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Variational measures and extensions of the integral

Some concepts of integration theories are recollected and and used especially for the Kurzweil-Henstock integral.

The general integral. Put $E = [a, b], -\infty < a < b < +\infty$ and denote $\mathrm{Sub}(E)$ the set of compact subintervals in E.

A functional S is a set of pairs (f, γ) ($f : E \to \mathbb{R}$, $\gamma \in \mathbb{R}$ the value of S), γ is uniquely determined by f, $\gamma = S(f)$.

The set of all f for which the functional S is defined is $\mathrm{Dom}(S)$.

The functional S is *additive* if

A) $0 \in \mathrm{Dom}(S)$ and $S(0) = 0$,

B) if $c \in [a, b] = E$, $I_1 = [a, c], I_2 = [c, b]$ then $f \in \mathrm{Dom}(S) \iff f \cdot \chi(I_1), f \cdot \chi(I_2) \in \mathrm{Dom}(S)$ and $S(f) = S(f, I_1) + S(f, I_2)$.

If S is an additive and $f \in \mathrm{Dom}(S)$, then $F : E \to \mathbb{R}$ is an *S-primitive* to f provided $F[I] = F(d) - F(c) = S(f, I) = S(f \cdot \chi_I), \forall I = [c, d] \in \mathrm{Sub}(E)$.

An additive S is an *integral* in E if all S-primitive functions to $f \in \mathrm{Dom}(S)$ are continuous in E ($F \in C(E)$).

Denote *the set of all integrals in* E by $\mathfrak{S}$. If $S \in \mathfrak{S}$ and $f \in \mathrm{Dom}(S)$ then f is called S- *integrable.*

Comparing integrals. If $T, S \in \mathfrak{S}$ then T *includes* S ($S \sqsubset T$) if $\mathrm{Dom}(S) \subset \mathrm{Dom}(T)$ and if for $f \in \mathrm{Dom}(S)$ and $I \in \mathrm{Sub}(E)$ the equality $T(f, I) = S(f, I)$ holds.

By the relation $\sqsubset$ a partial ordering in $\mathfrak{S}$ is given.

The Henstock-Kurzweil integral. A *division* is a finite system $D = \{I_j \in \mathrm{Sub}(E); j \in \Gamma\}$, where $\mathrm{Int}(I_j) \cap I_k = \emptyset$ for $j \neq k$.

A map τ from $\mathrm{Sub}(E)$ into E is a *tag* if $\tau(I) \in I$ for $I \in \mathrm{Dom}(\tau)$.

A *tagged system* is a pair (D, τ), where $D = \{I_j; j \in \Gamma\}$ is a division and τ is a tag defined for all I_j, $j \in \Gamma$. $\tau_j = \tau(I_j)$.

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For $M \subset E$ the tagged system $(\{I_j; j \in \Gamma\}, \tau)$ is M-tagged if $\tau_j = \tau(I_j) \in M$. The tagged system $(\{I_j; j \in \Gamma\}, \tau)$ is a division of E if $\bigcup_{j\in\Gamma} I_j = E$.

A *gauge* is any function on E with values in $(0, +\infty)$.

If δ is a gauge, then a tagged system (D, τ) $(D = \{I_j; j \in \Gamma\})$ is *δ-fine* if $|I_j| < \delta(\tau_j)$ for $j \in \Gamma$.

Cousin's lemma. For any gauge δ there exists a δ-fine division of E.

Denote K the set of all pairs (f, γ), $f : E \to \mathbb{R}$, $\gamma \in \mathbb{R}$, for which to any $\varepsilon > 0$ there exists a gauge δ such that

$$|\sum_{j\in\Gamma} f(\tau_j)|I_j| - \gamma| < \varepsilon$$

for any δ-fine division $(\{I_j; j \in \Gamma\}, \tau)$ of E.

$\gamma \in \mathbb{R}$ is the *Kurzweil-Henstock integral* of f over E and $K(f) = \gamma = (K)\int_E f$.

If $f \in \mathrm{Dom}(K)$, then the K-primitive $F : E \to \mathbb{R}$ is continuous. $F[I] = K(f, I)$ for $I \in \mathrm{Sub}(E)$. We have $K \in \mathfrak{S}$.

It is known that

$$N \sqsubset K$$

and

$$R \sqsubset L \sqsubset K,\ K = P = D_*,$$

where N, R, L, P, D_* stands for the Newton, Riemann, Lebesgue, Perron and Denjoy-Perron integrals, respectively.

Let $F \in C(E)$ and $M \subset E$. The function F is called to be of *negligible variation on* M if for any $\varepsilon > 0$ there is a $\delta \in \Delta(E)$ such that

$$|\sum_{j\in\Gamma} F[I_j]| < \varepsilon$$

for any δ-fine M-tagged division $(\{I_j; j \in \Gamma\}, \tau)$.

The Kurzweil-Henstock integral has the following properties:

⊛ If $\mu(N) = 0$ then $f \cdot \chi(N) \in \mathrm{Dom}(K)$ and $K(f, N) = 0$.

⊛ If $f \in \mathrm{Dom}(K)$ then F is of negligible variation on $N \subset E$ with $\mu(N) = 0$.

⊛ If $f \in \mathrm{Dom}(K)$ then f is measurable.

⊛ If $f \in \mathrm{Dom}(K)$, F its K-primitive. Then

$$W_F(M) \le 2|E||f|_M = 2|E| \sup_{x\in M} |f(x)|$$

for $M \subset E$.

⊛ K is a linear functional, i.e. $K(\alpha f + \beta g) = \alpha K(f) + \beta K(g)$.

Variational measures. The *oscillation* of $F \in C(E)$ on an interval $I \in \mathrm{Sub}(E)$ is given by

$$\omega(F, I) = \sup\{|F(x) - F(y)|; x, y \in I\} = \sup\{|F[J]|; J \in \mathrm{Sub}(I)\}.$$

For $F \in C(E)$ and a division $D = \{I_j; j \in \Gamma\}$ define

$$\Omega(F, D) = \sum_{j\in\Gamma} \omega(F, I_j) \quad \text{and} \quad A(F, D) = \sum_{j\in\Gamma} |F[I_j]|.$$

If $F \in C(E)$ and $M \subset E$ then for any gauge δ set

$$W_\delta(F,M) = \sup\{\Omega(F,D);\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\},$$
$$V_\delta(F,M) = \sup\{A(F,D);\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}$$

and put

$$W_F(M) = \inf\{W_\delta(F,M);\ \delta \in \Delta(E)\} = \inf_\delta\{\sup_D\{\Omega(F,D)\}\},$$
$$V_F(M) = \inf\{V_\delta(F,M);\ \delta \in \Delta(E)\} = \inf_\delta\{\sup_D\{A(F,D)\}\}.$$

For $F \in C(E)$, $W_F(\cdot)$ and $V_F(\cdot)$ are set functions. Using the terms presented by **B.S.Thomson** (Mem. AMS, 1991) we identify $W_F(M)$ and $V_F(M)$ as *full variational measures* generated by the continuous interval functions given for $I \in \mathrm{Sub}(E)$ by $\omega(F,I)$, $F[I]$, respectively. ($W_F(\cdot)$ and $V_F(\cdot)$ are metric outer measures (B. Thomson).)

It is easy to show that

$$V_F(M) \le W_F(M) \le 2V_F(M)$$

and this means that F is of negligible variation on M if and only if $W_F(M) = 0$.

Denote by $C^*(E)$ the set of all continuous functions on E which are of negligible variation on sets of Lebesgue measure zero (the *strong Luzin condition*)

$$C^*(E) = \{F \in C(E);\ W_F(N) = 0 \text{ whenever } \mu(N) = 0\}.$$

A descriptive characterization of the Kurzweil-Henstock integral given by **B. Bongiorno, L. Di Piazza** and **V. Skvortsov**:

$F : E \to \mathbb{R}$ is a K-primitive function to some $f : E \to \mathbb{R}$ if and only if $F \in C^*(E)$.

This means that $F \in C^*(E) \iff F \in ACG_*$ (see the book of **S. Saks**).

The class $\mathfrak{T}$. $\mathfrak{T}$ denotes the set of integrals $S \in \mathfrak{S}$ fulfilling the following ($N, A \subset E$, $\mu(A)$ is the Lebesgue measure of a set A, f is a function on E and F is an S-primitive function to f):

1. If $\mu(N) = 0$ then $f \cdot \chi(N) \in \mathrm{Dom}(S)$ and $S(f,N) = 0$.
2. If $f \in \mathrm{Dom}(S)$ then $F \in C(E)^*$.
3. If $f \in \mathrm{Dom}(S)$ then f is measurable.
4. There exists $\lambda < \infty$ such that
$$W_F(A) \le \lambda |f|_A = \lambda \sup_{x\in A} |f(x)|$$
if $f \in \mathrm{Dom}(S)$ and A is a closed set.
5. S is a linear functional.

Looking at the properties above, marked by $\circledast$, we get for the Henstock-Kurzweil integral $K \in \mathfrak{T}$ and, of course, the same holds for any $S \in \mathfrak{S}$, $S \sqsubset K$.

Extension of integrals. A mapping $Q : \mathfrak{S} \to \mathfrak{S}$ defined on $\mathrm{Dom}(Q) \subset \mathfrak{S}$ is an *extension* if for every $S \in \mathrm{Dom}(Q)$ we have $S \sqsubset Q(S)$, $Q(S) \in \mathrm{Dom}(Q)$ and if $S_1, S_2 \in \mathrm{Dom}(Q) \subset \mathfrak{S}$ then $S_1 \sqsubset S_2 \implies Q(S_1) \sqsubset Q(S_2)$.

The extension Q is called *effective* if $Q^2 = Q$, i.e. $Q(Q(S)) = Q(S)$ for every $S \in \mathrm{Dom}(Q)$.

An integral S is called *invariant with respect to an extension* Q if $S \in \mathrm{Dom}(Q)$ and $Q(S) = S$.

$\mathfrak{R} \subset \mathfrak{S}$ is some set of integrals.

If $S \in \mathfrak{R}$ and $\mathcal{P}$ is a set of extensions with $\mathrm{Dom}(P) = \mathfrak{R}$ for $P \in \mathcal{P}$, then $\mathrm{Min}(\mathcal{P}; S)$ is the *minimal* $\mathcal{P}$*-invariant integral containing* S (if it exists), i.e. $T = \mathrm{Min}(\mathcal{P}; S)$ if and only if

(i) $S \sqsubset T$ and

(ii)$T \sqsubset R$ whenever $R \in \mathfrak{R}$ is such that $S \sqsubset R$ and $P(R) = R$ for all $P \in \mathcal{P}$.

Extension of integrals. If f is a function on E and $S \in \mathfrak{S}$, then $x \in E$ is called an *S-regular point of* f if there is a $I \in \mathrm{Sub}(E)$ such that $x \in \mathrm{Int}(I)$ and $f \cdot \chi(I) \in \mathrm{Dom}(S)$.

The set of all S-regular points of f is denoted by $\rho(f, S)$.

The complement $\sigma(f, S) = E \backslash \rho(f, S)$ of $\rho(f, S)$ in E is the set of *S-singular points of* f. The set $\sigma(f, S)$ is closed.

Cauchy extension. For $S \in \mathfrak{S}$ denote by S_C the set of all pairs (f, γ) for which $\sigma(f, S)$ is a finite set and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$ and for every interval $I \subset \rho(f, S)$ we have $f \cdot \chi(I) \in \mathrm{Dom}\ (S)$ and $F[I] = S(f, I)$.

The set $\{(S, S_C); S \in \mathfrak{S}, S_C \text{ exists }\}$ is denoted by P_C.

$S \in \mathfrak{S}$ is P_C-invariant ($P_C(S) = S_C \sqsubset S$) if and only if for S the following statement holds.

Hake's Theorem. If $I = [c, d] \in \mathrm{Sub}(E)$ then $f \cdot \chi(I) \in \mathrm{Dom}(S)$ if and only if for every $c < \alpha < \beta < d$ we have $f \cdot \chi_{[\alpha,\beta]} \in \mathrm{Dom}(S)$ and

$$\lim_{\alpha \to c+, \beta \to d-} S(f \cdot \chi_{[\alpha,\beta]}) = A \in \mathbb{R}.$$

In this case $S(f, I) = A$.

Harnack extension. For $S \in \mathfrak{S}$ denote by S_H the set of all pairs (f, γ) for which there is a closed set $Q \subset E$ such that $f \cdot \chi(Q) \in \mathrm{Dom}(S)$ and $f \cdot \chi(U_j) \in \mathrm{Dom}(S)$ for $j \in \Gamma$ where $\{U_j; j \in \Gamma\} = \mathrm{Comp}(E, Q)$ and for which there is a function $F \in C(E)$ such that $\gamma = F[E] = F(b) - F(a)$,

$$\sum_{U \in \mathrm{Comp}(E,Q)} \omega(F, \overline{U}) < \infty$$

and

$$F[I] = S(f, I \cap Q) + \textstyle\sum_{j \in \Gamma} S(f, I \cap \overline{U_j})$$

for any $I \in \mathrm{Sub}(E)$.

The set $\{(S, S_H); S \in \mathfrak{S},\ S_H \text{ exists }\}$ is denoted by P_H.

Using the tools given before it can be shown that

$K = D_* = \mathrm{Min}(\{P_C, P_H\}; L)$, L is the Lebesgue integral.

Extensions Q_X ***and*** Q_Z**.** For $S \in \mathfrak{T}$ denote by S_X the set of all (f, γ) for which there exist $F \in C(E)^*$, measurable sets $N_1, N_2 \subset E$ with $\mu(N_1) = \mu(N_2) = 0$, a sequence (f_j) in $\mathrm{Dom}(S)$, $j \in \mathbb{N}$ and a sequence (M_k) of measurable subsets of E such that $\gamma = F[E]$ and

$$f(x) = \lim_{j\to\infty} f_j(x) \text{ for } x \in E \setminus N_1,$$
$$M_k \nearrow E \setminus N_2,$$
$$\text{if } k \in \mathbb{N} \text{ then } W_{F-F_j}(M_k) \to 0 \text{ for } j \to \infty,$$
$$F_j \text{ is an } S\text{-primitive to } f_j.$$

The set $\{(S, S_X); S \in \mathfrak{T}, S_X \text{ exists }\}$ is denoted by Q_X.

$Q_X : \mathfrak{T} \to \mathfrak{T}$ is an effective extension.

If $S \in \mathfrak{T}$ then S_Z denotes the set of all pairs (f, γ) for which there exists a function $F \in C(E)^*$ and a sequence (A_k) of closed subsets of E such that $\gamma = F[E]$ and

$$A_k \nearrow E,$$
$$f_j = f \cdot \chi(A_j) \in \mathrm{Dom}(S) \text{ for } j \in \mathbb{N},$$
$$W_{F-F_j}(A_k) = 0 \text{ for } j \geq k,$$

$$\text{if } k \in \mathbb{N} \text{ then } \sum_{U \in \mathrm{Comp}(E, A_k} \omega(F - F_j, \overline{U}) \to 0 \text{ for } j \to \infty$$

where F_j is an S-primitive function to f_j.

The set $\{(S, S_Z); S \in \mathfrak{T}, S_Z \text{ exists }\}$ is denoted by Q_Z.

$Q_Z : \mathfrak{T} \to \mathfrak{T}$ is an extension and $Q_Z(S) \sqsubset Q_X(S)$.
$Q_Z(S)$ is P_C- and P_H-invariant for $S \in \mathfrak{T}$.

We have

$$Q_Z(L) = Q_X(L) = K.$$

This result makes it possible to describe the Kurzweil-Henstock integral via sequences of Lebesgue integrable functions.
The extensions Q_X and Q_Z are close to the method of describing the Perron-Denjoy (=Kurzweil-Henstock) integral by successive approximation given by **S. Nakanishi** (*Math. Japonica* 41 (1995)). In fact S. Nakanishi presents four slightly different extensions of this type. The examples given there concern mainly extensions of L which lead to the Kurzweil-Henstock integral.

There is another extension of L given by **B. Bongiorno, L. DiPiazza** and **D. Preiss**, the so called C-integral, which is strictly between L and K and is taylored to integrate derivatives. Other possibilities of integrals between L and K presents **J. Kurzweil** in his 2002 book.

The way how to describe formally extensions leading to such intermediate integrals, using the abstract concepts presented, is at this moment open.