) in Theorem 3.10 we have

$$W_F(\bigcup_{k=1}^{\infty} M_k) \leq \sum_{k=1}^{\infty} W_F(M_k)$$

and the assertion follows.

□

Theorem 3.14 shows that if $F \in C^*(E)$ then the variational measure $W_F(\cdot)$ generated by F is countably additive on the σ-algebra of measurable subsets of E.

4 The Kurzweil-Henstock integral K

Let us start with the basic definition of the integral.

Definition 4.1. K denotes the set of all pairs (f,γ), where f is a function on E and $\gamma \in \mathbb{R}$, for which to any $\varepsilon > 0$ there exists a gauge δ such that

$$|\sum_{j\in\Gamma} f(\tau_j)|I_j| - \gamma| < \varepsilon$$

for any δ-fine division $(\{I_j; j \in \Gamma\}, \tau)$ of the interval E.

The value $\gamma \in \mathbb{R}$ is called the *Kurzweil-Henstock integral* of f over E and it will be denoted by $K(f)$ or $(K)\int_E f$.

K is in fact a mapping from a set of functions on E into $\mathbb{R}$ (a functional).

Denote by $\mathrm{Dom}(K)$ the set of all f for which the functional K is defined.

If $f \in \mathrm{Dom}(K)$ then f is called *K-integrable over E*.

Denote the *characteristic function* of a set $M \subset E$ by $\chi(M)$, i.e. $\chi(M) = 1$ on M and $\chi(M) = 0$ on $E \setminus M$.

The characteristic function of the empty set $\emptyset$ may be denoted simply by 0 if no confusion can arise.

If the product $f \cdot \chi(M)$ belongs to $\mathrm{Dom}(K)$, then $K(f,M)$ (or $(K)\int_M f$) denotes the value of the functional K on $f \cdot \chi(M)$, i.e. $K(f,M) = K(f \cdot \chi(M))$ and of course $K(f,E) = K(f)$.

Definition 4.2. If $f \in \mathrm{Dom}(K)$, then a function $F : E \to \mathbb{R}$ is called a *K-primitive* (or the *indefinite K-integral*) to f provided

$$F[I] = K(f,I)$$

holds for every $I \in \mathrm{Sub}(E)$.

Now we present a collection of basic properties of the Kurzweil-Henstock integral which will be used in the framework of this paper and in subsequent work.

Proposition 4.3.

$$0 \in \mathrm{Dom}(K) \;\; \textit{and} \;\; K(0) = 0. \tag{4.1}$$

If $c \in [a,b] = E$ and $I_1 = [a,c], I_2 = [c,b]$ then $f \in \mathrm{Dom}(K)$ if and only if $f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(K)$ and

$$K(f) = K(f,I_1) + K(f,I_2). \tag{4.2}$$

If $f = 0$ almost everywhere (with respect to the Lebesgue measure) then

$$f \in \mathrm{Dom}(K) \;\; \textit{and} \;\; K(f) = 0. \tag{4.3}$$

$$\textit{If } \; f \in \mathrm{Dom}(K) \textit{ and } \; F \; \textit{ is a } \; K\textit{-primitive to } \; f \; \textit{ then } \; F \in C^*(E). \tag{4.4}$$

$$\textit{If } f \in \mathrm{Dom}(K) \textit{ then } f \textit{ is (Lebesgue) measurable.} \tag{4.5}$$

K is a linear functional, i.e. if $f, g \in \mathrm{Dom}(K)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \mathrm{Dom}(K)$ and

$$K(\alpha f + \beta g) = \alpha K(f) + \beta K(g). \tag{4.6}$$

Proof. The properties (4.1), (4.2) and (4.6) are easy to prove.

In [3, Theorem 9.5] it is shown that (4.3) holds.

In [7, Theorem 3.9.2] it is proved that a K-primitive function F to $f \in \mathrm{Dom}(K)$ is continuous and of negligible variation on sets of zero (Lebesgue) measure and this means that (4.4) is satisfied (cf. Definition 3.11).

The Lebesgue measurability of every $f \in \mathrm{Dom}(K)$ is proved e.g. in [3, Theorem 9.12]). □

Let us mention that a K-primitive function to $f \in \mathrm{Dom}(K)$ always exists (e.g. $F(x) = K(f, [a, x])$ for $x \in E = [a, b]$ is a K-primitive to f) and it is determined uniquely up to a constant.

If $M \in \mathrm{Alg}(E)$ and $\{I_j; j \in \Gamma\}$ is a division of M, then $f \cdot \chi(M) \in \mathrm{Dom}(K)$ if and only if $f \cdot \chi(I_j) \in \mathrm{Dom}(K)$ for all $j \in \Gamma$ and

$$K(f, M) = \sum_{j\in\Gamma} K(f, I_j).$$

In connection with the property (4.4) from Proposition 4.3 the following beautiful descriptive characterization of the Kurzweil-Henstock integral presented by Bongiorno, Di Piazza an Skvortsov in [1, Theorem 3] should be mentioned.

Theorem 4.4. *A function $F : E \to \mathbb{R}$ is a K-primitive function to some $f : E \to \mathbb{R}$ if and only if $F \in C^*(E)$.*

In other words the class of all functions $F : E \to \mathbb{R}$ which are K-primitive to some f coincides with the class of all $F \in C(E)$ for which $W_F(N) = 0$ if $N \subset E$ and $\mu(N) = 0$.

For more detail see [1] and also [8], [9].

From Gordon's book [3] it is known that a function $F : E \to \mathbb{R}$ is K-primitive to some $f : E \to \mathbb{R}$ if and only if F is an ACG_* function on E. This leads immediately to the conclusion of Theorem 4 in [1] which says that the class of all ACG_* functions on E coincides with the class $C^*(E)$ of functions satisfying the strong Luzin condition.

Similar problems are dealt with also in the posthumous paper [2] of Vasile Ene in connection with an older result of Jarník and Kurzweil from [4].

The following assertion known as the Saks-Henstock lemma plays an important role in the theory (see e.g. [3, Lemma 9.11], [5, Lemma 5.3] , etc.).

Proposition 4.5. *Let $f \in \mathrm{Dom}(K)$. Then to any $\varepsilon > 0$ there is a gauge δ such that for any δ-fine tagged division $(\{I_j; j \in \Gamma\}, \tau)$ in E the inequality*

$$|\sum_{j\in\Gamma} f(\tau_j)|I_j| - K(f, \bigcup_{j\in\Gamma} I_j)| < \varepsilon \tag{4.7}$$

holds.

In other words (F being the K-primitive to f) we have

$$|\sum_{j\in\Gamma} f(\tau_j)|I_j| - \sum_{j\in\Gamma} F[I_j]| < \varepsilon. \tag{4.8}$$

In [3, Theorem 9.21] the following is presented.

Theorem 4.6 (Hake). *Let $f : E \to \mathbb{R}$ be given. Suppose that $f \cdot \chi([c, d]) \in \mathrm{Dom}(K)$ for each $[c, d] \subset E$, $a < c < d < b$. If $K(f, [c, d])$ has a finite limit as $c \to a+$ and $d \to b-$ then $f \in \mathrm{Dom}(K)$ and*

$$K(f) = \lim_{c\to a+,\ d\to b-} K(f, [c, d]).$$

Now we give another property of the Kurzweil-Henstock integral.

Lemma 4.7. *Assume that $f \in \mathrm{Dom}(K)$ and let F be its K-primitive function. Then*

$$W_F(M) \le 2|E||f|_M \tag{4.9}$$

holds for $M \subset E$.

Proof. Proof Let $\varepsilon > 0$ be given. Let $\delta \in \Delta(E)$ be such that (4.8) holds. Assume that $(\{I_j; j \in \Gamma\}, \tau)$ is a δ-fine M-tagged division and let $J_j \subset I_j$ be such that $\tau_j \in J_j$ and $\omega(F, I_j) \le 2|F[J_j]|$ for $j \in \Gamma$ (see Lemma 3.3).

Assume that $\Gamma_1 = \{j \in \Gamma; F[J_j] \ge 0\}$ and set $\Gamma_2 = \Gamma \setminus \Gamma_1$. Evidently $(\{I_j; j \in \Gamma_1\}, \tau)$ and $(\{I_j; j \in \Gamma_2\}, \tau)$ are δ-fine divisions in E.

We have

$$\sum_{j\in\Gamma} \omega(F, I_j) \le 2 \sum_{j\in\Gamma} |F[J_j]| = 2|\sum_{j\in\Gamma_1} F[J_j]| + 2|\sum_{j\in\Gamma_2} F[J_j]|$$

and by (4.8)

$$\sum_{j\in\Gamma_1} |F[J_j]| = \sum_{j\in\Gamma_1} F[J_j] = \sum_{j\in\Gamma_1} f(\tau_j)|J_j| + \sum_{j\in\Gamma_1} (F[J_j] - f(\tau_j)|J_j|) \le$$

$$\le |\sum_{j\in\Gamma_1} f(\tau_j)|J_j|| + |\sum_{j\in\Gamma_1} (F[J_j] - f(\tau_j)|J_j|)| < \sum_{j\in\Gamma_1} |f(\tau_j)|J_j| + \varepsilon.$$

Similarly

$$\sum_{j\in\Gamma_2} |F[J_j]| = -\sum_{j\in\Gamma_2} F[J_j] = \sum_{j\in\Gamma_2} f(\tau_j)|J_j| - \sum_{j\in\Gamma_2} (F[J_j] - f(\tau_j)|J_j|) \le$$

$$\le |\sum_{j\in\Gamma_2} f(\tau_j)|J_j|| + |\sum_{j\in\Gamma_2} (F[J_j] - f(\tau_j)|J_j|)| < \sum_{j\in\Gamma_1} |f(\tau_j)||J_j| + \varepsilon.$$

Therefore

$$\sum_{j\in\Gamma} \omega(F, I_j) < 2 \sum_{j\in\Gamma} |f(\tau_j)||J_j| + 4\varepsilon \le$$

$$\le 2|f|_M \sum_{j\in\Gamma} |J_j| + 4\varepsilon \le 2|f|_M|E| + 4\varepsilon$$

and

$$W_\delta(F, M) < 2|f|_M|E| + 4\varepsilon.$$

Hence

$$W_F(M) < 2|f|_M|E| + 4\varepsilon$$

for every $\varepsilon > 0$ and this implies (4.9). □

Definition 4.8. If $I \in \mathrm{Sub}(E)$ and $A \subset E$ is closed then $\mathrm{Comp}(I, A)$ denotes the set of all (maximal and nonempty) connected components of the set $I \setminus A$.

The set $\mathrm{Comp}(I, A)$ is always at most countable and any element

$$U \in \mathrm{Comp}(I, A)$$

is an interval, i.e. $\overline{U} \in \mathrm{Sub}(E)$.

Lemma 4.9. *Let $A \subset E$ be a closed set, $f, F : E \to \mathbb{R}$.*

Assume that

1) *$f = 0$ on A,*

2) *for every* $[c,d] \subset U \in \mathrm{Comp}(E,A)$ *we have* $f \cdot \chi([c,d]) \in \mathrm{Dom}(K)$ *and*

$$K(f,[c,d]) = F(d) - F(c),$$

3) $F \in C(E)$,

4) $W_F(A) = 0$.

Then $f \in \mathrm{Dom}(K)$ *and* F *is a* K*-primitive to* f.

Proof. By 4) to any $\varepsilon > 0$ there is a $\delta_0 \in \Delta(E)$ such that

$$\Omega(F,D) = \sum_{j\in\Gamma} \omega(F,I_j) < \varepsilon$$

for every δ_0-fine A-tagged division $(\{I_j; j \in \Gamma\}, \tau)$. Therefore

$$|\sum_{j\in\Gamma} F[I_j]| \leq \sum_{j\in\Gamma} |F[I_j]| \leq \sum_{j\in\Gamma} \omega(F,I_j) < \varepsilon$$

for every δ_0-fine A-tagged division $(\{I_j; j \in \Gamma\}, \tau)$.

The conditions 2) and 3) together with Hake's Theorem 4.6 yield

$$f \cdot \chi(\overline{U}) \in \mathrm{Dom}(K)$$

for every $U \in \mathrm{Comp}(E,A)$ and

$$K(f,\overline{U}) = F[\overline{U}] = F(r(\overline{U})) - F(l(\overline{U}))$$

by the continuity of F which is required by 3).

$\mathrm{Comp}(E,A)$ is at most countable, $\mathrm{Comp}(E,A) = \{U_j; j \in \mathbb{N}\}$, because A is closed.

Since $f \cdot \chi(\overline{U_j}) \in \mathrm{Dom}(K)$ for every $j \in \mathbb{N}$ and $K(f,I) = F[I]$ for every $I \in \mathrm{Sub}(\overline{U_j})$, there is a $\delta_j \in \Delta(\overline{U_j})$ such that

$$|\sum_{l\in\Gamma_j} (f(\tau_l)|I_l| - F[I_l])| < \frac{\varepsilon}{2^j}$$

holds for every δ_j-fine division $(\{I_l; l \in \Gamma_j\}, \tau)$ in $\overline{U_j}$, $j \in \mathbb{N}$. This follows from the Saks-Henstock lemma 4.5.

Define

$$\delta(t) = \begin{cases} \min\{\delta_j(t), \frac{1}{2}\mathrm{dist}(t,A)\} & \text{for } t \in \overline{U_j}, j \in \mathbb{N}, \\ \delta_0(t) \text{ for } t \in A. \end{cases}$$

Clearly $\delta \in \Delta(E)$. Assume that $(\{J_k; k \in \Phi\}, \tau)$ is a δ-fine division of E. Denote $\Gamma_0 = \{k \in \Phi; \tau_k \in A\}$, $\Gamma_j = \{k \in \Phi; \tau_k \in \overline{U_j}\}$. By the definition of $\delta \in \Delta(E)$ we have $J_k \subset U_j$ for $k \in \Gamma_j$ and

$$|\sum_{k\in\Phi} f(\tau_k)|J_k| - F[E]| = |\sum_{k\in\Phi} (f(\tau_k)|J_k| - F[J_k]| \leq$$

$$\leq |\sum_{k\in\Gamma_0} F[J_k]| + \sum_{j\in\mathbb{N}} |\sum_{k\in\Gamma_j} (f(\tau_k)|J_k| - F[J_k]| < \varepsilon + \sum_{j\in\mathbb{N}} \frac{\varepsilon}{2^j} = 2\varepsilon.$$

Hence $f \in \mathrm{Dom}(K)$ and $K(f) = F[E]$.

If $I \in \mathrm{Sub}(E)$ then the same procedure can be used for the interval I and the closed set $A \cap I \subset E$ to show that $f \cdot \chi([I]) \in \mathrm{Dom}(K)$ and that $K(f,I) = F[I]$. This yields the statement.

□

Corollary 4.10. *Let $A \subset E$ be a closed set, $f, F : E \to \mathbb{R}$.*
Assume that

1) $f = 0$ *on* A,

2) *for every interval* $I = [c, d] \subset U \in \mathrm{Comp}(E, A)$ *we have* $f \cdot \chi(I) \in \mathrm{Dom}(K)$ *and*

$$K(f, I) = F[I] = F(d) - F(c),$$

3) $F \in C(E)$.

Then $f \in \mathrm{Dom}(K)$ *and* F *is a* K*-primitive to* f *if and only if* $W_F(A) = 0$.

Proof. Lemma 4.9 gives one of the implications an therefore it suffices to show that if $f \in \mathrm{Dom}(K)$ and F is a K-primitive to f then $W_F(A) = 0$. But this is clear by (4.9) from Lemma 4.7 because by 1) we have $|f|_A = 0$. □

Theorem 4.11. *Let $A \subset E$ be a closed set, $g, F : E \to \mathbb{R}$.*
Assume that

1) $g \cdot \chi(A) \in \mathrm{Dom}(K)$,

2) *for every interval* $I \subset U \in \mathrm{Comp}(E, A)$ *we have* $g \cdot \chi(I) \in \mathrm{Dom}(K)$ *and*

$$K(g, I) = F[I],$$

3) $F \in C(E)$.

Then $g \in \mathrm{Dom}(K)$ *if and only if* $W_F(A) = 0$ *and in this case we have*

$$K(g) = K(g, A) + F[E] = K(g, A) + F(b) - F(a).$$

Proof. Let us set $f = g - g \cdot \chi(A)$. Then clearly $f = 0$ on A and $f = g$ on every $U \in \mathrm{Comp}(E, A)$. By 2) we obtain that $f \cdot \chi(I) \in \mathrm{Dom}(K)$ for every $I \subset U \in \mathrm{Comp}(E, A)$ and

$$K(f, I) = F[I].$$

This together with 3) implies by Corollary 4.10 that $f \in \mathrm{Dom}(K)$ if and only if $W_F(A) = 0$ and F is a K-primitive to f. This implies also $K(f) = F[E]$.

By (4.6) and by the definition of f we obtain $g \in \mathrm{Dom}(K)$ if and only if $W_F(A) = 0$ and

$$K(g) = K(g \cdot \chi(A)) + K(f) = K(g, A) + F[E].$$

The theorem is proved. □

Remark. Let us mention that if G is a K-primitive to $g \cdot \chi(M) \in \mathrm{Dom}(K)$, then $G + F$ is a K-primitive to g.

In [7, Theorem 3.4.1] the following statement was proved.

Theorem 4.12. *If g is K-integrable over $I \in \mathrm{Sub}(E)$ and G is its K-primitive then $|g|$ is K-integrable over I if and only if $V(G, I) < \infty$ and*

$$V(G, I) = K(|g|, I).$$

In this situation we have $G \in C(E)$ and using Lemma 3.2 we get the following.

Lemma 4.13. *If g is K-integrable over $I \in \mathrm{Sub}(E)$ and G is its K-primitive then $|g|$ is K-integrable over I if and only if $W(G, I) < \infty$ and*

$$W_G(I) = K(|g|, I)$$

in this case.

Lemma 4.14. *If $M \subset E$ and $f, g = f \cdot \chi(M) \in \mathrm{Dom}(K)$ where F, G are K-primitives to f, g then*

$$W_F(M) = W_G(M) \tag{4.10}$$

Proof. Proof Since $f - g \in \mathrm{Dom}(K)$ and $F - G$ is a K-primitive to $f - g$ we have by (4.9) in Lemma 4.7

$$W_{F-G}(M) \le 2|E||f - g|_M = 0.$$

Hence by (3.14) from Theorem 3.10 we get

$$W_F(M) = W_{F-G+G}(M) \le W_{F-G}(M) + W_G(M) = W_G(M).$$

Similarly also $W_G(M) \le W_F(M)$ and (4.10) holds. □

Lemma 4.15. *Assume that $f \in \mathrm{Dom}(K)$ with F being its K-primitive, $M \subset E$ (Lebesgue) measurable and $g = |f| \cdot \chi(M) \in \mathrm{Dom}(K)$ with the K-primitive G. Then*

$$W_F(M) = K(|f|, M) = K(g). \tag{4.11}$$

Proof. Proof By (4.5) f is measurable and therefore $f \cdot \chi(M)$ is measurable as well.

Since $|f \cdot \chi(M)| = |f| \cdot \chi(M) \in \mathrm{Dom}(K)$ we have $f \cdot \chi(M) \in \mathrm{Dom}(K)$ (see e.g. [7, Theorem 3.11.2]).

Hence by Lemma 4.14 we have $W_F(M) = W_G(M)$.

Since $M \subset E$ we have $W_G(M) \le W_G(E)$ by (3.13) and on the other hand by (3.14) we get

$$W_G(E) \le W_G(M) + W_G(E \setminus M) = W_G(M)$$

because by Lemma 4.7 we have $W_G(E \setminus M) \le 2|E||g|_{E\setminus M} = 0$. This yields $W_G(M) = W_G(E)$ and therefore

$$W_F(M) = W_G(E).$$

By Lemma 4.13 we have

$$W_G(E) = K(g) = K(|f| \cdot \chi(M)) = K(|f|, M)$$

because $g = |g|$ and (4.11) is proved. □

For $f \in \mathrm{Dom}(K)$, $M \subset E$ measurable, denote

$$\begin{aligned} \overline{K}(|f|, M) &= K(|f|, M) \quad \text{if} \quad |f| \cdot \chi(M) \in \mathrm{Dom}(K), \\ \overline{K}(|f|, M) &= \infty \quad \text{otherwise.} \end{aligned}$$

Using Lemma 4.15 we have

$$W_F(M) \le \overline{K}(|f|, M) \tag{4.12}$$

for every $f \in \mathrm{Dom}(K)$ with F being its K-primitive.

Proposition 4.16. *If $f \in \mathrm{Dom}(K)$, F a K-primitive to f and $M \subset E$ measurable, then*

$$W_F(M) = \overline{K}(|f|, M). \tag{4.13}$$

Proof. Since (4.12) holds, the equality (4.13) holds for the case when $W_F(M) = \infty$.

Assume that $W_F(M) < \infty$. By (4.12) for proving (4.13) it suffices to show that

$$\overline{K}(|f|, M) \le W_F(M). \tag{4.14}$$

Denote $g = |f| \cdot \chi(M)$ and assume that $\varepsilon > 0$ is given.

Since $f \in \mathrm{Dom}(K)$, by the Saks-Henstock lemma (Proposition 4.5) there is a $\delta_1 \in \Delta(E)$ such that

$$|\sum_{j\in\Gamma}(f(\tau_j)|I_j| - FI_j)| < \varepsilon \tag{4.15}$$

for any δ_1-fine division $(\{I_j, j \in \Gamma\}, \tau)$ in E. Using the definition of $W_F(M)$ assume further that $\delta_2 \in \Delta(E)$ is such that

$$\sum_{j\in\Gamma}\omega(F, I_j) < W_F(M) + \varepsilon \tag{4.16}$$

for every δ_2-fine M-tagged division $(\{I_j, j \in \Gamma\}, \tau)$ in E and put

$$\delta = \min\{\delta_1, \delta_2\}.$$

Let $(\{I_j, j \in \Gamma\}, \tau)$ be an arbitrary δ-fine division in E. Denote $\widetilde{\Gamma} = \{j \in \Gamma; g(\tau_j) \ne 0\}$. For $j \in \widetilde{\Gamma}$ we have clearly $\tau_j \in M$ and $\{I_j, j \in \widetilde{\Gamma}\}$ forms an M-tagged division in E which is both δ_1- and δ_2-fine.

Then

$$\sum_{j\in\Gamma} g(\tau_j)|I_j| = \sum_{j\in\widetilde{\Gamma}} g(\tau_j)|I_j| = \sum_{j\in\widetilde{\Gamma}} f(\tau_j)|I_j|$$
$$= \sum_{j\in\Gamma_+} f(\tau_j)|I_j| - \sum_{j\in\Gamma_-} f(\tau_j)|I_j|$$

where $\Gamma_+ = \{j \in \widetilde{\Gamma}, f(\tau_j) > 0\}$, $\Gamma_- = \{j \in \widetilde{\Gamma}, f(\tau_j) < 0\}$.

Hence by (4.15) and (4.16) we obtain

$$\sum_{j\in\Gamma} g(\tau_j)|I_j| \le$$
$$|\sum_{j\in\Gamma_+} f(\tau_j)|I_j| - F[I_j]| + |\sum_{j\in\Gamma_-} f(\tau_j)|I_j| - F[I_j]| +$$
$$|\sum_{j\in\Gamma_+} F[I_j]| + |\sum_{j\in\Gamma_-} F[I_j]| <$$
$$2\varepsilon + \sum_{j\in\widetilde{\Gamma}} |F[I_j]| \le 2\varepsilon + \sum_{j\in\widetilde{\Gamma}} \omega(F, I_j) < W_F(M) + 3\varepsilon.$$

Since all the integral sums corresponding to the nonnegative function $g = |f| \cdot \chi(M)$ and to the δ_1-fine tagged division $(\{I_j, j \in \Gamma\}, \tau)$ are bounded by $W_F(M) + 3\varepsilon$ we obtain that the integral $K(g) = K(|f|, M)$ exists and satisfies the estimate

$$K(g) = K(|f|, M) < W_F(M) + 3\varepsilon$$

for an arbitrary $\varepsilon > 0$. Hence

$$K(|f|, M) \le W_F(M)$$

and (4.14) holds.

□

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