ON NON-ABSOLUTELY CONVERGENT INTEGRALS

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Abstract. The influence of Jan Mařík in the field of non absolute integration is described in the plane of Czech mathematics. A short historical account on the development of integration theory in the Czech region is presented in this connection together with the recent Riemann sum approach to the general Perron integral.

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From the viewpoint of the theory of integration, our century can be called without any exaggeration the Lebesgue century. The first announcement that a new integral, which is stronger than the Riemann integral, was created, appeared in 1901 in *Comptes Rendus* in an article by H. Lebesgue and in a more detailed form later in 1902 in Lebesgue's dissertation *Int grale, Longeur, Aire* (Ann. di Matem. (3), 7, 231–359). In 1904 Lebesgue wrote the book *Leçons sur l'integration et la recherche des fonctions primitives*, Gauthier-Villars, Paris, 1904, which appeared in the second edition in 1928.

The Lebesgue integral presented indisputable advances in comparison with the Riemann integral.

a) *For a function $f\colon [a,b] \to \mathbb{R}$ to be integrable in the sense of Lebesgue, this function need not be continuous at any point of the interval $[a,b]$.*

A function $f\colon [a,b] \to \mathbb{R}$ which is integrable in the sense of Riemann is necessarily continuous almost everywhere in the interval $[a,b]$.

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b) *If $f_n\colon [a,b] \to \mathbb{R}$ is a sequence of functions integrable in the Lebesgue sense, which converges pointwise to a function $f\colon [a,b] \to \mathbb{R}$ and $|f_n| \leqslant g$, where $g\colon [a,b] \to \mathbb{R}$ is integrable in the Lebesgue sense, then $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

holds.

c) *If a function F has a bounded derivative F' in the interval $[a,b]$, then the function F' is integrable in the Lebesgue sense in $[a,b]$ and*

$$\int_a^x F' = F(x) - F(a)$$

holds for every $x \in [a,b]$.

Slightly different is the following proposition.

d) *If the function F is continuous in the interval $[a,b]$ and is differentiable to F' in $[a,b]$ everywhere except a countable set, and if F' is Lebesgue integrable, then*

$$\int_a^x F' = F(x) - F(a)$$

holds for every $x \in [a,b]$.

In the last proposition d) the requirement that F' has to be Lebesgue integrable is somewhat surprising. But this requirement cannot be omitted there.

Indeed, if we set

$$F(x) = x^2 \sin\left(\frac{\pi}{x^2}\right), \quad 0 < x \leqslant 1, \quad F(0) = 0,$$

we get a function which has a derivative

$$F'(x) = -\frac{2\pi}{x}\cos\left(\frac{\pi}{x^2}\right) + 2x\sin\left(\frac{\pi}{x^2}\right) = f(x) + g(x), \quad 0 < x \leqslant 1, \quad F'(0) = 0$$

at all points of the interval $[0,1]$.

It is not difficult to check that the function F is not absolutely continuous in the interval $[0,1]$ and therefore the function F' cannot be integrable in the Lebesgue sense in $[0,1]$.

Nevertheless, the function F' possesses in the interval $[0,1]$ the Newton integral and we have

$$(N)\int_0^1 F' = F(1) - F(0) = 0.$$

Looking closely at the function F', we establish easily that the summand

$$g(x) = 2x \sin\left(\frac{\pi}{x^2}\right), \quad 0 < x \leqslant 1, \quad g(0) = 0$$

is Lebesgue integrable and that the problems with integrability are caused by the function

$$f(x) = -\frac{2\pi}{x} \cos\left(\frac{\pi}{x^2}\right), \quad 0 < x \leqslant 1, \quad f(0) = 0,$$

because $\int_0^1 |f| = \infty$.

This classical example shows that the Lebesgue integral is generally not qualified to reconstruct a function if the derivative of this function is known, i.e. the relation

$$\int_a^x F' = F(x) - F(a)$$

need not hold for every $x \in [a,b]$ even in the case when the finite derivative F' exists everywhere in the interval $[a,b]$.

Moreover, it can be seen from this example that although for every $\varepsilon \in (0,1]$ the Lebesgue integral

$$\int_\varepsilon^1 F' = F(1) - F(\varepsilon) = -F(\varepsilon)$$

exists and also the proper limit

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^1 F' = -\lim_{\varepsilon \to 0+} F(\varepsilon) = 0$$

exists, the Lebesgue integral $\int_0^1 F'$ does not exist, i.e., for the Lebesgue integral the so called Hake Theorem is not valid.

These drawbacks of the Lebesgue integral—which is very powerful in other respects—led immediately after Lebesgue's work at the beginning of the century to attempts to create an integration theory in which a proposition of type d) would hold without the assumption of integrability of the function F'. In other words, an integration theory was needed in the frame of which the integrability of a derivative would be ensured provided this derivative exists in some reasonable sense.

A theory satisfying this desire was developed in 1912 by A. Denjoy. This was the Denjoy total produced by a relatively complicated process based on the use of transfinite numbers. Shortly after this A. Luzin connected the new Denjoy integration with the concept of generalized absolute continuity (ACG_*). The result was the following proposition:

A function $f\colon [a,b] \to \mathbb{R}$ is integrable in the sense of Denjoy if there is an ACG_* function $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere in $[a,b]$.

This corresponds to the following known assertion for the Lebesgue integral:

A function $f\colon [a,b] \to \mathbb{R}$ is integrable in the Lebesgue sense if there is an absolutely continuous function (an AC function) $F\colon [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere in $[a,b]$.

Let us shortly follow the development of the modern views of the concept of the integral in Bohemia in the 20th century.

Modern integration theory in this geographical region goes back to Professor Karel Petr who wrote a textbook *Integral Calculus* (Počet integrální). Riemann's approach to integration is presented there in full extent and precisely without mentioning Lebesgue integration. The first edition of this voluminous book appeared in 1915 and the second in 1931.

The new 1931 edition of Petr's book was thoroughly revised and considerably extended. An appendix *Introduction into the theory of sets* was written by Vojtěch Jarník.

Petr's exposition was based on the Newton and Riemann concepts of integration, and it has to be noted that it contains a good deal of art of calculation techniques and even numerical methods. This was a strong and rich part of K. Petr's mathematical knowledge. In the appendix to Petr's book Prof. V. Jarník mentions the work of H. Lebesgue, E. Kamke, L. Schlesinger and A. Plesner, Ch. de la Vallée-Poussin devoted to the Lebesgue theory of integral and he notes that this theory is not included in the appendix. In a certain sense the second edition of Petr's book was the reason for the long delay of presenting Lebesgue's theory in a Czech book.

In the year 1936 Eduard Čech published the book *Point Sets. Part one* (Bodové množiny. Část první) with an appendix *On derivation numbers of real functions* (O derivovaných číslech funkcí jedné proměnné) again written by V. Jarník. The fourth chapter of Čech's *Point Sets* has the title *Measure and integral* and is very extensive, representing approximately one half of the whole book, i.e. 220 pages.

The chapter on integration in Čech's book is the first Czech presentation of the theory of the Lebesgue integral in the form of a book. Namely, in *Point Sets* Čech presents a very profound exposition of the theory. Let us mention shortly some of the topics: algebras of sets, σ-algebras, Borel sets, additive and σ-additive set functions, general theory of measure, general theory of integral (measurable functions, Fubini's theorem, the special case of the Lebesgue measure), set functions of bounded variation (Vitali's covering, derivatives of set functions, metric density), point functions of bounded variation, the Stieltjes integral.

Čech's chapter on integration is a detailed and comprehensive explanation of the theory of the integral. The calculus part of integration is missing there: these things which are so important for students were available at that time in the book of Karel Petr. In the preface to his *Point Sets* E. Čech mentioned that he was influenced and inspired by the French version of the book of Stanislaw Saks *Th orie de l'integrale.* The approach used in the text of Čech gives a clear evidence of this fact.

Saks' book on integration theory played a very important and decisive role in the thirties not only within the theory of integral, but also in the theory of real functions in general.

Professor V. Jarník was also influenced by the book of S. Saks and Saks mentioned this in the preface to the English translation of the book from 1937 in connection with some inaccuracies in the French version which had been detected and corrected by V. Jarník.

Professor V. Jarník evidently had for a long time the intention of writing a book on integral calculus based on the Lebesgue integral. He began to prepare such a text during the World War II. His book *Integral Calculus II* (Integrální počet II) was published in 1955. (Earlier he published *Integral Calculus I* (Integrální počet I) which was based on the Riemann integral and was essentially shorter than Petr's book.) V. Jarník himself characterizes his book as follows: ... *this book, even if it is based on the modern concept of integral, is rather an "Integral Calculus" than a "Theory of integration".* Concerning the more general theoretical approach, Jarník refers the reader to Čech's *Point Sets.* This statement of Jarník is very modest, his *Integral Calculus II* is both theoretical and calculational. And this makes the book very instructive and useful for students. Nowadays this book of V. Jarník is more than forty years old and in spite of that it has not been replaced by another Czech book of equal importance.

In the above part we overjumped in time the description how the views of integration developed in our country. Soon after WW II a group of mathematicians was growing up, their density among the population of postwar students and postgraduates being unusually high. In the field of integration theory Jan Mařík was one of them. After he ended his graduate studies in Mathematics, he became assistent at the Technical University in Prague and evidently had contact with students in the course of which he encountered rather unsatisfactory approaches to integration theory. In the year 1952 J. Mařík published in *asopis pro p Ξstov n matematiky* (77 (1952)) a long paper divided into three parts *Foundations of the theory of integral in Euclidean spaces* (Základy theorie integrálu v Euklidových prostorech). The paper consists of 107 pages and is also very comprehensive. It was published before the appearence of the above mentioned book of V. Jarník. Jarník mentions in the preface to his book on the Lebesgue integral the existence of Mařík's article. (Let us note that before this paper in 1951 Mařík published in Časopis pro pěstování matematiky another Czech paper *The Lebesgue integral in abstract spaces* (Časopis Pěst. Mat. 76 (1951)).

Mařík's big work *Foundations of the theory of integral in Euclidean spaces* is of educational character. The presentation is independent of other sources, selfcontained, accessible to a medium educated and patient mathematician. All the necessary concepts are contained in the article with the necessary careful and economized explanation. Especially the introductory part is educating the reader, all the known integration theories are assessed, both their advantages and disadvantages being pointed out.

Let us present some of Mařík's ideas from the introduction to the paper:

We will mostly consider the Perron integral; why we will not start "as usual" with the Riemann integral? This has sufficiently serious reasons. The Riemann integral possesses some merits; its definition—especially in the one dimensional case—is simple and sufficiently "instructive"; in the more dimensional case it is a well fitted tool for introducing some physical quantities ...

Further, it can be said that the majority of functions for which we do "calculations" have a proper or improper integral.

However, this is probably the end of the list of good properties of the Riemann integral.

Further Mařík says:

By having pronounced a certain definition (e.g. the definition of the Riemann integral) in fact nothing is done; merely some notation is introduced. To have a theory of integral which "is of any use", the theory has to provide not only definitions, but above all theorems; especially theorems helping in really calculating or at least estimating the value of the integral in individual cases. Of course we would be glad if the theorems were as general as possible, if their formulation was not too complicated and, last but not least, it should be also taken into account whether it is possible to deduce them in a simple and relatively elementary way.

This quotation in fact describes the program of Mařík's work.

Mařík is also very critical in the case of the Lebesgue integral:

The theory is lucid and definitive; it is its advantage that it can be used in abstract spaces in which there has been nothing heard even of topology.

However, it is quite credible that using such a too general approach we cannot get sufficiently deep into what we need in the case of Euclidean spaces.

The main defect of the Lebesgue integral is that it covers only absolutely convergent integrals; hence the Lebesgue integral is a generalization neither of the improper Riemann integral nor the Newton integral. (For example the derivative of the function $x^2 \sin(1/x^2)$ *completed at the point zero by the corresponding limit does not possess the Lebesgue integral over the interval* $\langle -1, 1\rangle$ *though it has the improper Riemann integral as well as the Newton integral over this interval.) This example shows also that for the Lebesgue integral the following theorem does not hold:*

If the (Perron) integral of the function f exists over every interval $\langle a, b-\varepsilon\rangle$, where ε is an arbitrary positive number less than $b-a$, and if the proper limit $\lim\limits_{\varepsilon\to 0+}\int_a^{b-\varepsilon} f(x)\,\mathrm{d}x$ exists, then also the integral $\int_a^b f(x)\,\mathrm{d}x$ exists and is equal to this limit.

(Nowadays this theorem is called Hake's theorem and Mařík is mentioning at this place the known deficiency of the Lebesgue integral which we have already mentioned above.)

It can be presumed that especially for beginners the Perron theory of integral is more suited than the Lebesgue theory. Indeed, the Perron theory of integral can be constructed in such a manner that we work only with the concepts of the limit of a sequence and of the more dimensional interval; it is not necessary to speak simultaneously about measure or topology. The proofs of theorems look also more natural than in Lebesgue's theory and they are usually much simpler.

The relation between the Perron and Lebesgue integral is simple.

A function f possesses the Lebesgue integral in an interval K, if and only if both the functions f and $|f|$ possess the Perron integral.

Mařík constructs the integral step by step without skipping anything necessary for his reasoning. Let us recall shortly his way to the definition:

He is working in an m-dimensional interval K with functions of an interval. The function F is in the interval K *superadditive* (*subadditive*), if it is defined on the set of all intervals $I \subset K$ and if

$$F(I) + F(J) \leqslant F(I \dotplus J)$$
$$\big(F(I) + F(J) \geqslant F(I \dotplus J)\big)$$

holds provided the sum on the left hand side makes sense and $I \dotplus J \subset K$. I and J are nonoverlapping intervals (i.e. their interiors are disjoint) and $I \dotplus J$ means that $I + J$ is also an interval. (Note that $+$ is used here for the union of sets.)

If F and G are functions in the interval K, G is finite and $x \in K$ then the *upper derivative of the function F with respect to the function G at the point x with respect to the interval K* is denoted by $\overline{F}(G, x, K)$ and means the supremum of the set of all $t \in E_1^*$ of the form (we use here the symbols used by J. Mařík; let us mention only that E_1^* means "the set E_1 ($= \mathbb{R}$) enlarged by the elements ∞ and $-\infty$", for E_1^* the symbol $\mathbb{R}^*$ will be used in the sequel)

$$t = \lim(F(I_n) : G(I_n)),$$

while the *lower derivative of the function F with respect to the function G at the point x with respect to the interval K* is denoted by $\underline{F}(G, x, K)$ and means the infimum of the set of all $t \in E_1^*$ of the form

$$t = \lim(F(I_n) : G(I_n)),$$

where the limits are taken for $I_n \to x$, $I_n \subset K$ and $I_n \to x$, which means that I_n is a sequence of intervals with $x \in I_n$, $n = 1, 2, \ldots$ and $d(I_n) \to 0$, where $d(I)$ denotes the length of the largest edge of the interval I.

Note. When defining the upper derivative $\overline{F}(G, x, K)$ of the function F with respect to the function G at the point x with respect to the interval K and the lower derivative $\underline{F}(G, x, K)$ of the function F with respect to the function G at the point x with respect to the interval K we have in fact to do with the supremum and the infimum of the set of numbers

$$S(x) = \{t \in \mathbb{R}^*;\, t = \lim(F(I_n) : G(I_n))\},$$

where $\lim(F(I_n) : G(I_n))$ has the meaning stated above.

Assume that $B(x, \delta) = \{y \in \mathbb{R}^m;\, \|y - x\| < \delta\}$ is the ball in $\mathbb{R}^m$ with its center at the point $x \in \mathbb{R}^m$ and radius $\delta > 0$.

Denote by $\widetilde{S}(x)$ the set of all $t \in \mathbb{R}^*$ such that for every $\varepsilon > 0$ there is a $\delta(x) > 0$ such that

$$\left|\frac{F(I)}{G(I)} - t\right| < \varepsilon$$

provided $I \subset \mathbb{R}^m$ is an interval, $x \in I \subset B(x, \delta(x))$ with the usual convention in the case when $t = \infty$ or $t = -\infty$, i.e. $\frac{F(I)}{G(I)} > \frac{1}{\varepsilon}$ in the former case and $\frac{F(I)}{G(I)} < -\frac{1}{\varepsilon}$ in the latter.

By the Bolzano-Weierstrass Theorem it is easy to see that

$$S(x) = \widetilde{S}(x).$$

Let us follow further how Mařík proceeds with the definition of the integral:

Let G be a *finite, nonnegative additive* function in an interval K; let f be a point function in K.

M is called a *majorant of the function f with respect to the function G in the interval K*, if

a) M is in K superadditive,

b) $-\infty \neq \underline{M}(G, x, K) \geqslant f(x)$ for every $x \in K$.

Similarly, m is a *minorant of the function f with respect to the function G in the interval K* if $(-m)$ is a majorant to $(-f)$, i.e. if

a′) m is in K subadditive,

b′) $\infty \neq \overline{m}(G, x, K) \leqslant f(x)$ for every $x \in K$.

The *upper* (Perron-Stieltjes) *integral of the function f with respect to the function G in the interval K* is

$$\overline{\int_K} f \,\mathrm{d}G = \inf M(K),$$

where the infimum is taken over all majorants M of the function f.

The *lower* (Perron-Stieltjes) *integral of the function f with respect to the function G in the interval K* is

$$-\overline{\int_K}(-f)\,\mathrm{d}G,$$

or in other words

$$\underline{\int_K} f\,\mathrm{d}G = \sup m(K),$$

where the supremum is taken over all minorants m of the function f.

Using these definitions of the upper and lower integrals Mařík presents the following definition.

Definition 1. We say that the *function f has an integral* (or that *the integral of the function f exists*) *with respect to the function G in the interval K*, if

$$\underline{\int_K} f\,\mathrm{d}G = \overline{\int_K} f\,\mathrm{d}G \in E_1$$

holds.

In this case we write

$$\underline{\int_K} f\,\mathrm{d}G = \overline{\int_K} f\,\mathrm{d}G = \int_K f\,\mathrm{d}G$$

and denote by $\mathfrak{P} = \mathfrak{P}(G,K)$ the set of all such functions.

The approach used by Mařík for defining the Perron-Stieltjes integral over more dimensional intervals can be found already (in the situation when the function G has the meaning of the volume) in the paper of H. Bauer: *Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen* (Monatshefte Math. Phys. 26) from 1915.

Further Mařík introduces the following notation:

$\mathfrak{P}_0 = \mathfrak{P}_0(G,K)$ is the set of all bounded functions in $\mathfrak{P}$;

$\mathfrak{P}_A$ is the set of all functions $f \in \mathfrak{P}$ for which also $|f| \in \mathfrak{P}$;

$\mathfrak{P}_R$ is the set of functions for which

$$\underline{\int_K} f\,\mathrm{d}G = \overline{\int_K} f\,\mathrm{d}G$$

holds, where the values $-\infty$ or ∞ are also allowed.

He shows that

$$f \in \mathfrak{P} \Rightarrow f_+, f_-, |f| \in \mathfrak{P}_R$$

and that we have

$$\mathfrak{P}_0 \subset \mathfrak{P}_A \subset \mathfrak{P} \subset \mathfrak{P}_R.$$

In the subsequent parts of the article Mařík investigates relations of the integral introduced by him to other types of integrals (Riemann-Stieltjes, Newton, etc.) and he derives theorems which are important for calculating the integral (the integration by parts theorem for the case when K is a one dimensional interval, the change of variables theorem for functions belonging to $\mathfrak{P}_A, \dots$). Mařík completes his program and shows that simple notions allow to construct an integral which has sufficiently nice properties, and also that his construction involves all known and commonly used integrals.

The idea that ... *especially for beginners the Perron theory of integral is more suited than the Lebesgue theory* ... was practically never realized by Mařík in a separate text for students. In the years 1960–61 he published two volumes of a university text *Integral Calculus I, II* together with I. Černý. This text was mainly devoted to an alternative presentation (in comparison with the older Czech textbook *Integral Calculus II* by Vojtěch Jarník) of the construction of the Lebesgue integral in the first part and to some problems concerning k-dimensional integrals in m-dimensional spaces in the second volume. Nevertheless in the second volume of this university text the Perron integral appears in a relatively simple form, which serves to deduce a strong version of the Gauss theorem in the textbook. In the sixties J. Mařík wrote papers *A non-absolutely convergent integral in E_m and the theorem of Gauss* (Czechoslovak Math. J. 15 (90), 1965, 253–260) and *On representations of some Perron integrable functions* (Czechoslovak Math. J. 19 (94), 1969, 745–749) on nonabsolutely convergent integrals and related questions jointly with Karel Karták. At this point also the paper *On a generalization of the Lebesgue integral in E_m* (Czechoslovak Math. J. 15 (90), 1965) written together with Jiří Matyska and *Continuous additive mappings* (Czechoslovak Math. J. 15 (90), 1965) with Jaroslav Holec should be mentioned. Almost at the same time Jan Mařík published the paper *Extensions of additive mappings* (Czechoslovak Math. J. 15 (90), 1965) where a method is described which allows to extend the Lebesgue integral. This series of Mařík's papers is connected with his former work *The surface integral* (Czechoslovak Math. J. 6 (81), 1956).

At this place also his work on the generalized integral and trigonometric series should be mentioned. This is a carefully written text for university students, which was prepared during Mařík's exile in the USA and which came to us in the form of a copy of his handwritten manuscript. At the beginning of this text the Perron integral over one dimensional intervals is described in full detail.

Let us add a simple consideration to the topics presented above. Take into account Mařík's Definition 1 of the integral $\int_K f \, \mathrm{d}G \in \mathbb{R}$ and functions f for which $f \in \mathfrak{P} = \mathfrak{P}(G, K)$.

By Definition 1 we have

$$\int_K f \, \mathrm{d}G = \sup m(K) = \inf M(K),$$

where the supremum is taken over all minorants m of the function f and the infimum over all majorants M of the function f. By the Theorem on the Supremum (Infimum) we know that for every $\varepsilon > 0$ there exists a minorant m^* and a majorant M^* to the function f such that

$$(*) \qquad M^*(K) - \varepsilon < \int_K f \, \mathrm{d}G < m^*(K) + \varepsilon.$$

Since m^* is a minorant to the function f, we know that m^* is subadditive in K and that

$$\infty \neq \overline{m^*}(G, x, K) \leqslant f(x) \quad \text{for every } x \in K$$

holds.

Taking into account the above Note, we obtain that the upper derivative $\overline{m^*}(G, x, K)$ is the supremum of the set of all $t \in \mathbb{R}^*$ such that for every $\varepsilon > 0$ there is a $\delta(x) > 0$ such that if $I \subset \mathbb{R}^m$ is an interval, $x \in I \subset B(x, \delta(x))$, then

$$\left| \frac{m^*(I)}{G(I)} - t \right| < \varepsilon.$$

But this means by the definition of the supremum that for given $x \in K$ and $\varepsilon > 0$ there exists a $\delta(x) > 0$ such that if $x \in I \subset B(x, \delta(x))$ then also

$$\frac{m^*(I)}{G(I)} - \varepsilon < \overline{m^*}(G, x, K) \leqslant f(x),$$

or, in other words, the inequality

$$(**) \qquad m^*(I) - \varepsilon G(I) < f(x) G(I)$$

holds. For the majorant M^* we get analogously the inequality

$$f(x) G(I) < M^*(I) + \varepsilon G(I)$$

provided $x \in I \subset B(x, \delta(x))$.

Assume now for a moment that there is a finite system of intervals

$$J_1, J_2, \ldots, J_k \subset K$$

such that

$$K = J_1 \dotplus J_2 \dotplus \ldots \dotplus J_k,$$

and points $x_i \in K$, $i = 1, 2, \ldots, k$ such that $x_i \in J_i \subset B(x_i, \delta(x_i))$. Then, because G is finite and additive, m^* is subadditive and $(**)$ holds for $x_i \in J_i$, $i = 1, 2, \ldots, k$, we get

$$\begin{aligned} m^*(K) &\leqslant \sum_{i=1}^{k} m^*(J_i) < \sum_{i=1}^{k} \big[f(x_i)G(J_i) + \varepsilon G(J_i)\big] \\ &= \sum_{i=1}^{k} f(x_i)G(J_i) + \varepsilon G(K). \end{aligned}$$

By virtue of the inequality $(*)$, we obtain from this relation the inequality

$$\begin{aligned} \int_K f \, \mathrm{d}G &< m^*(K) + \varepsilon < \sum_{i=1}^{k} f(x_i)G(J_i) + \varepsilon G(K) + \varepsilon \\ &= \sum_{i=1}^{k} f(x_i)G(J_i) + \varepsilon[G(K) + 1]. \end{aligned}$$

Working in a similar way with the majorant, we arrive at the inequality

$$\sum_{i=1}^{k} f(x_i)G(J_i) - \varepsilon[G(K) + 1] < \int_K f \, \mathrm{d}G,$$

which together with the previous one gives

$$(***) \qquad \Big| \sum_{i=1}^{k} f(x_i)G(J_i) - \int_K f \, \mathrm{d}G \Big| < \varepsilon[G(K) + 1].$$

The inequality $(***)$, together with the fact that $\varepsilon > 0$ can be taken arbitrarily small, means that the value of the integral $\int_K f \, \mathrm{d}G$ as it was defined by J. Mařík can be approximated by a Riemann (-Stieltjes) type integral sum of the form $\sum_{i=1}^{k} f(x_i)G(J_i)$.

This reasoning depends of course on the fact whether for the positive function $\delta(x)$, which comes from the definition of the upper derivative of the function m^* and the lower derivative of the function M^*, there really exists a finite system of intervals

$$J_1, J_2, \dots, J_k \subset K$$

such that

$$K = J_1 \dotplus J_2 \dotplus \dots \dotplus J_k,$$

and points $x_i \in K$, $i = 1, 2, \dots, k$ such that $x_i \in J_i \subset B(x_i, \delta(x_i))$.

This problem is answered in the affirmative by the so called *Cousin Lemma*, which states that for an arbitrary function $\delta(x) > 0$ given on the interval K there is a finite system of "point-interval" couples (x, J) which has the properties required above.

After this consideration we introduce the following definition:

Definition 2. The function f has an integral with respect to the function G in the interval K if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a function $\delta\colon K \to (0, +\infty)$ such that

$$\Big| \sum_{i=1}^{k} f(x_i) G(J_i) - I \Big| < \varepsilon$$

holds for every finite system of intervals

$$J_1, J_2, \dots, J_k \subset K,$$

for which

$$K = J_1 \dotplus J_2 \dotplus \dots \dotplus J_k,$$

where $x_i \in J_i \subset B(x_i, \delta(x_i))$ for $i = 1, 2, \dots, k$.

We denote the set of all such functions by $\mathfrak{K} = \mathfrak{K}(G, K)$ and the number I by $\int_K f \, \mathrm{d}G$.

The function $\delta\colon K \to (0, +\infty)$ is called a *gauge* on K and the system $\{(x_i, J_i), i = 1, 2, \dots, k\}$ with the properties described in the definition is called a *partition of the interval* K, which is δ*-fine* with respect to the gauge δ.

This definition of the Stieltjes integral belongs to Jaroslav Kurzweil, who presented a similar definition in the year 1957 in his paper *Generalized ordinary differential equations and continuous dependence on a parameter* (Czechoslovak Math. J. 7(82), 1957).

It is remarkable that the following statement is true:

Proposition. *The function f has an integral with respect to the function G in the interval K in Maflk's sense if and only if it has the integral in the sense of Kurzweil. In such a case the two integrals have the same value, i.e. the equality*

$$\mathfrak{P} = \mathfrak{K}$$

holds.

Let us go back for a moment to Mařík's assessment of the Riemann integral. He stated ... *its definition—especially in the one dimensional case—is simple and sufficiently "instructive"; in the more dimensional case it is a well fitted tool for introducing some physical quantities* ... Evidently the "instructiveness" has to be understood in the way that the Riemann integral sum closely approximates, under the supposition that the partition is sufficiently fine, e.g. the area which is expressed by the integral, or that it describes in a transparent way the physical quantity which is to be determined.

Proposition stated above shows that the integral which was presented by Mařík in his work from 1952 can be introduced via Riemann-Stieltjes integral sums and that it has therefore also one of the few advantages of the Riemann integral which had been pointed out by Mařík. Moreover, we can follow Mařík's steps of criticism and claim that for introducing the Perron integral by Definition 2 much less sophisticated tools are needed in comparison with those used by Mařík (it is not necessary to present upper and lower derivatives, we need not worry about majorants and minorants, suprema and infima, etc.).

At this stage we have to close the eulogy of the Perron concept of integration. It has to be said that among other it is very discomfortable when integrating over moredimensional domains especially from the point of view of transformations (this concerns the Change of Variables Theorem). Mařík did not mention this defect of the theory although in his lectures at Charles University in Prague he presented many times an example of a function which is integrable in the Perron sense over a twodimensional interval but after a simple transformation it loses the property of integrability.

Comparing Kurzweil's definition of the Perron integral with the definition of the Riemann integral, we almost cannot recognize the difference although we get the nonabsolutely convergent Perron integral which includes the Riemann, Newton and also the Lebesgue integral. The essence of this matter is in the concept how the fineness of a partition has to be understood. We are working with partitions of an interval K which are δ-fine with respect to a gauge δ which is represented by a positive function on K. In the case of the Riemann integral this gauge is also a positive function but this function is required to be constant.

The integral given by Definition 2 is nowadays called the Kurzweil-Henstock integral. The name of the Northern Ireland's mathematician Ralph Henstock appears in this connection because he discovered in 1960 the same kind of integral independently of the work of J. Kurzweil. R. Henstock himself says that the first time he heard about Kurzweil's work was in 1963, when he was informed about this fact in a letter from K. Karták.

This independence of discoveries is not a great surprise. The new theory of integral presented in the work of J. Kurzweil in 1957 was not the aim of the paper, it was in fact a tool for explaining some convergence effects in the theory of ordinary differential equations and for introducing the concept of generalized differential equations. Generalized differential equations complete in some sense the family of ordinary differential equations with respect to topologies given by continuous dependence theorems with the weakest possible assumptions on the convergence of the right hand sides. Papers of this kind are usually not followed carefully by specialists in integration theory, since they do not expect important new relevant results in papers of this kind. And a specialist in integration theory would probably not search for an essentially new definition of the Perron integral in a paper on continuous dependence theorems for ordinary differential equations.

J. Kurzweil himself was devoted to problems in the theory of differential equations and he did not paid too much attention to the development of his new integral at the time. He got interested in integration theory again in the seventies, and this period ended by his German book *Nichtabsolut konvergente Integrale*, published in 1980 by Teubner in Leipzig.

In 1981 in the Czechoslovak Math. Journal J. Mawhin published the paper *Generalized multiple Perron integrals and the Green-Goursat theorem for vector fields.* In the paper J. Mawhin showed a more dimensional analog of the theorem on the integral of a derivative under relatively weak conditions. His integral had some unpleasant properties which a proper integral should not have (Mawhin's integral is not additive with respect to intervals).

This was the starting moment for another period of investigation of the integral over more dimensional sets. The main point consists in modifying the definitions in such a way that the resulting integral have the possibly best properties. In particular, validity of the Divergence Theorem and the Change of Variables Theorem are the main requirements on a new theory. The work in this direction continued in the Czech Republic up to now and in this way we also can feel the influence of Jan Mařík's ideas on present Czech mathematics.

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