Henstock-Kurzweil and McShane product integration; Descriptive definitions

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Let an interval $[a,b] \subset \mathbb{R}$, $-\infty < a < b < +\infty$ be given. A pair (τ, J) of a point $\tau \in [a,b]$ and a compact interval $J \subset [a,b]$ is called a *tagged interval*, where τ is the *tag* of J.

A finite collection $\{(\tau_j, J_j),\ j = 1, \dots, k\}$ of tagged intervals is called an *M-system* if

$$Int(J_i) \cap Int(J_j) = \emptyset \text{ for } i \neq j$$

(where $Int(J)$ denotes the interior of interval J). *M-partition* is an M-system which moreover satisfies

$$\bigcup_{j=1}^{k} J_j = [a,b].$$

An M-system (M-partition) $\{(\tau_j, J_j),\ j = 1, \dots, k\}$ for which

$$\tau_j \in J_j, \quad j = 1, \dots, k$$

is called a *K-system (K-partition)* on $[a,b]$. Clearly every K-system is also an M-system.

In the sequel we assume that every system of tagged intervals $\{(\tau_i, J_i)\}_{i=1}^{k}$ is ordered in such a way that

$$\sup J_i \leq \inf J_{i+1}, \quad i = 1, \dots, k-1.$$

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In other words, the notation $\{(\tau_i, [\alpha_{i-1}, \alpha_i])\}_{i=1}^k$ implies

$$a \le \alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le b.$$

Given a positive function $\delta : [a, b] \to (0, +\infty)$ called a *gauge* on $[a, b]$, a tagged interval (τ, J) is said to be *δ-fine* if

$$J \subset [\tau - \delta(\tau), \tau + \delta(\tau)].$$

Using this concept we can speak about *δ-fine systems* and *δ-fine partitions* $\{(\tau_j, J_j),\ j = 1, \dots, k\}$ of the interval $[a, b]$ whenever (τ_j, J_j) is δ-fine for every $j = 1, \dots, k$.

It is a well-known fact that given a gauge $\delta : [a, b] \to (0, +\infty)$ there exists a δ-fine K-partition of $[a, b]$. This result is called *Cousin's lemma,* see e.g. [11, Theorem on p. 119].

Assume that Y is a real Banach space with the norm $\|\cdot\|_Y$. Let us consider a function $f : [a, b] \to Y$ and assume that μ is the Lebesgue measure on the real line.

Definition 1. Assume that $f : [a, b] \to Y$ is given. The function f is called *McShane integrable* if there is an element $M_f \in Y$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^k f(t_i)\mu(J_i) - M_f \right\|_Y < \varepsilon$$

for every δ-fine M-partition $\{(t_i, J_i),\ i = 1, \dots, k\}$ of $[a, b]$.

The operator M_f is called *the McShane integral* of f over $[a, b]$ and we use the notation $M_f = (M) \int_a^b f$.

Definition 2. Assume that $f : [a, b] \to Y$ is given. The function f is called *Henstock-Kurzweil integrable* if there is an element $K_f \in Y$ such that for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^k f(t_i)\mu(J_i) - K_f \right\|_Y < \varepsilon$$

for every δ-fine K-partition $\{(t_i, J_i),\ i = 1, \dots, k\}$ of $[a, b]$.

The operator K_f is called *the Henstock-Kurzweil integral* of f over $[a, b]$ and we use the notation $K_f = (HK) \int_a^b f$.

1 Henstock-Kurzweil and McShane product integrals

Assume now that X is a real Banach space. Denote by $L(X)$ the Banach space of bounded linear operators on X with the usual operator norm given by

$$\|A\| = \|A\|_{L(X)} = \sup_{\|x\|=1} \|Ax\|_X$$

for $A \in L(X)$. By I the identity operator in $L(X)$ will be denoted.

Let $\mathfrak{J}$ be the set of all compact subintervals in $[a,b]$. Assume that a point-interval function $V : [a,b] \times \mathfrak{J} \to L(X)$ is given.

For a given M-partition $D = \{(t_i, J_i),\, i = 1, \dots, k\}$ of $[a,b]$ define

$$P(V,D) = \prod_{i=k}^{1} V(t_i, J_i) = V(t_k, J_k)V(t_{k-1}, J_{k-1}) \dots V(t_1, J_1).$$

Definition 3. A function $V : [a,b] \times \mathfrak{J} \to L(X)$ is called *McShane product integrable over* $[a,b]$ if there exists $Q \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta : [a,b] \to (0, +\infty)$ on $[a,b]$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine M-partition $D = \{(t_i, J_i),\, i = 1, \dots, k\}$ of $[a,b]$.

The operator Q is called *the McShane product integral* of V over $[a,b]$ and we use the notation $Q = (M) \prod_a^b V(t, dt)$.

Definition 4. A function $V : [a,b] \times \mathfrak{J} \to L(X)$ is called *Henstock-Kurzweil product integrable over* $[a,b]$ if there exists $Q \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta : [a,b] \to (0, +\infty)$ on $[a,b]$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine K-partition $D = \{(t_i, J_i),\, i = 1, \dots, k\}$ of $[a,b]$.

The operator Q is called *the Henstock-Kurzweil product integral* of V over $[a,b]$ and we use the notation $Q = (HK) \prod_a^b V(t, dt)$.

Remark 5. A similar concept of product integration was introduced by J. Jarník and J. Kurzweil in [6] (see also [9]) for the case of $n \times n$-matrix valued point-interval functions V with K-partitions. The corresponding product integral was called the *Perron product integral* in [6]. This terminology originates in the well known fact that a real function $g : [a, b] \to \mathbb{R}$ is Perron integrable to the value $\int_a^b g(t)\,dt \in \mathbb{R}$ if and only if to every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\left| \sum_{i=1}^{k} g(t_i)\mu(J_i) - \int_a^b g(t)\,dt \right| < \varepsilon$$

for every δ-fine K-partition $D = \{(t_i, J_i),\ i = 1, \dots, k\}$ of $[a, b]$.

Since evidently every δ-fine K-partition is also a δ-fine M-partition we obtain the following statement.

Proposition 6. *If $V : [a, b] \times \mathfrak{J} \to L(X)$ is McShane product integrable then V is also Henstock-Kurzweil product integrable and*

$$(HK) \prod_a^b V(t, dt) = (M) \prod_a^b V(t, dt).$$

Let us mention that a similar statement holds also for the integrals based on integral sums presented in Definitions 1 and 2.

Proposition 7. *Let $V : [a, b] \times \mathfrak{J} \to L(X)$ be given. Then V is McShane (Henstock-Kurzweil) product integrable if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that*

$$\|P(V, D_1) - P(V, D_2)\| < \varepsilon \tag{1}$$

for every δ-fine M-partitions (K-partitions) D_1, D_2 of $[a, b]$.

Proof. If V is McShane product integrable then the condition (1) is clearly satisfied.

Assume that (1) holds. Let $\delta_n : [a, b] \to (0, +\infty)$ be the gauge on $[a, b]$ which corresponds to $\varepsilon = 1/n$ ($n \in \mathbb{N}$) by (1). Denote

$$P_n = \{P(V, D) \in L(X); D \text{ is a } \delta_n\text{-fine } M\text{-partition}\}.$$

Clearly $P_{n+1} \subset P_n$ for every n and by (1) also

$$\operatorname{diam} P_n = \sup\{\|A - B\|; A, B \in P_n\} < \frac{1}{n}.$$

Since the space $L(X)$ is complete, the intersection $\bigcap_{n=1}^{\infty} \overline{P_n}$ consists of exactly one point $Q \in L(X)$ ($\overline{P_n}$ is the closure of the set P_n in $L(X)$) and

$$\|P(V,D) - Q\| \le \frac{1}{n}$$

for every δ_n-fine M-partition D of $[a,b]$. This proves the statement. $\square$

Let us introduce the following condition concerning the point-interval function $V : [a,b] \times \mathfrak{J} \to L(X)$.

Condition (C). *For every $t \in [a,b]$ and $\zeta > 0$ there exists $\sigma = \sigma(t) > 0$ such that*

$$\|V(t,J) - I\| < \zeta$$

for any interval $J \subset [a,b]$ such that $J \subset (t-\sigma, t+\sigma)$.

Typical cases of V satisfying condition **(C)** are for example

$$V_1(t,J) = I + A(t)\mu(J)$$

or

$$V_2(t,J) = e^{A(t)\mu(J)}$$

where $A : [a,b] \to L(X)$, μ being the Lebesgue measure on the real line. We denote the corresponding product integrals $\prod_a^b (I + A(t)\,dt)$ and $\prod_a^b e^{A(t)\,dt}$, respectively.

The following result was proved in [8] for McShane product integral and in [6] for Henstock-Kurzweil product integral (in the case $X = \mathbb{R}^n$).

Theorem 8. *Let $V : [a,b] \times \mathfrak{J} \to L(X)$ be McShane (Henstock-Kurzweil) product integrable over $[a,b]$ with $\prod_a^b V(t,dt) = Q \in L(X)$ where Q is an invertible operator. Assume that V satisfies condition* **(C)**.

Then for every $s \in [a,b]$ the McShane (Henstock-Kurzweil) product integrals

$$\prod_a^s V(t,dt),\ \prod_s^b V(t,dt)$$

exist, the equality

$$\prod_s^b V(t,dt) \prod_a^s V(t,dt) = \prod_a^b V(t,dt)$$

holds and there exists a constant $K > 0$ such that

$$\|\prod_a^s V(t, dt)\| \leq K, \;\; \|(\prod_a^s V(t, dt))^{-1}\| \leq K$$

for $s \in [a, b]$.

2 The indefinite product integral

Let us start with the following lemma (see [6] and [9]).

Lemma 9. *Let $A_1, A_2, \dots, A_k \in L(X)$ with $\sum_{i=1}^k \|A_i\| \leq 1$. Put*

$$B = (I + A_k)(I + A_{k-1}) \dots (I + A_1) - I$$

and

$$C = B - \sum_{i=1}^k A_i.$$

Then

$$\|B\| \leq 2 \sum_{i=1}^k \|A_i\|$$

and

$$\|C\| \leq (\sum_{i=1}^k \|A_i\|)^2.$$

Proof. Put $\lambda_i = \|A_i\|$ for $i = 1, \dots, k$ and $\lambda = \sum_{i=1}^k \lambda_i \leq 1$. Then

$$(1 + \lambda_k)(1 + \lambda_{k-1}) \dots (1 + \lambda_1) =$$

$$= 1 + \sum_{i=1}^k \lambda_i + \sum_{j_2 > j_1} \lambda_{j_2}\lambda_{j_1} + \sum_{j_3 > j_2 > j_1} \lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \dots + \lambda_k \lambda_{k-1} \dots \lambda_1 \leq$$

$$\leq e^{\lambda_k} e^{\lambda_{k-1}} \dots e^{\lambda_1} = e^{\lambda}.$$

Hence

$$\sum_{i=1}^k \lambda_i + \sum_{j_2 > j_1} \lambda_{j_2}\lambda_{j_1} + \sum_{j_3 > j_2 > j_1} \lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \dots + \lambda_k \lambda_{k-1} \dots \lambda_1 \leq$$

$$\leq e^{\lambda} - 1 < 2\lambda \tag{2}$$

and

$$\sum_{j_2>j_1} \lambda_{j_2}\lambda_{j_1} + \sum_{j_3>j_2>j_1} \lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \cdots + \lambda_k\lambda_{k-1}\ldots\lambda_1$$
$$\leq e^{\lambda} - 1 - \lambda \leq \lambda^2 \tag{3}$$

because $\lambda \leq 1$. Now

$$B = (I + A_k)(I + A_{k-1})\ldots(I + A_1) - I =$$

$$= \sum_{i=1}^{k} A_i + \sum_{j_2>j_1} A_{j_2}A_{j_1} + \sum_{j_3>j_2>j_1} A_{j_3}A_{j_2}A_{j_1} + \cdots + A_kA_{k-1}\ldots A_1$$

and

$$C = B - \sum_{i=1}^{k} A_i = \sum_{j_2>j_1} A_{j_2}A_{j_1} + \sum_{j_3>j_2>j_1} A_{j_3}A_{j_2}A_{j_1} +$$
$$+ \cdots + A_kA_{k-1}\ldots A_1.$$

Therefore by (2) we get

$$\|B\| \leq \sum_{i=1}^{k} \|A_i\| + \sum_{j_2>j_1} \|A_{j_2}\|\|A_{j_1}\| + \sum_{j_3>j_2>j_1} \|A_{j_3}\|\|A_{j_2}\|\|A_{j_1}\| +$$
$$+ \cdots + \|A_k\|\|A_{k-1}\|\ldots\|A_1\| =$$
$$= \sum_{i=1}^{k} \lambda_i + \sum_{j_2>j_1} \lambda_{j_2}\lambda_{j_1} + \sum_{j_3>j_2>j_1} \lambda_{j_3}\lambda_{j_2}\lambda_{j_1} +$$
$$+ \cdots + \lambda_k\lambda_{k-1}\ldots\lambda_1 < 2\lambda = 2\sum_{i=1}^{k} \|A_i\|$$

and similarly by (3) we obtain

$$\|C\| \leq \sum_{j_2>j_1} \|A_{j_2}\|\|A_{j_1}\| + \sum_{j_3>j_2>j_1} \|A_{j_3}\|\|A_{j_2}\|\|A_{j_1}\| +$$
$$+ \cdots + \|A_k\|\|A_{k-1}\|\ldots\|A_1\| < \lambda^2 = (\sum_{i=1}^{k} \|A_i\|)^2.$$

$\square$

Assume that $V : [a,b] \times \mathfrak{J} \to L(X)$ satisfies condition **(C)** and that $(M)\prod_a^b V(t,dt) \in L(X)$ $((HK)\prod_a^b V(t,dt) \in L(X))$ is an invertible operator. Denote

$$U_M(s) = (M)\prod_a^s V(t,dt) \in L(X),\ s \in (a,b],\ U_M(a) = I \tag{4}$$

and

$$U_{HK}(s) = (HK)\prod_a^s V(t,dt) \in L(X),\ s \in (a,b],\ U_{HK}(a) = I \tag{5}$$

the "indefinite" product integrals of V defined for $s \in [a,b]$. By Theorem 8 the above notions make sense.

First let us note that

$$(M)\prod_\alpha^\beta V(t,dt) = U_M(\beta)U_M^{-1}(\alpha),$$

$$(HK)\prod_\alpha^\beta V(t,dt) = U_{HK}(\beta)U_{HK}^{-1}(\alpha).$$

Secondly, let us mention that by Proposition 6 we have $U_M(s) = U_{HK}(s)$ if U_M is defined by (4) and the properties of U_{HK} hold also for U_M.

Theorem 10. *Let $V : [a,b] \times \mathfrak{J} \to L(X)$ be McShane (Henstock-Kurzweil) product integrable over $[a,b]$ with $\prod_a^b V(t,dt) = Q$ where $Q \in L(X)$ is an invertible operator. Assume that V satisfies the condition* **(C)***.*

For $\varepsilon > 0$ find a gauge $\delta : [a,b] \to (0,+\infty)$ on $[a,b]$ such that

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine M-partition (K-partition) D of $[a,b]$. Let $\{\tau_j, [\xi_j,\eta_j]\}_{j=1}^r$ be a δ-fine M-system (K-system). Define

$$U^{-1}(\eta_j)V(\tau_j,[\xi_j,\eta_j])U(\xi_j) = I + Z_j,\ j = 1,\ldots,r.$$

Then

$$\|(I+Z_r)(I+Z_{r-1})\ldots(I+Z_1) - I\| \le \|Q^{-1}\|\varepsilon. \tag{6}$$

Proof. Denote $\eta_0 = a$ and $\xi_{r+1} = b$. Since the product integral exists over all intervals of the form $[\eta_j, \xi_{j+1}]$, $j = 0, \dots, r$, for any $\omega > 0$ there exist gauges δ_j on $[\eta_j, \xi_{j+1}]$ such that $\delta_j(t) < \delta(t)$ and

$$\|P(V, \Delta_j) - \prod_{\eta_j}^{\xi_{j+1}} V(t, dt)\| = \|P(V, \Delta_j) - U(\xi_{j+1})U^{-1}(\eta_j)\| < \omega \tag{7}$$

for every δ_j-fine M-partition (K-partition) Δ_j of $[\eta_j, \xi_{j+1}]$.

Composing the partitions, we obtain that

$$D = \Delta_0 \circ (\tau_1, [\xi_1, \eta_1]) \circ \Delta_1 \circ \dots \Delta_{r-1} \circ (\tau_r, [\xi_r, \eta_r]) \circ \Delta_r$$

is a δ-fine M-partition (K-partition) of the interval $[a, b]$ and therefore

$$\begin{gathered} \|P(V, D) - Q\| \\ = \|Q - P(V, \Delta_r)V(\tau_r, [\xi_r, \eta_r])P(V, \Delta_{r-1}) \dots \\ P(V, \Delta_1)V(\tau_1, [\xi_1, \eta_1])P(V, \Delta_0)\| < \varepsilon. \end{gathered}$$

This yields

$$\begin{gathered} \|I - Q^{-1}P(V, \Delta_r)V(\tau_r, [\xi_r, \eta_r])P(V, \Delta_{r-1}) \dots \\ P(V, \Delta_1)V(\tau_1, [\xi_1, \eta_1])P(V, \Delta_0)\| \\ = \|Q^{-1}Q(I - Q^{-1}P(V, \Delta_r)V(\tau_r, [\xi_r, \eta_r])P(V, \Delta_{r-1}) \dots \\ P(V, \Delta_1)V(\tau_1, [\xi_1, \eta_1])P(V, \Delta_0))\| \\ \le \|Q^{-1}\|\|Q - P(V, \Delta_r)V(\tau_r, [\xi_r, \eta_r])P(V, \Delta_{r-1}) \dots \\ P(V, \Delta_1)V(\tau_1, [\xi_1, \eta_1])P(V, \Delta_0)\| < \|Q^{-1}\|\varepsilon. \end{gathered} \tag{8}$$

The inequality (8) can be written in the form

$$\begin{gathered} \|I - U(b)^{-1}P(V, \Delta_r)U(\eta_r)U^{-1}(\eta_r)V(\tau_r, [\xi_r, \eta_r]) \\ U(\xi_r)U^{-1}(\xi_r)P(V, \Delta_{r-1})U(\eta_{r-1})U^{-1}(\eta_{r-1}) \dots \\ U(\xi_2)U^{-1}(\xi_2)P(V, \Delta_1)U(\eta_1)U^{-1}(\eta_1)V(\tau_1, [\xi_1, \eta_1]) \\ U(\xi_1)U^{-1}(\xi_1)P(V, \Delta_0)\| < \|Q^{-1}\|\varepsilon. \end{gathered} \tag{9}$$

Now we take

$$U^{-1}(\xi_{j+1})P(V, \Delta_j)U(\eta_j) - I = W_j$$

for $j = 0, 1, \ldots, r$. Then using (7) we have

$$\|W_j\| = \|U^{-1}(\xi_{j+1})P(V, \Delta_j)U(\eta_j) - I\|$$

$$\leq \|U^{-1}(\xi_{j+1})\|\|P(V, \Delta_j) - U(\xi_{j+1})U^{-1}(\eta_j)\|\|U(\eta_j)\|$$

$$\leq \|U^{-1}(\xi_{j+1})\|\|U(\eta_j)\|\omega$$

and by Theorem 8 we obtain

$$\|W_j\| \leq K^2\omega \tag{10}$$

for $j = 0, 1, \ldots, r$.

Looking at the definition of Z_j (in the formulation of the Theorem) and at the definition of W_j we rewrite the inequality (9) as follows

$$\|I - (I + W_r)(I + Z_r)(I + W_{r-1})(I + Z_{r-1}) \ldots$$

$$(I + W_1)(I + Z_1)(I + W_0)\| \leq \|Q^{-1}\|\varepsilon.$$

Now we have

$$\|I - (I + Z_r)(I + Z_{r-1}) \ldots (I + Z_1)\|$$

$$\leq \|I - (I + W_r)(I + Z_r)(I + W_{r-1})(I + Z_{r-1}) \ldots$$

$$(I + W_1)(I + Z_1)(I + W_0)\|$$

$$+\|(I + W_r)(I + Z_r)(I + W_{r-1})(I + Z_{r-1}) \ldots (I + W_1)(I + Z_1)(I + W_0)$$

$$-(I + Z_r)(I + Z_{r-1}) \ldots (I + Z_1)\| \leq \|Q^{-1}\|\varepsilon$$

because (10) implies that

$$\|(I + W_r)(I + Z_r)(I + W_{r-1})(I + Z_{r-1}) \ldots (I + W_1)(I + Z_1)(I + W_0)$$

$$-(I + Z_r)(I + Z_{r-1}) \ldots (I + Z_1)\|$$

is arbitrarily small if $\omega > 0$ is small enough. □

Theorem 11. *Let $V : [a, b] \times \mathfrak{J} \to L(X)$ be McShane product integrable over $[a, b]$ with $(M)\prod_a^b V(t, dt) = Q \in L(X)$ where $Q \in L(X)$ is an invertible operator. Assume that V satisfies the condition* **(C)**.

Then the function $U_M : [a, b] \to L(X)$ given by (4) is continuous at every point $s \in [a, b]$.

The same statement holds if the McShane case is replaced by the Henstock-Kurzweil one.

Proof. We present the proof for the case of McShane product integral only; the proof for the Henstock-Kurzweil case is similar and was given in [6] for the case $X = \mathbb{R}^n$, i.e. for the case of $n \times n$ matrices.

Looking at the Definition 3 let for $\varepsilon > 0$ the gauge $\delta : [a, b] \to (0, +\infty)$ on $[a, b]$ be such that

$$\|P(V, D) - Q\| < \varepsilon$$

for every δ-fine M-partition $D = \{(t_i, J_i),\ i = 1, \dots, k\}$ of $[a, b]$.

By the condition (C) for every $s \in [a, b]$ and $\varepsilon > 0$ there exists $\sigma = \sigma(s) > 0$ such that

$$\|V(s, J) - I\| < \varepsilon$$

for any interval $J \subset [a, b]$ for which $J \subset (s - \sigma, s + \sigma)$.

Assume that $s \in [a, b)$ is given and let $t \in (s, b]$ satisfies $s < t < s + \delta_0(s)$ where $0 < \delta_0(s) < \min(\delta(s), \sigma(s))$.

Let D_1 be a δ-fine M-partition of $[a, s]$ and let us set

$$D_2 = D_1 \circ (s, [s, t]).$$

Then D_2 is evidently a δ-fine M-partition of $[a, t]$.

Assume that $D_1 = \{(t_i, [\alpha_{i-1}, \alpha_i])\}$, $i = 1, \dots, l$. Then $\alpha_0 = a$ and $\alpha_l = s$. We have

$$U^{-1}(s)P(V, D_1) = U^{-1}(\alpha_l)P(V, D_1)$$
$$= U^{-1}(\alpha_l)V(t_l, [\alpha_{l-1}, \alpha_l])V(t_{l-1}, [\alpha_{l-2}, \alpha_{l-1}]) \dots V(t_1, [\alpha_0, \alpha_1])$$
$$= U^{-1}(\alpha_l)V(t_l, [\alpha_{l-1}, \alpha_l])U(\alpha_{l-1})U^{-1}(\alpha_{l-1})V(t_{l-1}, [\alpha_{l-2}, \alpha_{l-1}])U(\alpha_{l-2})$$
$$U^{-1}(\alpha_{l-2}) \dots U(\alpha_1)U^{-1}(\alpha_1)V(t_1, [\alpha_0, \alpha_1])$$

and

$$U^{-1}(s)P(V, D_1) - I$$
$$= U^{-1}(\alpha_l)V(t_l, [\alpha_{l-1}, \alpha_l])U(\alpha_{l-1})U^{-1}(\alpha_{l-1})V(t_{l-1}, [\alpha_{l-2}, \alpha_{l-1}])U(\alpha_{l-2})$$
$$U^{-1}(\alpha_{l-2}) \dots U(\alpha_1)U^{-1}(\alpha_1)V(t_1, [\alpha_0, \alpha_1])U(\alpha_0) - I$$

because $U(\alpha_0) = U(a) = I$.

Denote

$$U^{-1}(\alpha_j)V(t_j, [\alpha_{j-1}, \alpha_j])U(\alpha_{j-1})$$

for $j = 1, \dots, l$. Then Theorem 10 and especially (6) implies

$$\|U^{-1}(s)P(V, D_1) - I\|$$

$$= \|(I+Z_l)(I+Z_{l-1})\dots(I+Z_1) - I\| \le \|Q^{-1}\|\varepsilon$$

and by Theorem 8 we get

$$\|P(V,D_1) - U(s)\| \|U(s)[U^{-1}(s)P(V,D_1) - I]\| \le$$
$$\|U(s)\| \|U^{-1}(s)P(V,D_1) - I\| \le K\|Q^{-1}\|\varepsilon.$$

In a fully analogous way we also get

$$\|P(V,D_2) - U(t)\| \le K\|Q^{-1}\|\varepsilon.$$

Now by the form of condition (C) from the beginning of the proof we have

$$\|U(t) - U(s)\| \le$$
$$\|P(V,D_2) - U(t)\| + \|P(V,D_1) - U(s)\| + \|P(V,D_2) - P(V,D_1)\|$$
$$\le 2K\|Q^{-1}\|\varepsilon + \|P(V,D_2) - P(V,D_1)\| =$$
$$2K\|Q^{-1}\|\varepsilon + \|V(s,[s,t])P(V,D_1) - P(V,D_1)\|$$
$$= 2K\|Q^{-1}\|\varepsilon + \|[V(s,[s,t]) - I]P(V,D_1)\| \le$$
$$2K\|Q^{-1}\|\varepsilon + \|V(s,[s,t]) - I\| \|P(V,D_1)\|$$
$$\le 2K\|Q^{-1}\|\varepsilon + K\varepsilon = K(2\|Q^{-1}\| + 1)\varepsilon$$

and this proves the continuity of U from the right at the point s. The left continuity of U at $s \in (a,b]$ can be shown analogously and the result is proved. □

3 Finite-dimensional case

At this point we switch to the case $X = \mathbb{R}^n$; the operators in $L(\mathbb{R}^n)$ are now represented by real $n \times n$ matrices.

For a matrix $A = (a_{i,j})_{i,j=1}^n$ define the special norm

$$\|A\|_\star = \max_{1\le i,j\le n} |a_{i,j}|. \tag{11}$$

Let us mention that all the norms on $L(\mathbb{R}^n)$ are equivalent. This means especially that if $\|\cdot\|$ is an arbitrary norm defined on the linear space of matrices, then there is a constant $L \ge 1$ such that

$$\frac{1}{L}\|A\|_\star \le \|A\| \le L\|A\|_\star.$$

The following important statement was presented in [6].

Lemma 12. *Let $0 < \theta < 1/9$. Assume that $Z_1, Z_2, \dots, Z_r \in L(\mathbb{R}^n)$ are such that for every p-tuple $\{j_1, \dots, j_p\} \subset \{1, 2, \dots, r\}$ with $j_1 < j_2 < \dots < j_p$ the inequality*

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \dots (I + Z_{j_1}) - I\|_\star \le \theta \tag{12}$$

holds. Then

$$\sum_{j=1}^{r} \|Z_j\|_\star \le 4n^2\theta \tag{13}$$

Proof. By (1) we have

$$\|Z_j\|_\star = \|(I + Z_j) - I\|_\star \le \theta \tag{14}$$

for $j = 1, \dots, r$.

If $Z_j = (z_{j;i,k})_{i,k=1,\dots,n}$, $j = 1, \dots, r$ denote $\varphi(j), \psi(j) \in \{1, \dots, n\}$ such that

$$\|Z_j\|_\star = \max_{i,k} |z_{j;i,k}| = |z_{j;\varphi(j),\psi(j)}|.$$

For $l, m \in \{1, 2, \dots, n\}$ set

$$J(l, m) = \{j \in \{1, \dots, r\};\ \varphi(j) = l, \psi(j) = m\}.$$

In case (13) is not valid we get that there is a couple $l, m \in \{1, 2, \dots, n\}$ such that

$$\sum_{j \in J(l,m)} \|Z_j\|_\star > 4\theta.$$

Put

$$J_+ = \{j \in J(l, m);\ z_{j;l,m} \ge 0\},\ J_- = J(l, m) \setminus J_+.$$

Then either

$$\sum_{j \in J_+} z_{j;l,m} > 2\theta$$

or

$$-\sum_{j \in J_-} z_{j;l,m} > 2\theta.$$

Assume e.g. that the first inequality occurs. By (14) we have $z_{j;l,m} = \|Z_j\|_\star \le \theta$ for $j \in J_+$ and therefore there is a subset $J_+^* \subset J_+$ such that

$$2\theta < \sum_{j \in J_+^*} z_{j;l,m} \le 3\theta. \tag{15}$$

Hence

$$2\theta < \| \sum_{j \in J_+^*} Z_j \|_* = \sum_{j \in J_+^*} \|Z_j\|_\bigstar = \sum_{j \in J_+^*} z_{j;l,m} \leq 3\theta \qquad (16)$$

and

$$\sum_{j \in J_+^*} \|Z_j\|_\bigstar < \frac{1}{3} < 1.$$

The matrices Z_j, $j \in J_+^*$ satisfy the assumptions of Lemma 9 and therefore

$$\| \prod_{j \in J_+^*} (I + Z_j) - I \|_\bigstar \leq 2 \sum_{j \in J_+^*} \|Z_j\|_\bigstar$$

and by (16)

$$\| \prod_{j \in J_+^*} (I + Z_j) - I - \sum_{j \in J_+^*} Z_j \|_\bigstar \leq (\sum_{j \in J_+^*} \|Z_j\|_\bigstar)^2 \leq 9\theta^2.$$

Hence by (12) we get

$$\| \sum_{j \in J_+^*} Z_j \|_\bigstar \leq \| \prod_{j \in J_+^*} (I + Z_j) - I - \sum_{j \in J_+^*} Z_j \|_\bigstar + \| \prod_{j \in J_+^*} (I + Z_j) - I \|_\bigstar \leq$$

$$\leq 9\theta^2 + \theta$$

and by (16) it is also

$$2\theta < \| \sum_{j \in J_+^*} Z_j \|_\bigstar \leq 9\theta^2 + \theta.$$

Therefore

$$\frac{1}{9} < \theta$$

and this contradicts the assumption. Hence (13) holds. □

At this moment it should be pointed out that an analog of the preceding Lemma 12 does not hold for infinite-dimensional Banach spaces. In [9], p. 389 it was shown by an example that even if the norm in (12) is arbitrarily small ($\leq \theta$) the inequality

$$\sum_{j=1}^{r} \|Z_j\| > M\theta$$

holds for arbitrarily large $M > 0$ when taking a sufficiently large number of factors in (1). The example from [9] concerns the infinite-dimensional Banach space $X = c_0$.

For this reason we restrict our considerations to the case $X = \mathbb{R}^n$ in the sequel. Using Lemma 12 we prove the next result.

Theorem 13. *Let $V : [a,b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ be McShane (Henstock-Kurzweil) product integrable with $\prod_a^b V(t,dt) = Q \in L(\mathbb{R}^n)$ where Q is a regular matrix. Assume that V satisfies condition* **(C)**.

Let $0 < \varepsilon < 1/(9 \cdot \|Q^{-1}\|_\star)$ and find a gauge $\delta : [a,b] \to (0,+\infty)$ on $[a,b]$ such that

$$\|P(V,D) - Q\|_\star < \varepsilon$$

for every δ-fine M-partition (K-partition) D of $[a,b]$. Let $\{\tau_j, [\xi_j, \eta_j]\}_{j=1}^r$ be a δ-fine M-system (K-system). Define

$$U^{-1}(\eta_j)V(\tau_j, [\xi_j, \eta_j])U(\xi_j) = I + Z_j, \ j = 1, \dots, r.$$

Then

$$\sum_{j=1}^r \|Z_j\|_\star \le 4n^2 \|Q^{-1}\|_\star \varepsilon \tag{17}$$

and

$$\sum_{j=1}^r \Big\| V(\tau_j, [\xi_j, \eta_j]) - \prod_{\xi_j}^{\eta_j} V(t,dt) \Big\|_\star \le 4K^2 n^2 \|Q^{-1}\|_\star \varepsilon \tag{18}$$

where K is the constant from Theorem 8.

Proof. By (6) from Theorem 10 we have the inequality

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \dots (I + Z_{j_1}) - I\|_\star \le \|Q^{-1}\|_\star \varepsilon < \frac{1}{9}$$

for every p-tuple $\{j_1, \dots, j_p\} \subset \{1, 2, \dots, r\}$ with $j_1 < j_2 < \dots < j_p$. Hence by Lemma 12 we obtain

$$\sum_{j=1}^r \|Z_j\|_\star \le 4n^2 \|Q^{-1}\|_\star \varepsilon$$

because all the assumptions of Lemma 12 are satisfied and (17) is proved.

To show (18) we take into account that for $j=1,\dots,r$ we have

$$V(\tau_j,[\xi_j,\eta_j]) - U(\eta_j)U^{-1}(\xi_j) = V(\tau_j,[\xi_j,\eta_j]) - \prod_{\xi_j}^{\eta_j} V(t,dt)$$

$$= U(\eta_j)Z_jU^{-1}(\xi_j)$$

because $U(\eta_j)U^{-1}(\xi_j) = \prod_{\xi_j}^{\eta_j} V(t,dt)$.

Hence

$$\|V(\tau_j,[\xi_j,\eta_j]) - \prod_{\xi_j}^{\eta_j} V(t,dt)\|_\star \le \|U(\eta_j)\|_\star \|Z_j\|_\star \|U^{-1}(\xi_j)\|_\star .$$

Therefore (17) and Theorem 8 imply (18). □

Theorem 14. *Let $V : [a,b]\times\mathfrak{J} \to L(\mathbb{R}^n)$ be Henstock-Kurzweil product integrable over $[a,b]$ with $(HK)\prod_a^b V(t,dt) = Q \in L(\mathbb{R}^n)$ where $Q \in L(\mathbb{R}^n)$ is an invertible operator (a regular $n\times n$ matrix). Assume that V satisfies condition* **(C)**.

Then there exists a set $E \subset [a,b]$, $\mu(E)=0$ such that for every $\varepsilon>0$, $t\in[a,b]\setminus E$ there is $\vartheta>0$ such that

$$\|V(t,[x,y]) - U_{HK}(y)U_{HK}^{-1}(x)\|_\star \le \varepsilon(y-x) \tag{19}$$

provided $t-\vartheta < x \le t \le y < t+\vartheta$, $x,y\in[a,b]$.

Proof. Assume that $T\subset[a,b]$ is the set of all $t\in[a,b]$ for which (19) holds for and set $E=[a,b]\setminus T$. For $t\in E$ the relation (19) is not satisfied.

Given $r\in\mathbb{N}$ denote by E_r the set of $t\in[a,b]$ such that there exist sequences $x_l = x_l(t)$, $y_l=y_l(t)$, $l\in\mathbb{N}$ with

$$x_l \le t \le y_l,\ y_l - x_l \to 0 \ \text{ as } \ l\to\infty$$

and

$$\|V(t,[x_l,y_l]) - U_{HK}(y_l)U_{HK}^{-1}(x_l)\|_\star \ge \frac{1}{r}(y_l-x_l). \tag{20}$$

Then $E=\bigcup_{r=1}^\infty E_r$.

Assume that $\mu_e(E)>0$, where $\mu_e(E)$ is the outer measure of the set $E\subset[a,b]$. Then there is an $r\in\mathbb{N}$ such that $\mu_e(E_r)>0$.

Let $\varepsilon > 0$ is such that $\varepsilon < \frac{1}{9}\|Q^{-1}\|_\star$ and

$$4K^2n^2\|Q^{-1}\|_\star\varepsilon < \frac{1}{2r}\mu_e(E_r) \tag{21}$$

($K > 0$ is the constant from Theorem 8) and according to Definition 4 find a gauge δ on $[a,b]$ such that

$$\|P(V,D) - Q\|_\star < \varepsilon$$

for every δ-fine K-partition D of $[a,b]$.

For $t \in E$ find $l_0 = l_0(t) \in \mathbb{N}$ such that

$$t - \delta(t) < x_l(t) \le t \le y_l(t) < t + \delta(t)$$

for all $l \ge l_0$. The system of intervals

$$\{[x_l(t), y_l(t)];\ t \in E, l \ge l_0(t)\}$$

is a Vitali cover of the set E and by the Vitali covering theorem it contains a finite subsystem of intervals

$$\{[\xi_j, \eta_j];\ j = 1, 2, \dots, s\}$$

for which

$$\tau_j - \delta(\tau_j) < \xi_j \le \tau_j \le \eta_j < \tau_j + \delta(\tau_j),\ \tau_j \in E,\ j = 1, 2, \dots, s,$$

$$\eta_j \le \xi_{j+1},\ j = 1, 2, \dots, s-1$$

and

$$\mu_e(E \setminus \bigcup_{j=1}^{s}[\xi_j, \eta_j]) < \frac{1}{2}\mu_e(E_r).$$

Hence

$$\sum_{j=1}^{s}(\eta_j - \xi_j) \ge \mu_e(E \cap \bigcup_{j=1}^{s}[\xi_j, \eta_j] > \frac{1}{2}\mu_e(E_r).$$

This inequality together with (20) and (21) yields

$$\sum_{j=1}^{s}\|V(\tau_j, [\xi_j, \eta_j]) - U_{HK}(\eta_j)U_{HK}^{-1}(\xi_j)\|_\star$$

$$= \sum_{j=1}^{s} \|V(\tau_j, [\xi_j, \eta_j]) - \prod_{\xi_j}^{\eta_j} V(t, dt)\|_\star$$

$$\geq \frac{1}{r} \sum_{j=1}^{s} (\eta_j - \xi_j) > \frac{1}{2r} \mu_e(E_r) \geq 4K^2 n^2 \|Q^{-1}\|_\star \varepsilon,$$

a contradiction to (18) from Theorem 13. Therefore $\mu_e(E_r) = 0$ for every $r \in \mathbb{N}$ and $\mu_e(E) = 0$ which yields $\mu(E) = 0$. □

Let us now turn our attention to the classical case when

$$V(t, J) = I + A(t)\mu(J) \tag{22}$$

with $A : [a, b] \to L(\mathbb{R}^n)$ and μ the Lebesgue measure on the real line. We consider the product integrals of the form $\prod_a^b (I + A(t)dt)$. As it was mentioned in Section 1 the function $V : [a, b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ given by (22) satisfies condition **(C)**.

The first thing is the following corollary of Theorem 14.

Corollary 15. *Assume that the Henstock-Kurzweil product integral* $(HK) \prod_a^b (I + A(t)dt) = Q \in L(\mathbb{R}^n)$ *exists and is invertible.*

Then for the indefinite product integral $U_{HK} : [a, b] \to L(\mathbb{R}^n)$ *given by*

$$\begin{aligned} U_{HK}(s) = (HK) \textstyle\prod_a^s (I + A(t)dt) \in L(\mathbb{R}^n),\ s \in (a, b], \\ U_{HK}(a) = I \end{aligned} \tag{23}$$

the derivative $\dot{U}_{HK}(t)$ *exists for almost all* $t \in [a, b]$ *and*

$$\dot{U}_{HK}(t) = A(t) U_{HK}(t) \tag{24}$$

for almost all $t \in [a, b]$.

Proof. Given an $\varepsilon > 0$, by Theorem 14 there exists a set $E \subset [a, b]$, $\mu(E) = 0$ such that for every $\varepsilon > 0$, $t \in [a, b] \setminus E$ there is $\vartheta > 0$ such that

$$\|I + A(t)(y - x) - U_{HK}(y) U_{HK}^{-1}(x)\|_\star \leq \varepsilon(y - x)$$

provided $t - \vartheta < x \leq t \leq y < t + \vartheta$, $x, y \in [a, b]$.

Take $t \notin E$. Then

$$\|I + A(t)(y - t) - U_{HK}(y) U_{HK}^{-1}(t)\|_\star \leq \varepsilon(y - t)$$

for $t < y < t + \vartheta$, $y \in [a, b]$. Hence

$$\left\| \frac{U_{HK}(t)U_{HK}^{-1}(t) - U_{HK}(y)U_{HK}^{-1}(t)}{y - t} + A(t) \right\|_{\star} \leq \varepsilon$$

and

$$\left\| \frac{U_{HK}(y) - U_{HK}(t)}{y - t} U_{HK}^{-1}(t) - A(t) \right\|_{\star} \leq \varepsilon$$

for $t < y < t + \vartheta$, $y \in [a, b]$. This means that

$$\left\| \frac{U_{HK}(y) - U_{HK}(t)}{y - t} - A(t)U_{HK}(t) \right\|_{\star} \leq \varepsilon \|U_{HK}(t)\|_{\star},$$

i.e. $\dot{U}_{HK}^{+}(t)$ (the derivative from the right of U_{HK} at the point t) exists and we have

$$\dot{U}_{HK}^{+}(t) = A(t)U_{HK}(t).$$

A similar relation for the derivative from the left leads to the conclusion that for $t \notin E$ the derivative $\dot{U}_{HK}(t)$ exists and

$$\dot{U}_{HK}(t) = A(t)U_{HK}(t).$$

This proves the corollary. □

Theorem 16. *Assume that the Henstock-Kurzweil product integral*

$$(HK) \prod_{a}^{b} (I + A(t)dt) = Q \in L(\mathbb{R}^n)$$

exists and is invertible.

Then $U_{HK} : [a, b] \to L(\mathbb{R}^n)$ from (23) satisfies the following condition:

(SL) *Let $\eta > 0$, $N \subset [a, b]$, $\mu(N) = 0$. Then there exists $\Delta : N \to (0, +\infty)$ such that if $\tau_j \in N$,*

$$\tau_j - \Delta(\tau_j) < \xi_j \leq \tau_j \leq \eta_j < \tau_j + \Delta(\tau_j)$$

for $j = 1, 2, \ldots, r$ and $\eta_j \leq \xi_{j+1}$ for $j = 1, 2, \ldots, r - 1$, then

$$\sum_{j=1}^{r} \|U_{HK}(\eta_j) - U_{HK}(\xi_j)\|_{\star} \leq \eta. \tag{25}$$

Proof. For $i \in \mathbb{N}$ put

$$N_i = \{t \in N;\ i - 1 \le \|A(t)\|_\star < i\}.$$

Then $N = \bigcup_i N_i$ and $N_i \cap N_j = \emptyset$ for $i \ne j$ while $\mu(N_i) = 0$ for every $i \in \mathbb{N}$. Therefore there exist functions $\delta_i : N_i \to (0, +\infty)$ such that

$$\mu(\bigcup_{t \in N_i} (t - \delta_i(t), t + \delta_i(t))) < \frac{\eta}{i 2^{i+1} K}, \tag{26}$$

where $K > 0$ is the constant from Theorem 8.

Assume that $\varepsilon > 0$ is so small that $0 < \varepsilon < \frac{1}{9\|Q^{-1}\|_\star}$ and

$$8K^3 n^2 \|Q^{-1}\|_\star \varepsilon < \eta.$$

Let $\Delta : [a, b] \to (0, +\infty)$ be a gauge on $[a, b]$ such that

$$\|P(V, D) - Q\|_\star < \varepsilon$$

holds by Definition 4 for every Δ-fine K-partition D of $[a, b]$. Without loss of generality we may assume that $\Delta(t) \le \delta_i(t)$ for $t \in N_i$, $i \in \mathbb{N}$. Using (18) from Theorem 13 we get

$$\sum_{j=1}^{r} \|I + A(\tau_j)(\eta_j - \xi_j) - U_{HK}(\eta_j) U_{HK}^{-1}(\xi_j)\|_\star \le 4K^2 n^2 \|Q^{-1}\|_\star \varepsilon.$$

By (26) and the choice of $\delta_i : N_i \to (0, +\infty)$ we have

$$\sum_{j=1}^{r} \|A(\tau_j)(\eta_j - \xi_j)\|_\star = \sum_{i=1}^{\infty} \sum_{\tau_j \in N_i} \|A(\tau_j)(\eta_j - \xi_j)\|_\star$$

$$\le \sum_{i=1}^{\infty} i \sum_{\tau_j \in N_i} (\eta_j - \xi_j) < \sum_{i=1}^{\infty} \frac{\eta}{2^{i+1} K} = \frac{\eta}{2K}.$$

Hence

$$\sum_{j=1}^{r} \|I - U_{HK}(\eta_j) U_{HK}^{-1}(\xi_j)\|_\star$$

$$\le \sum_{j=1}^{r} \|I + A(\tau_j)(\eta_j - \xi_j) - U_{HK}(\eta_j) U_{HK}^{-1}(\xi_j)\|_\star + \sum_{j=1}^{r} \| + A(\tau_j)(\eta_j - \xi_j)\|_\star$$

$$\leq 4K^2n^2\|Q^{-1}\|_\star\varepsilon + \frac{\eta}{2K} < \frac{\eta}{K}$$

because $\varepsilon < \dfrac{\eta}{8K^3n^2\|Q^{-1}\|_\star}$ by the assumption.

Since $U_{HK}(\xi_j) - U_{HK}(\eta_j) = U_{HK}(\xi_j)[I - U_{HK}(\eta_j)U_{HK}^{-1}(\xi_j)]$ and by Theorem 8 we have $\|U_{HK}(\xi_j)\|_\star \leq K$, we get

$$\sum_{j=1}^{r} \|U_{HK}(\xi_j) - U_{HK}(\eta_j)\|_\star \leq K \sum_{j=1}^{r} \|I - U_{HK}(\eta_j)U_{HK}^{-1}(\xi_j)\|_\star < \eta$$

and the statement is proved. □

Let us mention that the condition **(SL)** presented in Theorem 16 is the so called strong Lusin condition. By the results of Corollary 15 and by Theorem 16 we know that the indefinite product integral $U_{HK} : [a,b] \to L(\mathbb{R}^n)$ given by (23) possesses a derivative almost everywhere in $[a,b]$ and satisfies the strong Lusin condition on $[a,b]$. Since every McShane product integrable function is also Henstock-Kurzweil product integrable, the same can be stated also for the McShane indefinite product integral. For the McShane indefinite product integral we can state even more as it can be seen in the next theorem.

Theorem 17. *Assume that the McShane product integral*

$$(M)\prod_{a}^{b}(I + A(t)dt) = Q \in L(\mathbb{R}^n)$$

exists and is invertible.

Then the indefinite product integral $U_M : [a,b] \to L(\mathbb{R}^n)$ given by

$$\begin{gathered} U_M(s) = (M)\textstyle\prod_a^s(I + A(t)dt) \in L(\mathbb{R}^n),\ s \in (a,b], \\ U_M(a) = I \end{gathered} \tag{27}$$

satisfies the following condition:

(AC) *For every $\rho > 0$ there is a $\sigma > 0$ such that if $[\xi_j, \eta_j] \subset [a,b]$, $j = 1,\dots,r$ are non-overlapping intervals with $\sum_{j=1}^{r}(\eta_j - \xi_j) < \sigma$ then*

$$\sum_{j=1}^{r} \|U_M(\eta_j) - U_M(\xi_j)\|_\star < \rho.$$

Proof. Given a $\rho > 0$ take $\varepsilon > 0$ such that $\varepsilon < \dfrac{1}{9\|Q^{-1}\|_\bigstar}$ and

$$4K^3 n^2 \|Q^{-1}\|_\bigstar \varepsilon < \frac{\rho}{2},$$

where $K > 0$ is the constant from Theorem 8.

To this $\varepsilon > 0$ there is a gauge $\delta : [a,b] \to (0,+\infty)$ such that (by Definition 3) for $V(t,J) = A(t)\mu(J)$ we have

$$\|P(V,D) - Q\|_\bigstar < \varepsilon$$

provided

$$D = \{(t_i,[u_i,v_i]);\ i = 1,\dots,q\}$$

is a δ-fine M-partition of $[a,b]$.

Let us fix such an M-partition D and put

$$\sigma = \frac{\rho}{2K(\max_{i=1,\dots,q} \|A(t_i)\|_\bigstar + 1)}.$$

Assume that $[\xi_j,\eta_j] \subset [a,b]$, $j = 1,\dots,r$ are non-overlapping intervals with $\sum_{j=1}^r (\eta_j - \xi_j) < \sigma$ and consider the sum $\sum_{j=1}^r \|U_M(\eta_j) - U_M(\xi_j)\|_\bigstar$.

By subdividing the intervals $[\xi_j,\eta_j]$ if necessary, it can be assumed that every interval $[\xi_j,\eta_j]$ belongs to some interval $[u_i,v_i]$ of the fixed partition D. For each $i = 1,\dots,q$ let

$$M_i = \{j;\ 1 \le j \le r \ \text{ with } \ [\xi_j,\eta_j] \subset [u_i,v_i]\}$$

and take $\tau_j = t_i$ if $j \in M_i$. It is easy to check that for the points τ_j and the intervals $[\xi_j,\eta_j]$, $j = 1,\dots,r$ the assumption of Theorem 13 for the McShane case is satisfied if ordering ξ_j and η_j properly. Therefore we have (18) in our case.

Now we have

$$\begin{aligned}\|U_M(\xi_j) - U_M(\eta_j)\|_\bigstar &= \|[I - U_M(\eta_j)U_M^{-1}(\xi_j)]U_M(\xi_j)\|_\bigstar \\ &\le \|I + A(\tau_j)(\eta_j - \xi_j) - U_M(\eta_j)U_M^{-1}(\xi_j)\|_\bigstar \|U_M(\xi_j)\|_\bigstar \\ &\quad + \|I + A(\tau_j)(\eta_j - \xi_j)\|_\bigstar \|U_M(\xi_j)\|_\bigstar\end{aligned}$$

and by Theorem 8 we get

$$\sum_{j=1}^r \|U_M(\eta_j) - U_M(\xi_j)\|_\bigstar$$

$$\leq K \sum_{j=1}^{r} \|I + A(\tau_j)(\eta_j - \xi_j) - U_M(\eta_j)U_M^{-1}(\xi_j)\|_\bigstar$$
$$+K \max_{i=1,\dots,q} \|A(t_i)\|_\bigstar \sum_{j=1}^{r} (\eta_j - \xi_j).$$

Using (18) from Theorem 13 we obtain by the properties of ε and σ stated above the inequality

$$\sum_{j=1}^{r} \|U_M(\eta_j) - U_M(\xi_j)\|_\bigstar \leq 4K^3 n^2 \|Q^{-1}\|_\bigstar \varepsilon + K \max_{i=1,\dots,q} \|A(t_i)\|_\bigstar \sigma$$
$$< \frac{\rho}{2} + \frac{\rho}{2} = \rho$$

and the statement is proved. □

The condition **(AC)** in Theorem 17 says that the indefinite product integral $U_M : [a,b] \to L(\mathbb{R}^n)$ given by (27) is absolutely continuous in $[a,b]$.

The special norm $\|\cdot\|_\bigstar$ of matrices was used in the previous parts for technical reasons only. Note that according to (11) the proofs can be modified in a straightforward way for any norm of matrices.

4 Bochner product integral

Assume that $B : [a,b] \to L(X)$ is a step-function, i.e. that there is a finite system of points

$$a = s_0 < s_1 < \cdots < s_{m-1} < s_m = b$$

such that B is constant on each (s_{k-1}, s_k) with the value $B_k \in L(X)$, $k = 1, 2, \dots, m$.

The product integral of this step-function is equal to

$$E_B = e^{B_m(s_m - s_{m-1})} e^{B_{m-1}(s_{m-1} - s_{m-2})} \dots e^{B_1(s_1 - s_0)}.$$

Definition 18. A function $f : [a,b] \to Y$ is called Bochner integrable if there is a sequence of step functions $f_k : [a,b] \to Y$, $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} (L) \int_a^b \|f_k(x) - f(x)\| \, dx = 0,$$

where (L) denotes the Lebesgue integral.

In the monograph [1, p. 54] the following definition of product integral is given (in finite dimensional case).

Definition 19. Assume that $A : [a, b] \to L(X)$ is Bochner integrable. The Bochner product integral $(B) \prod_a^b e^{A(t)dt}$ is defined by

$$(B) \prod_a^b e^{A(t)dt} = \lim_{n \to \infty} E_{A_n} \tag{28}$$

where A_n, $n = 1, 2, \dots$ is any sequence of step-functions convergent to A in the L^1 sense, i.e.

$$\lim_{n \to \infty} (L) \int_a^b \|A_n(s) - A(s)\| ds = 0$$

and E_{A_n} is the product integral of the step-function A_n.

It is known that a function $f : [a, b] \to \mathbb{R}^n$ is Bochner integrable if and only if it is Lebesgue integrable. For $X = \mathbb{R}^n$ we have $L(X) = \mathbb{R}^{n \times n}$ and a function $A : [a, b] \to \mathbb{R}^{n \times n}$ is Bochner product integrable if and only if its components are Lebesgue integrable functions.

The following theorem was proved in [1]:

Theorem 20. *Let $A : [a, b] \to L(\mathbb{R}^n)$ be Bochner integrable. Then the Bochner product integral $(B) \prod_a^b e^{A(t)\, dt}$ is an invertible matrix.*

Definition 21. A function $f : [a, b] \to Y$ has the property $\mathcal{S}^*\mathcal{M}$ if for every $\varepsilon > 0$ there is a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\sum_{i=1}^k \sum_{j=1}^l \|f(t_i) - f(s_j)\| \mu(J_i \cap L_j) < \varepsilon$$

for any δ-fine M-partitions $\{(t_i, J_i)\}_{i=1}^k$ and $\{(s_j, L_j)\}_{j=1}^l$ of $[a, b]$.

Theorem 22. *Let Y be a finite dimensional Banach space, $f : [a, b] \to Y$. Then the following conditions are equivalent:*

1) f is Bochner integrable,
2) f is McShane integrable,
3) f has the property $\mathcal{S}^*\mathcal{M}$.

Proof. A function $f : [a,b] \to Y$ is Bochner integrable if and only if it has the property $\mathcal{S}^*\mathcal{M}$ (see Theorem 5.1.4 in [10]). Moreover, in a finite dimensional Banach space, a function is McShane integrable if and only if it has the property $\mathcal{S}^*\mathcal{M}$ (Proposition 5.2.1 in [10]). □

The following theorem was proved in [8]:

Theorem 23. *If $A : [a,b] \to L(X)$ has the property $\mathcal{S}^*\mathcal{M}$ then the Bochner product integral $(B)\prod_a^b e^{A(t)dt}$, the McShane product integral $(M)\prod_a^b e^{A(t)dt}$ and the McShane product integral $(M)\prod_a^b (I + A(t)dt)$ exist and*

$$(B)\prod_a^b e^{A(t)dt} = (M)\prod_a^b e^{A(t)dt} = (M)\prod_a^b (I + A(t)dt).$$

Be aware that the paper [8] uses a different terminology: Our McShane product integral is called Bochner product integral there, while our Bochner product integral is reffered to as Lebesgue-type product integral and is denoted by $(L)\prod_a^b e^{A(t)dt}$.

Corollary 24. *Let $A : [a,b] \to L(\mathbb{R}^n)$ be Bochner integrable. Then*

$$(B)\prod_a^b e^{A(t)dt} = (M)\prod_a^b e^{A(t)dt} = (M)\prod_a^b (I + A(t)dt)$$

(where the above product integrals are guaranteed to exist).

Theorem 25. *Let $A : [a,b] \to L(\mathbb{R}^n)$ be given. If there is an absolutely continuous function $U : [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$, then A is Bochner integrable.*

Proof. The function U^{-1} is measurable since the components u_{ij} $(i,j = 1,\dots,n)$ of U are measurable and

$$U^{-1}(s) = \left\{ \frac{(-1)^{i+j}\det U_{ji}(s)}{\det U(s)} \right\}_{i,j=1}^{n} \tag{29}$$

(where $U_{ji}(s)$ is the minor obtained from $U(s)$ by deleting j-th row and i-th column). Since U is continuous and invertible on $[a,b]$ we have

$$m := \min_{x\in[a,b]} |\det U(x)| > 0.$$

It is also possible to find a constant $M > 0$ such that $|u_{ij}(s)| \le M$ for every $s \in [a,b]$ and $i,j = 1,\ldots,n$. From (29) we get

$$\|U^{-1}\|_\star \le \frac{|\det U_{ji}(s)|}{|\det U(s)|} \le \frac{(n-1)!M^{n-1}}{m},$$

i.e. the function U^{-1} is bounded.

The components of U' are Lebesgue integrable (because U is absolutely continuous), U^{-1} is measurable and bounded. Therefore the components of $A(s) = U'(s)U^{-1}(s)$ are Lebesgue integrable and A is Bochner integrable. $\square$

The following theorem might be regarded as a descriptive definition of the McShane product integral.

Theorem 26. *Consider function $A : [a,b] \to L(\mathbb{R}^n)$. Then the McShane product integral $(M)\prod_a^b(I+A(t)\,dt)$ exists if and only if there is an absolutely continuous function $U : [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$; in this case*

$$(M)\prod_a^b(I + A(t)\,dt) = U(b)U^{-1}(a).$$

Proof. The first part of the theorem is easily proved combining the results from Corollary 15, Theorem 17, Theorem 25 and Corollary 24: The McShane product integral $(M)\prod_a^b(I+A(t)\,dt)$ exists if and only if there is an absolutely continuous function $U : [a,b] \to L(\mathbb{R}^n)$ such that $U^{-1}(s)$ exists for every $s \in [a,b]$ and $U'(s) = A(s)U(s)$ for almost all $s \in [a,b]$.

Now take an arbitrary function U which satisfies the conditions stated above. Define

$$V(s) = (M)\prod_a^s(I + A(t)\,dt)$$

and let $W(s) = U^{-1}(s)V(s)$ for $s \in [a,b]$. The functions U and V are absolutely continuous. Using again the formula

$$U^{-1}(s) = \left\{\frac{(-1)^{i+j}\det U_{ji}(s)}{\det U(s)}\right\}_{i,j=1}^n$$

we see that U^{-1} and consequently W are absolutely continuous functions. The equality $U'U^{-1} = V'V^{-1}$ almost everywhere implies

$$W' = (U^{-1}V)' = \left(U^{-1}\right)' V + U^{-1}V' = -U^{-1}U'U^{-1}V + U^{-1}V' =$$

$$= -U^{-1}U'U^{-1}V + U^{-1}V'V^{-1}V = U^{-1}(V'V^{-1} - U'U^{-1})V = 0$$

almost everywhere on $[a,b]$, i.e. W is a constant function. The proof is finished observing that

$$(M)\prod_a^b (I + A(t)\,dt) = V(b) = U(b)W(b) = U(b)W(a) = U(b)U^{-1}(a).$$

□

Theorem 27. *Consider function $A : [a,b] \to L(\mathbb{R}^n)$ such that the McShane product integral $(M)\prod_a^b(I + A(t)\,dt)$ exists and is invertible. Then A is also Bochner integrable and*

$$(M)\prod_a^b (I + A(t)\,dt) = (B)\prod_a^b e^{A(t)\,dt}.$$

Proof. Define

$$U(s) = (M)\prod_a^s (I + A(t)\,dt).$$

¿From Theorem 17 and Corollary 15 we know that U is absolutely continuous and $U'(s) = A(s)U(s)$ almost everywhere on $[a,b]$. According to Theorem 8 the matrix $U^{-1}(s)$ exists for every $s \in [a,b]$. To complete the proof apply Theorem 25. □

The following theorem describes the relation between McShane product integral and Bochner product integral.

Theorem 28. *For every $A : [a,b] \to L(\mathbb{R}^n)$ the following conditions are equivalent:*

1) A is Bochner integrable.
2) The McShane product integral $(M)\prod_a^b(I + A(t)\,dt)$ exists and is invertible.

If one of these conditions is fulfilled, then

$$(B)\prod_a^b (I + A(t)\,dt) = (M)\prod_a^b (I + A(t)\,dt).$$

Proof. An easy consequence of Theorem 24, Theorem 20 and Theorem 27. □

5 Examples

Example 29. We now demonstrate the existence of a function $A : [a, b] \to L(\mathbb{R}^n)$ such that the McShane product integral $(M) \prod_a^b e^{A(x)\,dx}$ is not invertible. According to Theorem 20 and Corollary 24 such a function cannot be Bochner integrable.

Define $f(x) = -1/x$ for $x \in (0, 1]$ and $f(0) = 0$. We will show that

$$(M) \prod_0^1 e^{f(x)\,dx} = 0.$$

Note that we have identified the real function $x \mapsto f(x)$ with a 1×1 matrix valued function $x \mapsto \{f(x)\}$. Choose arbitrary $N \in \mathbb{N}$ and define

$$\Delta(x) = \frac{1}{16} \cdot \frac{1}{2^N}, \quad x \in [0, 1].$$

This is a constant function and we can write Δ instead of $\Delta(x)$. Let

$$D = \{(\tau_j, [\alpha_{j-1}, \alpha_j]);\ j = 1, \dots, m\}$$

be a Δ-fine M-partition of $[0, 1]$, i.e.

$$\tau_j - \Delta < \alpha_{j-1} \le \alpha_j < \tau_j + \Delta$$

for $j = 1, \dots, m$. Since $\alpha_j - \alpha_{j-1} < 2 \cdot \Delta = 1/(8 \cdot 2^N)$, to every $i \in \{1, \dots, N\}$ we can find indices $j_1(i)$ and $j_2(i)$ such that

$$\alpha_{j_1(i)} \in \left(\frac{1}{2^i} + \frac{1}{8} \cdot \frac{1}{2^i}, \frac{1}{2^i} + \frac{2}{8} \cdot \frac{1}{2^i} \right],$$

$$\alpha_{j_2(i)} \in \left[\frac{1}{2^{i-1}} - \frac{2}{8} \cdot \frac{1}{2^i}, \frac{1}{2^{i-1}} - \frac{1}{8} \cdot \frac{1}{2^i} \right).$$

Consequently

$$1 < j_1(N) < j_2(N) < j_1(N-1) < j_2(N-1) < \cdots < j_1(1) < j_2(1) < m,$$

$$\alpha_{j_2(i)} - \alpha_{j_1(i)} \ge \frac{1}{2^{i-1}} - \frac{2}{8} \cdot \frac{1}{2^i} - \frac{1}{2^i} - \frac{2}{8} \cdot \frac{1}{2^i} = \frac{1}{2^{i+1}}$$

and for every $j \in \mathbb{N}$ such that $j_1(i) + 1 \leq j \leq j_2(i)$ we have

$$\tau_j > \alpha_{j-1} - \Delta \geq \alpha_{j_1(i)} - \Delta > \frac{1}{2^i} + \frac{1}{8} \cdot \frac{1}{2^i} - \frac{1}{16} \cdot \frac{1}{2^N} > \frac{1}{2^i},$$

$$\tau_j < \alpha_j + \Delta \leq \alpha_{j_2(i)} + \Delta < \frac{1}{2^{i-1}} - \frac{1}{8} \cdot \frac{1}{2^i} + \frac{1}{16} \cdot \frac{1}{2^N} < \frac{1}{2^{i-1}},$$

i.e.

$$2^{i-1} < \frac{1}{\tau_j} < 2^i.$$

Finally

$$-\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1}) \geq \sum_{i=1}^{N} \sum_{j=j_1(i)+1}^{j_2(i)} \frac{1}{\tau_j}(\alpha_j - \alpha_{j-1}) >$$

$$> \sum_{i=1}^{N} 2^{i-1}(\alpha_{j_2(i)} - \alpha_{j_1(i)}) \geq \sum_{i=1}^{N} 2^{i-1} \frac{1}{2^{i+1}} = \frac{N}{4}$$

and

$$0 < \prod_{j=1}^{m} e^{f(\tau_j)(\alpha_j - \alpha_{j-1})} = \exp\left(\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1})\right) \leq e^{-\frac{N}{4}}.$$

If we choose $N \in \mathbb{N}$ greater than $-4 \log \varepsilon$ we have

$$0 < \prod_{j=1}^{m} e^{f(\tau_j)(\alpha_j - \alpha_{j-1})} < \varepsilon$$

for every Δ-fine M-partition of $[0,1]$, which means that

$$(M) \prod_0^1 e^{f(x)\,dx} = 0.$$

Observe that because Δ was a constant function, the Riemann product integral exists as well and

$$(R) \prod_0^1 e^{f(x)\,dx} = 0.$$

Example 30. Define again $f(x) = -1/x$ for $x \in (0,1]$ and $f(0) = 0$. We will prove that

$$(M)\prod_0^1 (1 + f(x)\,dx) = 0.$$

This will confirm that the invertibility condition in the statement of Theorem 28 cannot be left out.

We have to show that to every $\varepsilon > 0$ there is a gauge $\Delta : [0,1] \to (0,\infty)$ such that

$$\left|\prod_{j=1}^{m}(1 + f(\tau_j)(\alpha_j - \alpha_{j-1}))\right| < \varepsilon$$

for every Δ-fine M-partition $D = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ of interval $[0,1]$.

The first condition that we impose on Δ is that $\Delta(x) < x/2$ for $x \in (0,1]$, which will guarantee that

$$1 + f(\tau_j)(\alpha_j - \alpha_{j-1}) > 0$$

for $j = 1, \ldots, m$. This is indeed true in case τ_j=0. Otherwise the inequality

$$\tau_j - \Delta(\tau_j) < \alpha_{j-1} \le \alpha_j < \tau_j + \Delta(\tau_j)$$

implies

$$1 + f(\tau_j)(\alpha_j - \alpha_{j-1}) = 1 - \frac{1}{\tau_j}(\alpha_j - \alpha_{j-1}) > 1 - \frac{2 \cdot \Delta(\tau_j)}{\tau_j} > 0.$$

The well-known inequality

$$x_1 \cdots x_m \le \left(\frac{x_1 + \cdots + x_m}{m}\right)^m$$

(which holds for non-negative numbers $x_1, \ldots, x_m$) yields the estimate

$$0 < \prod_{j=1}^{m}(1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) \le \left(\frac{\sum_{j=1}^m (1 + f(\tau_j)(\alpha_j - \alpha_{j-1}))}{m}\right)^m =$$

$$= \left(1 + \frac{\sum_{j=1}^m f(\tau_j)(\alpha_j - \alpha_{j-1})}{m}\right)^m.$$

If we now require

$$\Delta(x) < \frac{1}{16} \cdot \frac{1}{2^N}, \quad x \in [0,1],$$

where N is an arbitrary fixed natural number, we have (see Example 29)

$$\sum_{j=1}^{m} f(\tau_j)(\alpha_j - \alpha_{j-1}) < -N/4.$$

Since

$$\lim_{k\to\infty} \left(1 - \frac{N/4}{k}\right)^k = e^{-N/4},$$

there exists $k_0(N) \in \mathbb{N}$ such that

$$\left| \left(1 - \frac{N/4}{k}\right)^k - e^{-N/4} \right| < 1/N$$

for every $k \geq k_0(N)$. If $\Delta(x) < 1/(2 \cdot k_0(N))$, then every Δ-fine M-partition satisfies $\alpha_j - \alpha_{j-1} < 1/k_0(N)$ and therefore consists of $m \geq k_0(N)$ subintervals of $[0,1]$.

¿From these facts we conclude that

$$0 < \prod_{j=1}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) < \left(1 - \frac{N/4}{m}\right)^m < e^{-N/4} + 1/N.$$

It is now easy to complete the proof: Given $\varepsilon > 0$, the number N can be chosen to be greater than $\max(2/\varepsilon, -4\log(\varepsilon/2))$. The gauge $\Delta : [0,1] \to (0,\infty)$ is an arbitrary function such that

$$\Delta(x) < \min\left(\frac{x}{2}, \frac{1}{16} \cdot \frac{1}{2^N}, \frac{1}{2 \cdot k_0(N)}\right)$$

for $x \in (0,1]$ and

$$\Delta(0) < \min\left(\frac{1}{16} \cdot \frac{1}{2^N}, \frac{1}{2 \cdot k_0(N)}\right).$$

Then

$$0 < \prod_{j=1}^{m} (1 + f(\tau_j)(\alpha_j - \alpha_{j-1})) < \varepsilon$$

for every Δ-fine M-partition of $[0,1]$, which means that

$$(M)\prod_0^1(1+f(x)\,dx)=0.$$

It is perhaps interesting to note that the Riemann product integral

$$(R)\prod_0^1(1+f(x)\,dx)$$

does not exist. This follows from Masani's result (see [7]) that every Riemann product integrable function is bounded, but can be also easily verified directly: If the Riemann integral exists, it must be equal to the McShane integral which is zero. Therefore to every $\varepsilon>0$ there is a $\delta>0$ such that

$$\left|\prod_{j=1}^m(1+f(\tau_j)(\alpha_j-\alpha_{j-1}))\right|<\varepsilon$$

for every partition

$$0=\alpha_0\le\tau_1\le\alpha_1\le\cdots\le\tau_m\le\alpha_m=1$$

such that $\alpha_j-\alpha_{j-1}<\delta$, $j=1,\ldots,m$. Take such a partition which moreover satisfies $\alpha_1>0$,

$$1+f(\tau_j)(\alpha_j-\alpha_{j-1})\neq 0,\quad j=1,\ldots,m$$

(this can achieved by choosing $\tau_j\neq\alpha_j-\alpha_{j-1}$) and

$$0<\tau_1<\frac{\alpha_1}{\left|\prod_{j=2}^m(1+f(\tau_j)(\alpha_j-\alpha_{j-1}))\right|^{-1}+1}.$$

Then

$$|1+f(\tau_1)(\alpha_1-\alpha_0)|=\left|1-\frac{\alpha_1}{\tau_1}\right|=\frac{\alpha_1}{\tau_1}-1>\left|\prod_{j=2}^m(1+f(\tau_j)(\alpha_j-\alpha_{j-1}))\right|^{-1}$$

and therefore

$$\left|\prod_{j=1}^m(1+f(\tau_j)(\alpha_j-\alpha_{j-1}))\right|>1$$

which is a contradiction.

This example shows (together with Example 29) that the Riemann product integrals $\prod_a^b(I+A(x)\,dx)$ and $\prod_a^b e^{A(x)\,dx}$ do not always coincide.

6 Equivalent functions

Now we follow the procedure from [6] (presented for the case of Henstock-Kurzweil product integral) to show that the McShane product integrals $\prod_a^b (I + A(t)\,dt)$ and $\prod_a^b e^{A(t)\,dt}$ have the same value provided one of them exists and is invertible; in fact we prove a more general statement.

Theorem 31. *Consider function $V : [a,b] \times \mathfrak{J} \to L(\mathbb{R}^n)$. Assume there exists $K > 0$ and a function $W : [a,b] \to L(\mathbb{R}^n)$ which satisfy the following conditions:*

1) $\|W(t)\| \le K$ and $\|W(t)^{-1}\| \le K$ for every $t \in [a,b]$.
2) For every $\varepsilon > 0$ there is a gauge $\Delta : [a,b] \to (0,\infty)$ such that

$$\sum_{j=1}^{m} \|V(\tau_j, [\alpha_{j-1}, \alpha_j]) - W(\alpha_j)W(\alpha_{j-1})^{-1}\| < \varepsilon$$

for every Δ-fine M-partition $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ of $[a,b]$.

Then the function V is McShane product integrable and

$$(M)\prod_a^b V(t, dt) = W(b)W(a)^{-1}.$$

Proof. Choose $\varepsilon > 0$ and let $\Delta : [a,b] \to (0,\infty)$ be the corresponding gauge from the statement of the theorem. Take arbitrary Δ-fine M-partition $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ and define

$$A_j = W(\alpha_j)^{-1} V(\tau_j, [\alpha_{j-1}, \alpha_j]) W(\alpha_{j-1}) - I.$$

We calculate

$$\sum_{j=1}^{m} \|A_j\| \le$$

$$\le \sum_{j=1}^{m} \|W(\alpha_j)^{-1}\| \cdot \|V(\tau_j, [\alpha_{j-1}, \alpha_j]) - W(\alpha_j)W(\alpha_{j-1})^{-1}\| \cdot \|W(\alpha_{j-1})\| < K^2 \varepsilon.$$

Without loss of generality we can assume $\varepsilon \le 1/K^2$ and using Lemma 9 we obtain the estimate

$$\|(I + A_m) \cdots (I + A_1) - I\| \le \sum_{j=1}^{m} \|A_j\| + \left(\sum_{j=1}^{m} \|A_j\| \right)^2 < K^2 \varepsilon + K^4 \varepsilon^2.$$

Therefore

$$\left\| \prod_{j=m}^{1} V(\tau_j, [\alpha_{j-1}, \alpha_j]) - W(b)W(a)^{-1} \right\| =$$

$$= \|W(b) \left((I + A_m) \cdots (I + A_1) - I \right) W(a)^{-1}\| \leq K^2(K^2\varepsilon + K^4\varepsilon^2)$$

which implies

$$\prod_a^b V(t, dt) = W(b)W(a)^{-1}.$$

□

Definition 32. Functions $V_1, V_2 : [a, b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ are called equivalent if for every $\varepsilon > 0$ there is a gauge $\Delta : [a, b] \to (0, \infty)$ such that

$$\sum_{j=1}^{m} \|V_1(\tau_j, [\alpha_{j-1}, \alpha_j]) - V_2(\tau_j, [\alpha_{j-1}, \alpha_j])\| < \varepsilon$$

for every Δ-fine M-partition $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ of $[a, b]$.

Theorem 33. *Let $V_1, V_2 : [a, b] \times \mathfrak{J} \to L(\mathbb{R}^n)$ be equivalent functions. Assume that V_1 satisfies condition* **(C)** *and that the integral $(M) \prod_a^b V_1(t, dt)$ exists and is an invertible matrix. Then $(M) \prod_a^b V_2(t, dt)$ exists as well and both product integrals have the same value.*

Proof. Choose $\varepsilon > 0$ and let $\Delta : [a, b] \to (0, \infty)$ be a gauge such that

$$\left\| P(V_1, D) - (M) \prod_a^b V_1(t, dt) \right\| < \varepsilon$$

and

$$\sum_{j=1}^{m} \|V_1(\tau_j, [\alpha_{j-1}, \alpha_j]) - V_2(\tau_j, [\alpha_{j-1}, \alpha_j])\| < \varepsilon$$

for every Δ-fine M-partition $D = \{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ of $[a, b]$. Denote

$$U(s) = (M) \prod_a^s V_1(t, dt).$$

According to Theorem 13 there is a constant $C > 0$ such that

$$\sum_{j=1}^{m} \|V_1(\tau_j, [\alpha_{j-1}, \alpha_j]) - U(\alpha_j)U(\alpha_{j-1})^{-1}\| < C\varepsilon.$$

The triangle inequality yields

$$\sum_{j=1}^{m} \|V_2(\tau_j, [\alpha_{j-1}, \alpha_j]) - U(\alpha_j)U(\alpha_{j-1})^{-1}\| < C\varepsilon + \varepsilon$$

and using Theorem 31 we conclude

$$(M)\prod_a^b V_2(t, dt) = U(b)U(a)^{-1} = (M)\prod_a^b V_1(t, dt).$$

$\square$

Theorem 34. *Consider function* $A : [a, b] \to L(\mathbb{R}^n)$. *Then the following conditions are equivalent:*
1) $(M)\prod_a^b (I + A(t)\,dt)$ *exists and is invertible.*
2) $(M)\prod_a^b e^{A(t)\,dt}$ *exists and is invertible.*
If one of these conditions is fulfilled, then

$$(M)\prod_a^b (I + A(t)\,dt) = (M)\prod_a^b e^{A(t)\,dt}.$$

Proof. The functions

$$V_1(t, [x, y]) = I + A(t)(y - x),$$

$$V_2(t, [x, y]) = e^{A(t)(y-x)}$$

satisfy condition (**C**). According to Theorem 33 it is sufficient to show that V_1 and V_2 are equivalent. For $x < y$ we have

$$\|I + A(t)(y - x) - e^{A(t)(y-x)}\| = \left\| \sum_{k=2}^{\infty} \frac{A(t)^k (y - x)^k}{k!} \right\| \leq$$

$$\leq \|A(t)\|^2 (y - x)^2 e^{\|A(t)\|(y-x)}.$$

Let $\Delta : [a, b] \to (0, \infty)$ be an arbitrary function such that

$$\Delta(t) < \min\left(\frac{1}{2\|A(t)\|}, \frac{\varepsilon}{2e(b-a)\|A(t)\|^2}\right)$$

whenever $\|A(t)\| > 0$. Then for every Δ-fine M-partition $\{(\tau_j, [\alpha_{j-1}, \alpha_j])\}_{j=1}^m$ we have

$$\alpha_j - \alpha_{j-1} < 2\Delta(\tau_j)$$

and

$$\sum_{j=1}^m \|I + A(\tau_j)(\alpha_j - \alpha_{j-1}) - e^{A(\tau_j)(\alpha_j - \alpha_{j-1})}\| <$$

$$\leq \sum_{j=1}^m \|A(\tau_j)\|^2 (\alpha_j - \alpha_{j-1})^2 e^{\|A(\tau_j)\|(\alpha_j - \alpha_{j-1})} < \sum_{j=1}^m \frac{\varepsilon(\alpha_j - \alpha_{j-1})}{b-a} = \varepsilon.$$

□

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