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## INFERENCE IN CONDITIONAL PROBABILITY LOGIC

NIKI PFEIFER AND GERNOT D. KLEITER

An important field of probability logic is the investigation of inference rules that propagate point probabilities or, more generally, interval probabilities from premises to conclusions. Conditional probability logic (CPL) interprets the common sense expressions of the form “if . . . , then . . . ” by conditional probabilities and not by the probability of the material implication. An inference rule is *probabilistically informative* if the coherent probability interval of its conclusion is not necessarily equal to the unit interval  $[0, 1]$ . Not all logically valid inference rules are probabilistically informative and *vice versa*. The relationship between logically valid and probabilistically informative inference rules is discussed and illustrated by examples such as the MODUS PONENS or the AFFIRMING THE CONSEQUENT. We propose a method to evaluate the strength of CPL inference rules. Finally, an example of a proof is given that is purely based on CPL inference rules.

*Keywords:* probability logic, conditional, modus ponens, SYSTEM P

*AMS Subject Classification:* 68T37, 03B48, 03B65

## 1. INTRODUCTION

Following Hailperin [15], we distinguish verity logic and probability logic (PL). Verity logic is concerned with deductive inference and investigates the validity of arguments consisting of a set of premises and a conclusion, respectively. PL is an extension of verity logic that propagates probabilities from premises to conclusions. We argue that conditional probability,  $P(B|A)$ , is more appropriate than material implication,  $A \rightarrow B$ , to formalize common sense expressions of the form IF  $A$ , THEN  $B$ . First, we want to avoid the well known paradoxes of the material implication. Second, the material implication cannot deal with uncertainty and does not allow for exceptions, while the IF  $A$ , THEN  $B$  does.

Coherence is the most essential concept in the foundation of subjective probability theory. It was introduced by de Finetti [10]. In the framework of betting schemes a probability assessment is coherent if it guarantees the avoidance of sure losses. More formally, a probability assessment  $P$  on an arbitrary family  $\mathcal{F}$  of  $n$  conditional events,  $E_1|H_1, \dots, E_n|H_n$ , is coherent iff there is a probability space  $(\Omega, \mathcal{A}, P')$  with a Boolean algebra  $\mathcal{A}$ , such that  $\mathcal{F} \subseteq \mathcal{A}$  and  $P$  is a restriction of  $P'$  to  $\mathcal{F}$ . The probability assessment  $P$  can be point-valued,  $P = (p_1, \dots, p_n)$ , interval-valued,  $P = ([p'_1, p''_1], \dots, [p'_n, p''_n])$ , or mixed [9].

The motivating idea underlying the present paper is to use purely syntactical logical rules to produce deductively a proof, and then to propagate imprecise probabilities by using probability logical inference rules. This idea is not new. For related work see, for example, [1, 3, 9, 11, 12, 14, 15, 18]. By referring to coherence and to SYSTEM P [14, 16] the present paper is more systematic than earlier work like that of Frisch & Haddawy [12] who used an eclectic set of inference rules. Furthermore, coherence avoids problems with conditioning events that have zero probability. Some authors, [1, 12], suggest that  $P(B|A) = 1$ , if  $P(A) = 0$ . This is problematic since it follows that if  $P(A) = 0$ , then  $P(B|A) = 1 = P(\neg B|A)$ , which is incoherent of course, since  $P(B|A) + P(\neg B|A)$  should sum up to 1 [8, 14]. Examples of some inconsistencies in probabilistic approaches like [12] are discussed in [8].

Throughout the paper, we assume the probability assessments of the premises to be coherent. Usually, no stochastic independence assumptions are made. Upper case letters,  $A, B, C, \dots$ , are variables for atomic or compound sentences. If not stated otherwise, they are neither tautologies nor contradictions and are not logically related to each other.

## 2. CONDITIONAL PROBABILITY LOGIC

Assigning and processing probabilities of sentences containing negations, conjunctions, or disjunctions only, is straightforward. The assignment of probability to indicative common sense conditionals (IF  $A$ , THEN  $B$ ), though, raises problems. Interpreting IF  $A$ , THEN  $B$  conditionals in PL as probabilities of material implication (as defined in classical logic),

$$P(\text{IF } A \text{ THEN } B) = P(A \rightarrow B),$$

is problematic. First, the basic intuition that the commonsense expression of the form IF  $A$ , THEN  $B$  is a *conditional* (where something is *assumed*, indicated by phrases like “if”) but not a disjunction or negated conjunction gets lost, since

$$P(A \rightarrow B) = P(\neg A \vee B) = P(\neg(A \wedge \neg B)).$$

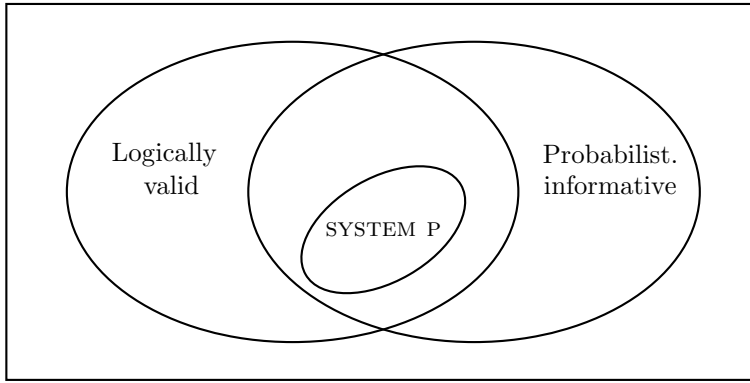
Compound sentences containing an IF  $A$ , THEN  $B$  cannot be re-written in disjunctive or conjunctive normal forms in the way it is done in the propositional calculus. Because of the special treatment of the conditional we call our approach *conditional probability logic*, CPL.

Second, and more important, interpreting the probability of the IF  $A$ , THEN  $B$  as the probability of a material implication would export the paradoxes of implication of verity logic to probability logic. For instance, consider the following list of obviously implausible but logically valid inference rules:

$$\neg A \quad \therefore \quad A \rightarrow B \tag{1}$$

$$B \quad \therefore \quad A \rightarrow B \tag{2}$$

$$A \rightarrow C \quad \therefore \quad A \wedge B \rightarrow C \tag{3}$$



**Fig. 1.** Deductive arguments can be logically valid or invalid and probabilistically informative or non-informative; arguments in SYSTEM P are both logically valid and probabilistically informative.

### 3. PROBABILISTIC INFORMATIVENESS AND VALIDITY

We call an inference rule *probabilistically informative* if the coherent probability interval of its conclusion is not necessarily equal to the unit interval  $[0, 1]$ . An inference rule is probabilistically non-informative, if the assignment of the unit interval to its conclusion is necessarily coherent. The premises of a probabilistically informative inference *inform* about the probability of the conclusion. Not all logically valid inference rules are probabilistically informative, and, *vice versa*, not all probabilistically informative inference rules are logically valid (see Figure 1). Typically, CPL involves *incomplete information* so that the probability of a sentence can be specified by a probability *interval* only, not by a point probability. We give examples of CPL inference rules below.

The paradoxes of the material implication (1), (2), (3) are probabilistically informative in a PL which interprets the IF  $A$ , THEN  $B$  as the probability of material implication, while probabilistically non-informative in CPL. This is an advantage of CPL, since the premises of a counterintuitive inference rule should provide no information about its conclusion. Let  $x'$  and  $x''$ ,  $0 \leq x' \leq x'' \leq 1$ , be the lower and upper probabilities of the premise, respectively. If  $P(\neg A) \in [x', x'']$  in the premise of (1) (or if  $P(B) \in [x', x'']$  in the premise of (2)), then

$$P(A \rightarrow B) \in [x', 1], \quad \text{while} \quad P(B|A) \in [0, 1]. \tag{4}$$

Likewise, (3) is probabilistically informative in the PL which interprets the IF  $A$ , THEN  $B$  as the probability of material implication, but (3) is non-informative in CPL (as desired),

$$P(A \rightarrow C) \in [x', x''] \quad \therefore \quad P(A \wedge B \rightarrow C) \in [x', 1], \text{ while} \tag{5}$$

$$P(C|A) \in [x', x''] \quad \therefore \quad P(C|A \wedge B) \in [0, 1]. \tag{6}$$

We therefore interpret the probability of the IF  $A$ , THEN  $B$  as a *conditional probability* [5, 6]:

$$P(\text{IF } A \text{ THEN } B) = P(B|A).$$

Thus, interpreting IF  $A$ , THEN  $B$  as a conditional probability is really different from interpreting IF  $A$ , THEN  $B$  as a probability of a material implication. As a consequence, the propagation of the probabilities from the premises to the conclusion differs substantially in both approaches. The propagation of the probabilities is driven by inference rules. We next discuss inference rules of CPL, such as probabilistic versions of MODUS PONENS. The Appendix gives an example of how to derive the inference rules, other examples are given in [15, 21, 22].

**Example 1.** The MODUS PONENS “if  $A \rightarrow B$  and  $A$ , then infer  $B$ ” is logically valid in verity logic, and probabilistically informative in CPL. The CPL version of the MODUS PONENS has the form:

$$P(B|A) \in [x', x''], P(A) \in [y', y''] \quad \therefore \quad P(B) \in [x'y', 1 - y' + x''y']. \quad (7)$$

**Example 2.** The MODUS TOLLENS “if  $A \rightarrow B$  and  $\neg B$ , then infer  $\neg A$ ” is logically valid in verity logic, and probabilistically informative in CPL. The CPL version of the MODUS TOLLENS has the form (adapted for interval values in the premises from [22, p. 751]):

$$P(B|A) \in [x', x''], P(\neg B) \in [y', y''], \quad \therefore \quad P(\neg A) \in [\max(z'_1, z'_2), 1], \text{ where,}$$

$$z'_1 = \begin{cases} \frac{1-x'-y''}{1-x'} & \text{if } x' + y'' \leq 1 \\ \frac{y''-x'}{x'} & \text{if } x' + y'' > 1, \end{cases} \quad (8)$$

$$z'_2 = \begin{cases} \frac{1-x''-y'}{1-x''} & \text{if } x'' + y' \leq 1 \\ \frac{x''+y'-1}{x''} & \text{if } x'' + y' > 1. \end{cases}$$

**Example 3.** The PREMISE STRENGTHENING “if  $A \rightarrow B$ , then infer  $A \wedge C \rightarrow B$ ” is logically valid in verity logic, but not probabilistically informative in CPL. The CPL version of the PREMISE STRENGTHENING has the form:

$$P(B|A) \in [x', x''] \quad \therefore \quad P(B|A \wedge C) \in [0, 1]. \quad (9)$$

The PREMISE STRENGTHENING reflects the *monotonicity* property of verity logic: a valid inference remains valid if further premises ( $C$ ) are added. Thus, in verity logic conclusions cannot be revised in the light of new evidence. Since common sense reasoning is nonmonotonic, the monotonicity property makes verity logic inappropriate for the formalization of common sense reasoning. PREMISE STRENGTHENING is probabilistically not informative in CPL, therefore CPL is appropriate for the formalization of nonmonotonic inferences, see [14].

**Example 4.** The HYPOTHETICAL SYLLOGISM (transitivity) “if  $A \rightarrow B$  and  $B \rightarrow C$ , then infer  $A \rightarrow C$ ” is logically valid in verity logic, but not probabilistically informative in CPL. The CPL version of the HYPOTHETICAL SYLLOGISM has the form:

$$P(B|A) \in [x', x''], P(C|B) \in [y', y''] \quad \therefore \quad P(C|A) \in [0, 1]. \quad (10)$$

**Example 5.** The AFFIRMING THE CONSEQUENT “if  $A \rightarrow B$  and  $B$ , then infer  $A$ ” is not logically valid in verity logic, but probabilistically informative in CPL. The CPL version of the AFFIRMING THE CONSEQUENT has the form:

If  $P(B|A) \neq P(B)$ , then

$$P(B|A) \in [x', x''], P(B) \in [y', y''] \quad \therefore \quad P(A) \in [0, \min(z''_1, z''_2)], \text{ where}$$

$$z''_1 = \begin{cases} \frac{1-y''}{1-x'} & \text{if } x' \leq y'' \\ \frac{y''}{x'} & \text{if } x' > y'' \end{cases} \quad (11)$$

$$z''_2 = \begin{cases} \frac{1-y'}{1-x''} & \text{if } x'' \leq y' \\ \frac{y'}{x''} & \text{if } x'' > y'. \end{cases}$$

In the special case if  $P(B|A) = P(B)$ , then  $P(A) = z' = z'' = 1$ .

**Example 6.** The DENYING THE ANTECEDENT “if  $A \rightarrow B$  and  $\neg A$ , then infer  $\neg B$ ” is not logically valid in verity logic, but probabilistically informative in CPL. The CPL version of the DENYING THE ANTECEDENT has the form:

$$P(B|A) \in [x', x''], P(\neg A) \in [y', y''] \quad \therefore \quad P(\neg B) \in [(1-x'')(1-y''), 1-x'(1-y'')]. \quad (12)$$

**Example 7.** The next inference scheme, “if  $A$ , then infer  $B$ ”, is neither logically valid in verity logic nor probabilistically informative in CPL:

$$P(A) \in [x', x''] \quad \therefore \quad P(B) \in [0, 1]. \quad (13)$$

Examples 1 and 2 are inference schemes that are both logically valid and probabilistically informative. Examples 3 and 4 are inference schemes that are both logically valid but not probabilistically informative. Example 5 and 6 are not logically valid but probabilistically informative. Finally, Example 7 is an inference scheme that is neither logically valid nor probabilistically informative. Thus, none of regions in Figure 1 is empty. The inference schemes that are in the intersection are of special interest since they share the desired properties logical validity and probabilistic informativeness. An example of a set of rules that are logically valid in verity logic and probabilistically informative in CPL is SYSTEM P [14], see below.

Important work on the probabilistic versions of MODUS PONENS and MODUS TOLLENS is contained in [15, 21, 22]. Second order probability distributions allow to generalize CPL by providing tools to express and reason from probability *distributions* instead of crisp lower and upper probability bounds [20]. For applications in psychology and probabilistic versions of AFFIRMING THE CONSEQUENT and DENYING THE ANTECEDENT see [19].

#### 4. STRENGTH OF PROBABILISTIC INFERENCE RULES

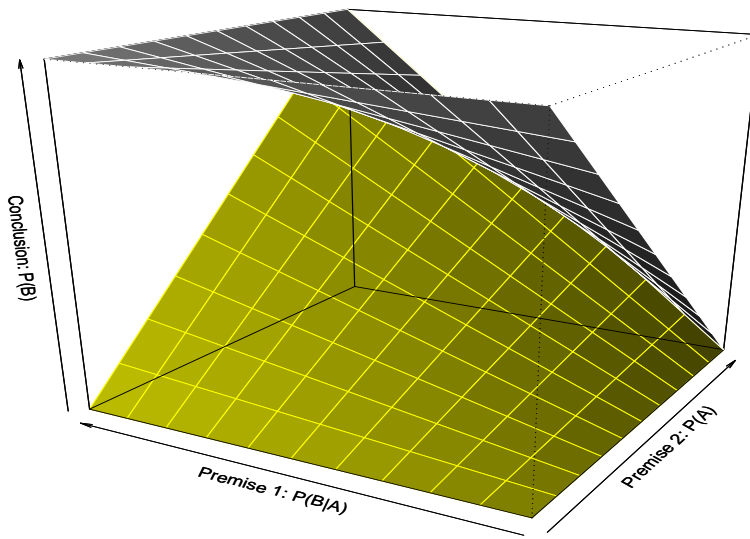
Upper and lower probabilities can be derived by (i) the Fourier–Motzkin elimination method for solving inequality systems, (ii) by linear programming methods, or (iii) by a system of inferences rules corresponding to a logical system. An example of such a logical system is SYSTEM P which is an important and broadly accepted system of nonmonotonic reasoning (see [16]). Its probabilistic interpretation has been given by Adams [1] and extended by Gilio [14]. Relationships between probabilistic logic under coherence in SYSTEM P and related systems, model-theoretic logic, and algorithmic aspects are investigated extensively in [3, 4, 18].

Each rule of SYSTEM P consists of a set of premises and a conclusion (see Section 5). If the conditional probability intervals of the premises are known, then the interval of the conclusion can be inferred.

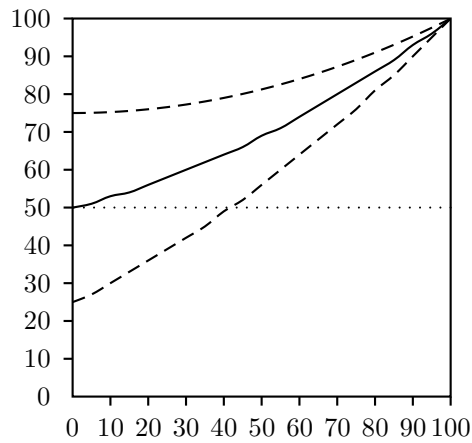
Figure 2 shows the relationship between the probabilities of the two premises and the conclusion for the MODUS PONENS in the three dimensions unit cube. The probabilities of the premises are plotted on the two axis at the bottom and the probability intervals for the conclusion are shown on the vertical axis. To each combination of the probabilities of the premises corresponds a point on the bottom square. The coherent upper and lower probabilities of the according conclusion are found on the two probability surfaces (bottom grid and top grid) “above” this point. For simplicity, the probabilities of the premises are assumed to be point probabilities.

The *strength* of an argument may be evaluated by the expected (mean) probability of its conclusion when the premises are more probable than a given value. Figures 3–6 show the mean probabilities together with their “confidence intervals”, the mean lower, and the mean upper probabilities, for the conclusions of four argument forms. The means are taken with respect to the probabilities of the premises in the range between  $x$  and 1. The means were calculated using uniform distributions of the premises probabilities. The values were determined numerically by averaging over a grid of  $1000 \times 1000$  values of the probabilities of the premises. Precise surfaces can be determined by double integrals. The figures show the difference between (i) the logically valid MODUS PONENS and the MODUS TOLLENS and (ii) the logically invalid AFFIRMING THE CONSEQUENT and the DENYING THE ANTECEDENT:

- In *logically valid* and probabilistically informative arguments (MODUS PONENS and MODUS TOLLENS) high probabilities in the premises give rise to highly probable conclusions with narrow confidence intervals. When the probabilities in the premises of these arguments are 1, then the probability of the according conclusion is also 1.
- When in *logically invalid* and probabilistically informative arguments (DENYING THE ANTECEDENT and AFFIRMING THE CONSEQUENT) the probabilities in the premises are 1, then the probability of the conclusion is in the interval between 0 and 1.

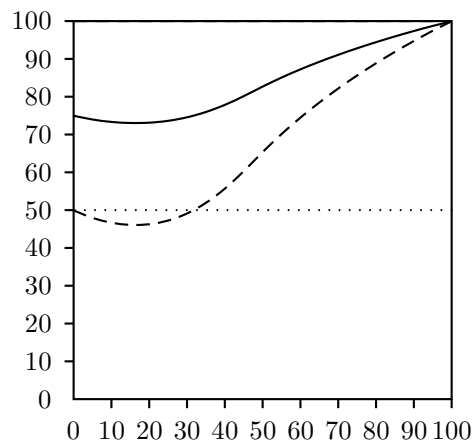


**Fig. 2.** Coherent intervals of the MODUS PONENS. The ordinates at the bottom indicate the probabilities of the premises,  $0 \leq P(B|A) \leq 1$  and  $0 \leq P(A) \leq 1$ . The coherent probabilities of the according conclusion  $0 \leq P(B) \leq 1$  are between the lower (bottom grid) and the upper (top grid) probability surfaces. The origin is at the front bottom corner.

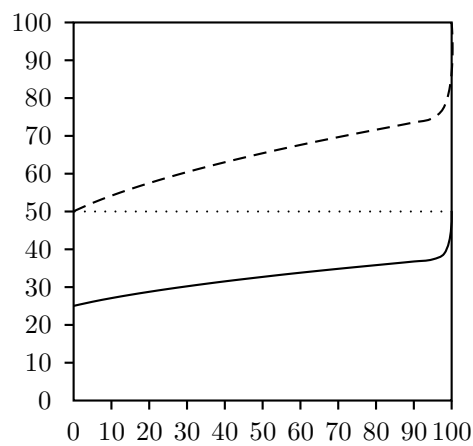


**Fig. 3.** MODUS PONENS. The mean probability of the conclusion  $P(B)$  (on the Y axis) if both premises are more probable than  $x$  (on the X axis),  $x \leq P(B|A) \leq 1$ ,  $x \leq P(A) \leq 1$ . The dashed lines give the mean lower and upper probabilities.

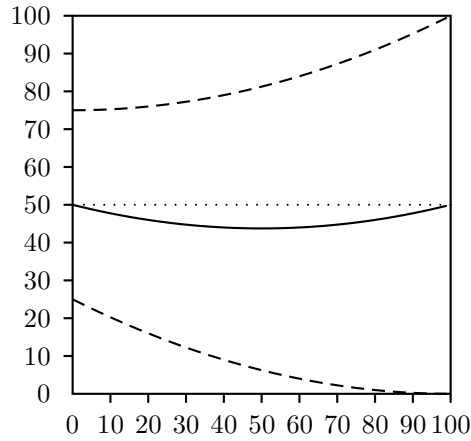




**Fig. 4.** MODUS TOLLENS. The mean probability of the conclusion  $P(\neg A)$  (on the Y axis) if both premises are more probable than  $x$  (on the X axis),  $x \leq P(B|A) \leq 1$ ,  $x \leq P(\neg B) \leq 1$ . The dashed line gives the mean lower probabilities, the upper probability is always 1.



**Fig. 5.** AFFIRMING THE CONSEQUENT. The mean probability of the conclusion  $P(A)$  (on the Y axis) if both premises are more probable than  $x$  (on the X axis),  $x \leq P(B|A) \leq 1$ ,  $x \leq P(B) \leq 1$ . The dashed line gives the mean upper probabilities, the lower probability is always 0.0.



**Fig. 6.** DENYING THE ANTECEDENT. The mean probability of the conclusion  $P(\neg B)$  (on the Y axis) if both premises are more probable than  $x$  (on the X axis),  $x \leq P(B|A) \leq 1, x \leq P(\neg A) \leq 1$ . The dashed lines give the mean lower and upper probabilities.

Both properties nicely correspond to our intuition. The mean conclusion probabilities of invalid arguments or probabilistically not informative arguments are close to 0.5 and the confidence intervals are wide. We note that if we know very little or “nothing” about the probability of our premises – we only know that they are more probable than zero – our conclusions obtain the guessing probability 0.5 in any argument form.

### 5. EXAMPLE OF A PROOF IN CPL

In this section we give an example of a proof in CPL. The CPL rules can produce proofs and explain by syntactical steps how conclusions are drawn. Table 1 shows the steps of a syntactical proof of the AND rule (from  $A \sim B$  and  $A \sim C$  infer  $A \sim B \wedge C$ ) by application of the rules of SYSTEM P. Table 2 shows the respective proof in CPL. For simplicity, the probabilities of the premises are assumed to be point probabilities. The generalization of the proof to interval probabilities in the premises is straightforward. Simple arithmetical transformations are omitted, the probabilistic interpretation of the SYSTEM P rules are based on coherence and can be found in [14].

The rules of SYSTEM P are (cf. [16]):

- REFLEXIVITY (axiom):  $\alpha \sim \alpha$
- LEFT LOGICAL EQUIVALENCE: from  $\models \alpha \leftrightarrow \beta$  and  $\alpha \sim \gamma$  infer  $\beta \sim \gamma$
- RIGHT WEAKENING: from  $\models \alpha \rightarrow \beta$  and  $\gamma \sim \alpha$  infer  $\gamma \sim \beta$

- OR: from  $\alpha \vdash \gamma$  and  $\beta \vdash \gamma$  infer  $\alpha \vee \beta \vdash \gamma$
- CUT: from  $\alpha \wedge \beta \vdash \gamma$  and  $\alpha \vdash \beta$  infer  $\alpha \vdash \gamma$
- CAUTIOUS MONOTONICITY: from  $\alpha \vdash \beta$  and  $\alpha \vdash \gamma$  infer  $\alpha \wedge \beta \vdash \gamma$

Proving the AND rule shows nicely the close analogy of the (syntactical) proof in SYSTEM P and the (semantical) proof in CPL. The steps of application of the inference rules are in both proofs analogous.

**Table 1.** Proof of the AND rule in SYSTEM P.

(1)	$A \vdash B$	(premise)
(2)	$A \vdash C$	(premise)
(3)	$\models A \wedge (B \wedge C) \rightarrow B \wedge C$	(tautology)
(4)	$A \wedge (B \wedge C) \vdash A \wedge (B \wedge C)$	(REFLEXIVITY)
(5)	$A \wedge (B \wedge C) \vdash B \wedge C$	(RW 1, 2)
(6)	$\models A \wedge (B \wedge C) \Leftrightarrow \models (A \wedge B) \wedge C$	(tautology)
(7)	$(A \wedge B) \wedge C \vdash B \wedge C$	(LLE 3, 4)
(8)	$A \wedge B \vdash C$	(CM 1, 2)
(9)	$A \wedge B \vdash B \wedge C$	(CUT 7, 8)
(10)	$A \vdash B \wedge C$	(CUT 9, 1)

**Table 2.** Proof of the AND rule in CPL. Compare the analogy to the proof in Table 1.

For the probability propagation rules of RW, LLE, CM, and CUT refer to [14].

(1)	$P(B A) = x$	(premise)
(2)	$P(C A) = y$	(premise)
(3)	$\models A \wedge (B \wedge C) \rightarrow B \wedge C$	(tautology)
(4)	$P(A \wedge (B \wedge C)   A \wedge (B \wedge C)) = 1$	(REFLEXIVITY)
(5)	$P(B \wedge C   A \wedge (B \wedge C)) = 1$	(RW 3, 4)
(6)	$\models A \wedge (B \wedge C) \Leftrightarrow \models (A \wedge B) \wedge C$	(tautology)
(7)	$P(B \wedge C   (A \wedge B) \wedge C) = 1$	(LLE 6, 5)
(8)	$P(C   A \wedge B) \in [\max(0, \frac{x+y-1}{x}), \min(\frac{y}{x}, 1)]$	(CM 1, 2)
(9)	$P(B \wedge C   A \wedge B)$ $\in [\max(0, \frac{x+y-1}{x}, \frac{y}{x}), 1]$	(CUT 7, 8)
(10)	$P(B \wedge C   A) \in [\max(0, x + y - 1), \min(x, y)]$	(CUT 9, 1)

## 6. FINAL REMARKS

The following list contains some final remarks, open problems and tentative answers:

- What are the conditions distinguishing probabilistically informative from non-informative arguments? Monotonicity, transitivity, and contraposition imply non-informativeness. Thus, it should be possible to find a proof for the following conjecture: Let  $\mathfrak{C}$  denote the conclusion of an CPL argument.

*For all arguments whose premises have probability 1:*

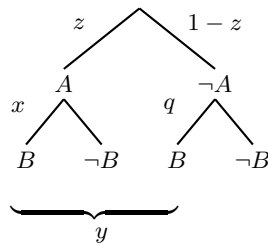
- *If the argument is probabilistically informative and logically valid, then  $P(\mathfrak{C}) = 1$  is coherent.*
  - *If the argument is probabilistically informative and not logically valid, then  $P(\mathfrak{C}) \in [0, 1]$  is coherent.*
  - *If the argument is probabilistically non-informative, then  $P(\mathfrak{C}) \in [0, 1]$  is coherent.*
- We observe that the MODUS PONENS and the DENYING THE ANTECEDENT differ only with respect to which sentences are affirmed, namely  $A$  and  $B$ , and which sentences are negated, namely  $\neg A$  and  $\neg B$ . In CPL this difference corresponds to the probabilities  $P(A)$  and  $P(B)$  and their complements  $1 - P(A)$  and  $1 - P(B)$ . The probabilities specified in the two argument forms imply lower and upper probabilities for the atoms  $\{A, B\}$ ,  $\{A, \neg B\}$ ,  $\{\neg A, B\}$  and  $\{\neg A, \neg B\}$ . A coherent probability assessment for the MODUS PONENS implies the existence of an assessment for the DENYING THE ANTECEDENT which leads to the same lower and upper probabilities of the atoms.
  - What are the relations between CPL and other logical systems (e. g. nonmonotonic reasoning)? All the rules of the nonmonotonic SYSTEM P are probabilistically informative [1, 14]. Conjectures: All the rules of most nonmonotonic systems that have a probabilistic interpretation are probabilistically informative. Every monotonic reasoning system has at least one probabilistically non-informative rule.

- What are the consequences of probabilistic conditional independence assumptions? Probabilistic conditional independence usually decreases uncertainty which results in tighter probability intervals. Consider, e. g., the AND rule,

$$P(B|A) \in [x', x''], P(C|A) \in [y', y''] \quad \therefore \quad P(B \wedge C|A) \in [z', z''],$$

where  $z' = \max(0, x' + y' - 1)$  and  $z'' = \min(x'', y'')$ . Under probabilistic conditional independence  $z'_{B \perp\!\!\!\perp C|A} = x'y'$  and  $z''_{B \perp\!\!\!\perp C|A} = x''y''$ .

- Which rules belong to a parsimonious set of CPL-rules?
- Are there efficient algorithms for the application of CPL-rules?
- How fast does the uncertainty increase or converge in the iterative application of CPL-rules?



**Fig. 7.** AFFIRMING THE CONSEQUENT. The precise probability assessments of the premises are  $P(B|A) = x$  and  $P(B) = y$ .  $P(B|\neg A) = q$  is not assessed and may have any value in the interval  $[0, 1]$ . The lower and upper bounds of the probability of the conclusion  $P(A) = z$  is derived in the text.

## APPENDIX

Finding the lower and upper probabilities for problems with two or three variables requires elementary algebra only. We derive, as an example, the boundary values for AFFIRMING THE CONSEQUENT. illustrates the notation. Because of the “missing” assessment of  $q$  the value of  $z$  can only be specified by an interval,  $0 \leq z' \leq z \leq z'' \leq 1$ .

By the theorem of total probability  $P(B) = P(A)P(B|A) + P(\neg A)P(B|\neg A)$  we have

$$y = zx + (1 - z)q.$$

Solving for  $z$  we obtain

$$z = \frac{y - q}{x - q}.$$

If  $q = 0$  and  $x = y$ , then  $z' = z'' = 1$ . If  $q = 0$  and  $x \neq y$ , then the minimum and maximum values of  $z$  result when  $q$  takes on the boundary values 0 or 1. In this case,  $x > y$  must hold (because if  $P(B|\neg A) = 0$ , then  $P(B) = P(A)P(B|A)$ ) and

$$z'' = \frac{y}{x}.$$

If  $q = 1$  and  $x = y$ , then again  $z' = z'' = 1$ . If  $q = 1$  and  $x \neq y$ , then  $x < y$  must hold (because if  $P(B|\neg A) = 1$ , then the theorem of total probability becomes  $P(B) = P(A)P(B|A) + P(\neg A)$  or  $P(B) - P(A)P(B|A) = P(\neg A)$  and as all probabilities are non-negative and between 0 and 1, we have  $0 \leq P(B|A) < P(B) \leq 1$ ) and

$$z'' = \frac{y - 1}{x - 1}.$$

We note that except for  $x = y$  the lower bound  $z'$  is not constrained and always 0.

If the assessment of the premises is not given in the form of point probabilities but in the form of intervals, then the various combinations of lower and upper

values of the intervals must be considered. The final results may be read-off more or less directly from the intervals already obtained for the precise probabilities of the premises.

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