A negative answer to a problem of Fremlin and Mendoza

Ye Guoju (叶国菊) *
College of Science, Hehai University, Nanjing, Jiangsu, 210098, P.R.China
Email: yegj@mailsvr.hhu.edu.cn
Stefan Schwabik†
Matematický ústav AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic
Email: schwabik@math.cas.cz

Abstract: *In this paper, we study some convergence results for the McShane integral of functions mapping the interval* $[0,1]$ *into a Banach space* X *from the point of view of an open problem proposed by D.H.Fremlin and J. Mendoza in [2], also we give a negative answer to this open problem..*
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1 Introduction

The McShane integral as it was described in [2] - [5] an [8] - [13] is a Riemann-type integral using "gauge-limit". It is equivalent to the Lebesgue integral for real functions. The Dunford, Pettis and Bochner integrals are generalizations of the Lebesgue integral to Banach-valued functions. The McShane integral of a vector-valued function and its relationship to the Bochner integral, Pettis integral were discussed in [2] - [4], [8] - [10], [13]. An interesting convergence theorem for the McShane integral was proved by D.H.Fremlin and J. Mendoza in [2]. The statement of this theorem is as follows (see Theorem 2I, p. 135 in [2]).

Theorem A. *Let* X *be a Banach space. Let* $f_n, n \in \mathbb{N}$ *be a sequence of McShane integrable functions from* $[0,1]$ *to* X*, and suppose that* $f(t) = \lim_{n\to\infty} f_n(t)$ *exists in* X *for every* $t \in [0,1]$*. If moreover the limit*

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X*, for the weak topology, for every measurable* $E \subset [0,1]$*, then* f *is McShane integrable and* $\int_0^1 f = F([0,1])$*.*

At the same time, an open problem was left in [2] (see Problem formulated on p. 138 in [2]):

In the above Theorem A it is supposed that $f(t) = \lim_{n\to\infty} f_n(t)$ in the norm topology for every t. Is it enough if $f(t)$ is the weak limit of $f_n(t)$, $n \in N$ for every t?

In other words the problem is if the following theorem holds?

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Theorem B. *Let X be a Banach space. Let f_n, $n \in N$ be a sequence of McShane integrable functions from $[0,1]$ to X, and suppose that the weak limit $f(t)$ of $f_n(t)$ exists in X for every $t \in [0,1]$. If moreover the limit*

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$, then f is McShane integrable and $\int_0^1 f = F([0,1])$?

Remark. In fact, in the theorem A the condition " $f(t) = \lim_{n\to\infty} f_n(t)$ exists in X for every $t \in [0,1]$" is instead of the condition "$f(t) = \lim_{n\to\infty} f_n(t)$ exists in X for almost every $t \in [0,1]$", then the result is still holds. Because the integrability and integral of a function f are invariant if we change its value on the set with measure zero. So the Theorem A can be stated as the following

Theorem A'. *Let X be a Banach space. Let $f_n, n \in \mathbb{N}$ be a sequence of McShane integrable functions from $[0,1]$ to X, and suppose that $f(t) = \lim_{n\to\infty} f_n(t)$ exists in X for almost every $t \in [0,1]$. If moreover the limit*

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$, then f is McShane integrable and $\int_0^1 f = F([0,1])$.

Corresponding the Theorem B can be stated as the following

Theorem B'. *Let X be a Banach space. Let f_n, $n \in N$ be a sequence of McShane integrable functions from $[0,1]$ to X, and suppose that the weak limit $f(t)$ of $f_n(t)$ exists in X for almost every $t \in [0,1]$. If moreover the limit*

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$, then f is McShane integrable and $\int_0^1 f = F([0,1])$?

Remark. Because Theorem A and Theorem A' are equivalent, so our question become that is the Theorem B' true?

In this paper we will concentrate on this problem using some results from our paper [9] and [6]. We answer the problem concerning the validity of Theorem B and Theorem B' in the case of a separable Banach space or a reflexive Banach space with an additional condition (P) concerning the unit ball of the dual X^*. On the other hand, in the general case we prove that the Theorem B' is not true. So we give a negative answer to Theorem B'.

2 Definitions and basic concepts

By X we denote a real Banach space with the norm $\|\cdot\|$ and by X^* its dual.

$[0,1]$ is the compact interval in $\mathbb{R}$, Σ is the set of all μ-measurable subsets of $[0,1]$, μ stands for the Lebesgue measure.

A *partial M-partition* D in $[0,1]$ is a finite collection of interval-point pairs (I,ξ) with non-overlapping intervals $I \subset [0,1]$, $\xi \in [0,1]$ being the associated point of I.

We write $D = \{(I,\xi)\}$.

A partial M-partition $D = \{(I,\xi)\}$ in $[0,1]$ is a *M-partition* of $[0,1]$ if the union of all the intervals I from D equals $[0,1]$.

Let δ be a positive function defined on the interval $[0,1]$ called a *gauge on* $[0,1]$. A partial M-partition $D = \{(I,\xi)\}$ is said to be δ-fine if for each interval-point pair $(I,\xi) \in D$ we have $I \subset B(\xi,\delta(\xi))$ where $B(\xi,\delta(\xi)) = (\xi - \delta(\xi), \xi + \delta(\xi))$.

Given a M-partition $D = \{(I,\xi)\}$ we write

$$f(D) = (D)\sum f(\xi)\mu(I)$$

for integral sum over D, whenever $f : [0,1] \to X$ and $\mu(I)$ is the length of the interval I.

Definition 1. An X-valued function f is said to be *McShane integrable* on $[0,1]$ if there exists an $S_f \in X$ such that for every $\varepsilon > 0$, there exists a gauge δ on $[0,1]$ such that for every δ-fine M-partition $D = \{(I,\xi)\}$ of $[0,1]$, we have

$$\|(D)\sum f(\xi)\mu(I) - S_f\| < \varepsilon.$$

We write $(M)\int_{[0,1]} f = S_f$ and S_f is the McShane integral of f over $[0,1]$.

f is McShane integrable on a set $E \subset [0,1]$ if the function $f \cdot \chi_E$ is McShane integrable on $[0,1]$, where χ_E denotes the characteristic function of E.

We write $(M)\int_E f = (M)\int_{[0,1]} f\chi_E = F(E)$ for the McShane integral of f on E and F is the primitive of f.

Denote the set of all McShane integrable functions $f : [0,1] \to X$ by $\mathcal{M}$.

The basic properties of the McShane integral, for example, the linearity of integrals, etc., can be found in [2] - [4], [8] - [9].

Definition 2. A set $K \subset \mathcal{M}$ is called *M-equiintegrable* if for every $\varepsilon > 0$ there is a $\delta : [0,1] \to (0,+\infty)$ such that

$$\|(D)\sum f(\xi)\mu(I) - \int_{[0,1]} f\| < \varepsilon$$

for every δ-fine M-partition $D = \{(I,\xi)\}$ of $[0,1]$ and every $f \in K$.

Using the concept of M-equiintegrability the following convergence result for the McShane integral holds (see e.g. [3]).

Theorem 3. *If the sequence of functions $f_n : [0,1] \to X$, $n \in \mathbb{N}$ is M-equiintegrable and*

$$\lim_{n\to\infty} f_n(t) = f(t) \ \ exists\ in\ X\ for\ \ every\ \ t \in [0,1]$$

then $f \in \mathcal{M}$ and

$$\lim_{n\to\infty} \int_{[0,1]} f_n = \int_{[0,1]} f.$$

Denoting by $\mathcal{L}$ the set of Lebesgue integrable real functions on $[0, 1]$ (with respect to the Lebesgue measure μ) let us mention that a real function f belongs to $\mathcal{L}$ if and only if it belongs to $\mathcal{M}$, i.e. $\mathcal{L} = \mathcal{M}$ (see e.g. [5]).

Definition 4. A function $f : [0, 1] \to X$ is called *measurable* if there is a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in [0, 1]$.

$f : [0, 1] \to X$ is called *weakly measurable* if for each $x^* \in X^*$ the real function $x^*(f) : [0, 1] \to \mathbb{R}$ is measurable.

In [9] it was shown that the following holds.

Theorem 5. *If $f : [0, 1] \to X$ is McShane integrable on $[0, 1]$, then*

(a) *for each x^* in X^*, $x^*(f)$ is McShane integrable on $[0, 1]$ and $\int_{[0,1]} x^*(f) = x^*(\int_{[0,1]} f)$,*

(b) *$\{x^*(f);\ x^* \in B(X^*)\}$ is M-equiintegrable on $[0, 1]$,*

(c) *f is weakly measurable,*

(d) *for every subinterval $I \subset [0, 1]$ and for every $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on I and*

$$\int_I x^*(f) = x^*(\int_I f),$$

(e) *if $E = \bigcup_{j=1}^p I_j$, where I_j are non-overlapping subintervals of $[0, 1]$, then f is McShane integrable on E with*

$$\int_E f = \sum_{j=1}^p \int_{I_j} f$$

and for every $x^* \in X^*$ we have

$$\int_E x^*(f) = \sum_{j=1}^p \int_{I_j} x^*(f) = x^*(\sum_{j=1}^p \int_{I_j} f) = x^*(\int_E f).$$

Finally, let us recall the concept of Pettis integral.

Definition 6. If $f : [0, 1] \to X$ is weakly measurable such that $x^*(f) \in \mathcal{L}$ for all $x^* \in X^*$ and if for every measurable $E \subset [0, 1]$ there is an element $x_E \in X$ such that

$$x^*(x_E) = \int_E x^*(f)$$

then f is called *Pettis integrable* and the Pettis integral of f over E is the element $x_E \in X$. We write $x_E = (P) \int_E f$ and denote by $\mathcal{P}$ the set of all Pettis integrable functions.

3 Some convergence results

In the situation presented in the previous parts of this paper the following Vitali-Hahn-Saks result holds (see Corollary 6 on p. 29 in [1]).

Theorem 7. *If F_n, $n \in \mathbb{N}$ is a sequence of X-valued μ-continuous measures on Σ and $\lim_{n\to\infty} F_n(E)$ exists for each $E \in \Sigma$, then*

$$\lim_{\mu(E)\to 0} F_n(E) = 0$$

uniformly in $n \in \mathbb{N}$.

Assuming $f_n : [0,1] \to \mathbb{R}$, $f_n \in \mathcal{L}$, $n \in \mathbb{N}$, put $F_n(E) = \int_E f_n$. Using Theorem 7 we obtain immediately the following version of the Vitali-Hahn-Saks theorem.

Corollary 8. *If* $f_n : [0,1] \to \mathbb{R}$, $f_n \in \mathcal{L} = \mathcal{M}$, $n \in \mathbb{N}$ *and the limit*

$$\lim_{n\to\infty} F_n(E) = \lim_{n\to\infty} \int_E f_n$$

exists for every $E \in \Sigma$, *then*

$$\lim_{\mu(E)\to 0} F_n(E) = \lim_{\mu(E)\to 0} \int_E f_n = 0$$

uniformly in $n \in \mathbb{N}$.

The following Vitali convergence theorem is well known (see Theorem 2, p. 168 in [7]).

Theorem 9. *If* $f_n : [0,1] \to \mathbb{R}$, $f_n \in \mathcal{L}$, $n \in \mathbb{N}$ *converges in measure to* $f : [0,1] \to \mathbb{R}$ *and*

$$\lim_{\mu(E)\to 0} \int_E f_n = 0$$

uniformly in $n \in \mathbb{N}$ *then* $f \in \mathcal{L}$ *and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

for every $E \in \Sigma$.

Using Corollary 8 we obtain from this the following.

Corollary 10. *If* $f_n : [0,1] \to \mathbb{R}$, $f_n \in \mathcal{L}$, $n \in \mathbb{N}$ *converges in measure to* $f : [0,1] \to \mathbb{R}$ *and the limit*

$$\lim_{n\to\infty} F_n(E) = \lim_{n\to\infty} \int_E f_n$$

exists for every $E \in \Sigma$ *then* $f \in \mathcal{L}$ *and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

for every $E \in \Sigma$.

Theorem 11. *Let* $f_n : [0,1] \to X$, $n \in N$ *be a sequence of McShane integrable functions and* $f : [0,1] \to X$. *Suppose that*

1) *for every* $x^* \in X^*$ *the limit* $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ *exists in measure in* $[0,1]$,

2) *the limit*

$$\lim_{n\to\infty} \int_E f_n(t) = x_E \in X$$

exists in X, *for the weak topology, for every measurable* $E \subset [0,1]$.

Then f *is Pettis integrable and* $(P)\int_0^1 f = x_{[0,1]}$.

Proof. Assume that $x^* \in X^*$ is arbitrary. Then for $x^*(f_n), x^*(f) : [0,1] \to \mathbb{R}$ we have by (a) in Theorem 5 with $x^*(f_n) \in \mathcal{L} = \mathcal{M}$ for $n \in \mathbb{N}$. By 1) and 2) the assumptions of Corollary 10 are satisfied and therefore $x^*(f) \in \mathcal{L}$, $x^*(f)$ is measurable and

$$\int_E x^*(f) = \lim_{n\to\infty} \int_E x^*(f_n) = \lim_{n\to\infty} x^*(\int_E f_n) = x^*(x_E)$$

for every $E \in \Sigma$. This yields the Pettis integrability of f and the theorem is proved.

A evident special case of Theorem 11 is the following.

Corollary 12. *Let $f_n : [0,1] \to X$, $n \in N$ be a sequence of McShane integrable functions and $f : [0,1] \to X$. Suppose that*

1) *for every $x^* \in X^*$ the limit* $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ *exists for every $t \in [0,1]$,*

2) *the limit*

$$\lim_{n\to\infty} \int_E f_n(t) = x_E \in X$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$.

Then f is Pettis integrable and $(P)\int_0^1 f = x_{[0,1]}$.

This Corollary shows that if the assumptions of Theorem B presented in the introduction are fulfilled, the function f is Pettis integrable.

Now a partial answer to the problem of Fremlin and Mendoza from [2] can be given using results concerning conditions under which a Pettis integrable function is also McShane integrable. Conditions of this type have been studied in [4] and [9]. We present them in the following statement.

Denote $B(X^*) = \{x^* \in X^*;\ \|x^*\| \leq 1\}$ the unit ball in X^*.

Theorem 13. *If $f : [0,1] \to X$ is Pettis integrable and one of the following conditions*

a) *f is measurable,*

b) *the Banach space X is separable,*

c) *the Banach space X is reflexive and X has the property (P), i.e., there exists a sequence $\{x_m^* \in B(X^*);\ m \in \mathbb{N}\}$ such that for every $x^* \in B(X^*)$ there exists a subsequence $\{x_k^* \in B(X^*);\ k \in \mathbb{N}\}$ of $\{x_m^* \in B(X^*);\ m \in \mathbb{N}\}$ such that*

$$x_k^*(x) \to x^*(x) \text{ forevery } x \in X \text{ if } k \to \infty,$$

holds, then f is McShane integrable.

Using Corollary 12 and Theorem 13 we obtain the following partial answer to the problem presented in the introduction to this paper.

Theorem 14. *Let $f_n : [0,1] \to X$, $n \in N$ be a sequence of McShane integrable functions and $f : [0,1] \to X$. Suppose that one of the conditions* a), b), c) *from Theorem 13 is satisfied and that*

1) *for every $x^* \in X^*$ the limit* $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ *exists for every $t \in [0,1]$,*

2) *the limit*

$$\lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$.

Then f is McShane integrable and $(M)\int_0^1 f = W - \lim_{n\to\infty} \int_{[0,1]} f_n(t)$.

4 A negative answer to the problem of Theorem B'

In the following, at first we will show an interesting example which is Pettis integrable and not McShane integrable. Then we will give a negative answer to the problem of Theorem B'.

Proposition 15. *The Theorem B' is not true.*

Let us look at the following example belonging to Prof. Di Piazza and L., Preiss, D..

Example 16. There exists a function $f; [0,1] \to X$, X is a Banach space, satisfying the following properties:

(1) f is Pettis integrable;

(2) f is not McShane integrable.

We take an example as the Example (CH) of [10] P.1184 which was given by Di Piazza, L. and Preiss, D. as following.

Let $X = l_\infty(\omega_1)$, where ω_1 is the first uncountable ordinal. Let $\{N_\alpha\}_{\alpha\in\omega_1}$ and $\{C_\alpha\}_{\alpha\in\omega_1}$ be two collections of subsets of $[0,1]$ satisfying the following properties:

(j_1) for each $\alpha \in \omega_1$, $\{N_\alpha\}_{\alpha\in\omega_1}$ is a set of zero Lebesgue measure'

(j_2) $N_\alpha \subset N_\beta$, if $\alpha < \beta$;

(j_3) every subset of $[0,1]$ of zero Lebesgue measure is contained in some set N_α;

(j_4) for each $\alpha \in \omega_1$, C_α is a countable set;

(j_5) $C_\alpha \subset C_\beta$, if $\alpha < \beta$;

(j_6) every countable subset of $[0,1]$ is contained in some set C_α.

Now define $f : [0,1] \to l_\infty(\omega_1)$ by

$$f(t)(\alpha) = \begin{cases} 1 & \text{if } t \in N_\alpha \setminus C_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the proof of the example (CH) in [10] that f is Pettis integrable and not McShane integrable. This means there exists a function f such that f is Pettis integrable and not McShane integrable.

Remark. Another more complicated example was given by Fremlin and Mendoza in [2]. Here we do not present it. For details, see the 3C example of [2] P.143.

In order to prove the Proposition 15, we first need well-known results.

Lemma 17. *If a function $f : [0,1] \mapsto X$ is a simple function, then f is McShane integrable.*

See [4].

Lemma 18. *If a function $f : [0,1] \mapsto X$ is Pettis integrable, the set $\{x^*f;\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable, i.e., $\lim_{\mu(E)\to 0} \int_E |x^*f| = 0$ uniformly in $x^* \in B(X^*)$.*

See the Proposition 17 in [9].

The following result is important and it is the Theorem 6 of [6].

Lemma 19. *The function $f : I_0 \mapsto X$ is Pettis integrable if and only if there is a sequence (f_n) of simple functions from I_0 into X such that*

(a) for each x^* in X^*, $\lim_{n\to\infty} x^*(f_n) = x^*(f)$ a.e. on I_0,

(b) the set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable.

Now we return our attention to the Proposition 15.

Proof of the Proposition 15. If the Theorem B' holds. This means that if f_n, $n \in N$ is a sequence of McShane integrable functions from $[0,1]$ to X, and suppose that the weak limit $f(t)$ of $f_n(t)$ exists in X for almost every $t \in [0,1]$. If moreover the limit

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$, then f is McShane integrable and $\int_0^1 f = F([0,1])$.

By the example 16, there is a function $f : [0,1] \to X$ such that f is Pettis integrable and not McShane integrable on $[0,1]$. So we choose the function f in the example 16, since f is Pettis integrable, by Lemma 19, there exists a sequence $\{f_n\}_n$ of simple functions f_n from $[0,1]$ to X such that

a) for every $x^* \in X^*$ $\lim_{n\to\infty} x^* f_n(t) = x^* f(t)$ exists for almost every $t \in [0,1]$;

b) the set $\{x^*(f_n);\ x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable.

By Theorem 9 and f is Pettis integrable, we have

2) the limit

$$F(E) = \lim_{n\to\infty} \int_E f_n(t)$$

exists in X, for the weak topology, for every measurable $E \subset [0,1]$ and $\int_0^1 f = F([0,1])$.

From Lemma 17 the simple function $f_n(t)$ is McShane integrable. So the sequence $\{f_n\}$ of McShane integrable functions satisfies the conditions of Theorem B'. Thus, if Theorem B' holds, f is McShane integrable. But f is not McShane integrable. This is a contradiction. Therefore, Theorem B' is not true.

Remark. a) The condition of the Theorem B' "the weak limit $f(t)$ of $f_n(t)$ exists in X for almost every $t \in [0,1]$" can be written as "for every $x^* \in X^*$ $\lim_{n\to\infty} x^* f_n(t) = x^* f(t)$ exists for each $t \in [0,1] - e$, here $e \subset [0,1]$ is a zero-measurable set and depends on x^* in X^*.

b) If the zero-measurable set $e \subset [0,1]$ doesn't depends on x^* in X^* the Theorem B' in fact is Theorem B. Because the integrability and integral of a function f over $[0,1]$ are invariant if we change its value on the set with measure zero.

Remark. D. H. Fremlin in [3] discussed the convergence theorem B' again, however, the example 16 shows it is not true.

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关于 Fremlin 与 Mendoza 的一个问题的否定回答

叶 国 菊

(河海大学理学院，南京，江苏， 210098 ，中国)

Stefan Schwabik

(捷克科学院数学研究所，布拉格，捷克)

摘　要：本文研究从 $[0,1]$ 到 Banach 空间 X 的抽象函数 McShane 积分的收敛性问题. 对 D.H.Fremlin 和 J. Mendoza 提出的一个公开问题给出了否定回答.

关键词：抽象函数； Pettis 积分； McShane 积分