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EQUIVALENT FUZZY SETS

Branimir Šešelja and Andreja Tepavčević

Necessary and sufficient conditions under which two fuzzy sets (in the most general, poset valued setting) with the same domain have equal families of cut sets are given. The corresponding equivalence relation on the related fuzzy power set is investigated. Relationship of poset valued fuzzy sets and fuzzy sets for which the co-domain is Dedekind–MacNeille completion of that posets is deduced.

Keywords: poset valued fuzzy set, cut, equivalent fuzzy sets, Dedekind–MacNeille completion

AMS Subject Classification: 03B52, 03E72, 06A15

1. INTRODUCTION

It is well known that fuzzy sets considered as functions are characterized by particular collections of crisp sets, known as cut sets. Among basic properties of fuzzy structures are cutworthy ones, i. e., those which are preserved under cuts. If fuzzy sets are endowed with some algebraic structure (group, ring, etc.), then the cuts are substructures (subgroups, subrings etc.). It is also known that different functions – fuzzy structures on the same domain can have equal collections of cut sets. Therefore, there were many attempts to investigate and somehow classify fuzzy structures on the same underlying set, which have equal collections cuts (see e. g., Murali and Makamba [8], Makamba [7], Alkhamees [1], Šešelja, Tepavčević [10, 11, 12, 15]; see also references in these). As it was pointed out in [10, 15], importance of the above mentioned classification is based on the following simple property of functions. There are uncountably many distinct functions (fuzzy sets) on the same (finite or infinite) domain and uncountable co-domain (e. g., the real interval $[0, 1]$). The same holds for algebraic structures: any group (even if it is of prim order) has uncountably many fuzzy subgroups. Not all of these functions (fuzzy sets, fuzzy subgroups) can be considered as essentially different and many of them have equal collections of cut sets. This leads to the natural classification of fuzzy structures by equality of cuts.

In the paper [15] the above equality of fuzzy sets was characterized for fuzzy sets whose domain is a (complete) lattice L. In the collection of all L-valued fuzzy sets on the same domain, necessary and sufficient conditions were given, under which two fuzzy sets from this collection have equal families of cuts. In particular, the notion of equality of fuzzy sets given in [8] was generalized.

In the present paper we consider the problem in its most general setting. Namely, we focus on the class of fuzzy sets as mappings from some fixed domain X into an arbitrary (also fixed) ordered set P. We present necessary and sufficient conditions under which fuzzy sets from the mentioned class have equal families of cut sets. Consequently, as special cases we deduce conditions concerning the corresponding equality for other kinds of fuzzy sets: lattice valued and real interval valued ones. It turns out that the solution of this problem is related to the well known Dedekind–MacNeille completion of ordered sets; this connection is also investigated and described.

Our investigation and results are formulated in terms of fuzzy sets, and not for fuzzy algebraic structures (like e. g., fuzzy groups). The reason is that the equality of cuts is essentially an order theoretic property. Therefore, our conditions are universal in the sense that they can easily, without changes, be formulated for any fuzzy algebraic or order-theoretic structure.

Notation and basic facts about L-valued fuzzy sets and lattices are given in Preliminaries; we refer also to the survey papers [13, 14].

2. PRELIMINARIES

We advance some definitions and notation concerning ordered structures. For more details, see e. g. [3].

If $(P, \leqslant)$ is a partially ordered set, poset, then infimum and supremum of $a, b \in P$ (if they exist) are denoted respectively by $a \wedge b$ and $a \vee b$. For imfimum or supremum of a subset or a family of elements of P, we use the notation $\bigwedge Q$, $\bigvee x_i$, and so on. For $a \in P$, we denote by $\downarrow a$ the *principal ideal generated by* a: $\downarrow a := \{x \in P \,|\, x \leqslant a\}$. Dually, a *principal filter generated by* a is defined by $\uparrow a := \{x \in P \,|\, a \leqslant x\}$. A poset in which every two-element subset has infimum and supremum is a *lattice*. A lattice L is *complete* if infimum and supremum exist for every subset of L.

Two posets $(P, \leqslant)$ and $(Q, \leq)$ are said to be *order isomorphic* if there is a bijection $f : P \to Q$ such that f and f^{-1} are isotone.

The *Dedekind–MacNeille completion* of a poset $(P, \leqslant)$ is a collection of subsets of P, defined by

$$DM(P) := \{X \subseteq P \,|\, X^{u\ell} = X\},$$

where for $X \subseteq P$,

$$X^u := \{y \in P \,|\, x \leqslant y, \text{ for every } x \in X\}$$

and

$$X^\ell := \{y \in P \,|\, y \leqslant x, \text{ for every } x \in X\}.$$

Obviously, by the consecutive application of the above two operators, we can get operators $u\ell$ and ℓu. Elementary properties of these operators are

$$X \subseteq X^{u\ell} \quad \text{and} \quad X \subseteq X^{\ell u}.$$

Let P be a poset and $Q \subseteq P$. Then Q is said to be *meet-dense* in P if for every $x \in P$ there is a subset R of Q such that $x = \bigwedge_P R$. A *join-dense* subset is defined dually.

We use some facts about DM-completions, given in the sequel.

Proposition 1. For any poset P, $DM(P)$ is a complete lattice under inclusion and P can be order embedded into $DM(P)$ by the map $\phi : x \mapsto{\downarrow}x$. In addition:

(i) $\phi(P)$ is both meet-dense and join-dense in $DM(P)$.

(ii) If P is a subset of a complete lattice L, in which it is both meet-dense and join-dense, then L is isomorphic with $DM(P)$ under the order-isomorphism whose restriction to P is ϕ.

Due to the order embedding ϕ, we sometimes consider P to be the sub-poset of $DM(P)$ (in the proof of Theorem 1 and further in the text).

Let $\mathcal{Z}$ be a collection of subsets of a nonempty set X satisfying:

(i) $\mathcal{Z}$ is closed under componentwise intersections, i. e., for every $x \in X$, $\bigcap(Y \in \mathcal{Z} \,|\, x \in Y) \in \mathcal{Z}$,

(ii) $\bigcup \mathcal{Z} = X$.

Then $\mathcal{Z}$ is called a *point closure system* on X.

Fuzzy sets

In this paper *fuzzy sets* are considered to be mappings from a non-empty set X (domain) into a poset P (co-domain). Special cases are obtained when P is a complete lattice or the unit interval $[0, 1]$ of real numbers. We sometimes use the term *P-fuzzy sets*, or *poset valued fuzzy sets*, but in majority of cases we simply refer to fuzzy sets.

If $\mu : X \to P$ is a fuzzy set on a set X then for $p \in P$, *p-cut*, or a *cut set*, (*cut*) is the well known subset μ_p of X:

$$\mu_p := \{x \in X \,|\, \mu(x) \geqslant p\}.$$

The collection of all cuts of μ is denoted by μ_P:

$$\mu_P := \{\mu_p \,|\, p \in P\}.$$

We are recalling some known properties of poset valued fuzzy sets (see e. g., [13, 14]).

Proposition 2. Let $\mu : X \to P$ be a fuzzy set on X. Then the collection $\mu_P = \{\mu_p \,|\, p \in P\}$ of cut subsets of μ is a point closure system on X under the set inclusion.

The image of the set X under μ is denoted as usual, by $\mu(X)$:

$$\mu(X) = \{p \in P \,|\, p = \mu(x), \quad \text{for some} \quad x \in X\}.$$

If a fuzzy set $\mu : X \to P$ is given on X, then we define the relation $\approx$ on P: for $p, q \in P$

$$p \approx q \text{ if and only if } \mu_p = \mu_q.$$

The relation $\approx$ is an equivalence on P, and it can be characterized as follows.

Proposition 3. If μ is a fuzzy set on X and $p, q \in P$, then

$$p \approx q \quad \text{if and only if} \quad \uparrow p \cap \mu(X) = \uparrow q \cap \mu(X).$$

The relation $\leq$ in the poset P induces an order on the set of equivalence classes modulo $\approx$, i. e., on $P/\approx$, in the following way: for p, $q \in L$, let

$$[p]_\approx \leq [q]_\approx \quad \text{if and only if} \quad \uparrow q \cap \mu(X) \subseteq \uparrow p \cap \mu(X). \tag{1}$$

The above relation $\leqslant$ is an ordering relation on $P/\approx$. This order is anti-isomorphic with the set inclusion among cut sets of μ, as follows.

Proposition 4. If μ is an P-fuzzy set on X, then:

$$[p]_\approx \leqslant [q]_\approx \quad \text{if and only if} \quad \mu_q \subseteq \mu_p.$$

3. RESULTS

Let P be a poset, X a nonempty set, and $\mathcal{F}_P(X)$ the collection of all fuzzy sets on X whose co-domain is P.

In terms of ordered sets and functions $\mathcal{F}_P(X)$ is the power denoted usually by P^X. This set can be ordered naturally, the order being induced by the one from the poset P:

$$\mu \leqslant \nu \quad \text{if and only if for each} \quad x \in X \quad \mu(x) \leqslant \nu(x).$$

The cardinality of the power $\mathcal{F}_P(X)$ depends on the cardinality of the poset P; if it is an uncountable poset (like the unit interval $[0, 1]$), then also there are uncountably many functions – fuzzy sets on X.

Recall that μ_P denotes the collection of all cuts of a fuzzy set $\mu \in \mathcal{F}_P(X)$:

$$\mu_P = \{\mu_p \,|\, p \in P\}.$$

Our aim is to find conditions under which different fuzzy sets (as functions) have equal these collections.

We begin with a necessary condition for the foregoing equality. It connects fuzzy sets having a poset for the domain with those for which the domain is a lattice.

Let $\mu : X \to P$ be a fuzzy set and let $L = DM(P)$ be a lattice which is the Dedekind–MacNeille completion of P (recall that this lattice consists of particular subsets of P, ordered by set inclusion). Then, we define a fuzzy set $\mu^{DM(P)} : X \to DM(P)$, where

$$\mu^{DM(P)}(x) = \downarrow (\mu(x)).$$

Proposition 5. If fuzzy sets $\mu : X \to P$ and $\nu : X \to P$ have equal families of cut sets then also families of cut sets of fuzzy sets $\mu^{DM(P)}$ and $\nu^{DM(P)}$ coincide.

P r o o f. First we note that cuts μ_p in μ and $\mu^{DM(P)}_{\phi(p)}$ in $\mu^{DM(P)}$ coincide (recall that $\phi : x \mapsto {\downarrow} x$, as in Proposition 1). Indeed, $x \in \mu_p$ if and only if $\mu(x) \geqslant p$ if and only if $\phi(\mu(x)) \geqslant \phi(p)$ if and only if $\mu^{DM(P)}(x) \geqslant \phi(p)$ if and only if $x \in \mu^{DM(P)}_{\phi(p)}$.

Suppose that μ and ν have equal families of cut sets. Let $p \in DM(P)$. Then, $p = \bigvee_{i\in I} x_i$, where $\{x_i \,|\, i \in I\}$ is a family of images of elements of P under ϕ (this is because of the density of $\phi(P)$ in $DM(P)$, see Proposition 1). Now,

$$\mu^{DM(P)}_p = \mu^{DM(P)}_{\vee x_i} = \bigcap(\mu^{DM(P)}_{x_i} \,|\, i \in I) = \bigcap(\nu^{DM(P)}_{y_i} \,|\, i \in I) = \nu^{DM(P)}_{\vee y_i} = \nu^{DM(P)}_p,$$

where for each $\mu^{DM(P)}_{x_i}$, $\nu^{DM(P)}_{y_i}$ is the corresponding (equal) cut in fuzzy set $\mu^{DM(P)}$. □

The converse of this statement is not valid, which is illustrated by the following example.

Example 1. Let P be the poset in Figure 1 and μ and ν fuzzy sets on $X = \{x, y, z, t\}$ defined by:

$$\mu = \begin{pmatrix} x & y & z & t \\ c & d & g & h \end{pmatrix} \qquad \nu = \begin{pmatrix} x & y & z & t \\ a & b & e & f \end{pmatrix}$$

These fuzzy sets do not have equal cuts, more precisely, $\{c, d\}$ is a cut set of μ but not of ν.

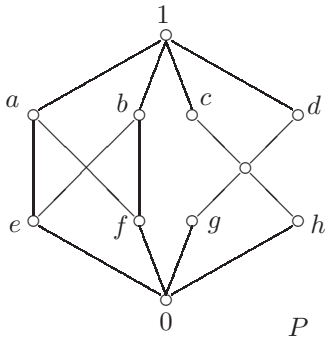
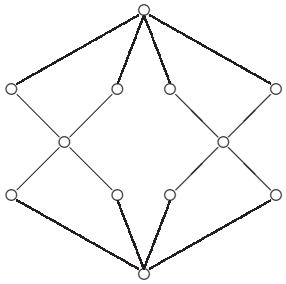


Fig. 1.

On the other hand, when we consider the extensions of these fuzzy sets to Dedekind–MacNeille completion of the poset P (the lattice $DM(P)$ is given in Figure 2), then the corresponding lattice valued fuzzy sets have equal families of cut sets, which can be easily checked.



$DM(P)$

Fig. 2.

Starting with a fuzzy set μ from $\mathcal{F}_P(X)$, we define a special poset ordered by set inclusion, whose elements are certain subsets of the set of all images of μ.

For $\mu \in \mathcal{F}_P(X)$, let

$$P_\mu := (\{\uparrow p \cap \mu(X) \,|\, p \in P\}, \subseteq).$$

In the following, the above collection is considered as a poset ordered by inclusion. Recall that μ_P denotes the collection of cut set of μ:

$$\mu_P := \{\mu_p \,|\, p \in P\}.$$

This poset is also ordered by set inclusion.

Proposition 6. If $\mu : X \to P$ is a fuzzy set on X, then there exists an order isomorphism from the poset P_μ to the poset μ_P of cuts of μ.

P r o o f. The function $f : \mu_p \mapsto \uparrow p \cap \mu(X)$ maps the collection μ_P of cuts of μ onto the poset P_μ. By the definition of the relation $\approx$ on P and by Proposition 3, f is a bijection. It is straightforward to prove that f and its inverse preserve the order (set inclusion), so f is an order isomorphism. □

Next we introduce our main definition by which we can classify fuzzy sets in the collection $\mathcal{F}_P(X)$.

Definition. Let $\sim$ be the relation on $\mathcal{F}_P(X)$, defined as follows: $\mu \sim \nu$ if and only if the correspondence $f : \mu(x) \mapsto \nu(x)$, $x \in X$ is a bijection from $\mu(X)$ onto $\nu(X)$ which has an extension to an isomorphism from P_μ onto the poset P_ν, given by the map

$$F(\uparrow p \cap \mu(X)) := \{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X), \quad p \in P. \qquad (*)$$

Remark. Within this definition we implicitly suppose that $\{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X)$ belongs to P_ν. Therefore, for every $p \in P$, there is an element $q \in P$, such that $\{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X) = \uparrow q \cap \nu(X)$.

Lemma 1. Let $\mu, \nu \in \mathcal{F}_P(X)$ and for $p \in P$, let $F(\uparrow p \cap \mu(X)) = \uparrow q \cap \nu(X)$. Now, if $\mu \sim \nu$, then for any $y \in X$,

$$\mu(y) \geqslant p \quad \text{if and only if} \quad \nu(y) \geqslant q.$$

P r o o f. By the definition of F, $\{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X) = \uparrow q \cap \nu(X)$.

Suppose that $\mu(y) \geqslant p$ for some $y \in X$. Since $\{\nu(x) \,|\, \mu(x) \geqslant p\} \cap \nu(X) \subseteq \{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X) = \uparrow q \cap \nu(X)$, we have that $\nu(y) \geqslant q$.

Conversely, let $\nu(y) \geqslant q$. By the definition, F is an extension of the bijection $f : \mu(x) \to \nu(x)$, i. e., extension of the mapping $\uparrow \mu(x) \cap \mu(X) \mapsto \uparrow \nu(x) \cap \nu(X)$. Since $\nu(y) \in \uparrow q \cap \nu(X)$, we have that $\uparrow \nu(y) \cap \nu(X) \subseteq \uparrow q \cap \nu(X)$. By the fact that F is an isomorphism, $\uparrow \mu(y) \cap \mu(X) \subseteq \uparrow p \cap \mu(X)$, therefore $\mu(y) \in \uparrow p \cap \mu(X)$, and hence $\mu(y) \geqslant p$. □

Theorem 1. The relation $\sim$ is an equivalence relation on $\mathcal{F}_P(X)$.

P r o o f. Reflexivity of $\sim$ holds by virtue of the identity map on P_μ.

In the sequel, symmetry of $\sim$ is proved. If $\mu \sim \nu$, then F is an isomorphism and

$$F(\uparrow p \cap \mu(X)) = \{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X) = \uparrow q \cap \nu(X).$$

Now, F^{-1} is also the isomorphism from P_ν to P_μ and we have to prove that

$$F^{-1}(\uparrow q \cap \nu(X)) = \{\mu(x) \,|\, \nu(x) \geqslant q\}^{\ell u} \cap \mu(X) \; (= \uparrow p \cap \mu(X)).$$

We consider the infimum of all elements $\mu(x)$ for which $\nu(x) \geqslant q$ in the Dedekind–MacNeille completion of P. This completion is a complete lattice, and the following is true:

$$\uparrow \bigwedge_{DM(P)} \{\mu(x) \,|\, \nu(x) \geqslant q\} \cap \mu(X) = \{\mu(x) \,|\, \nu(x) \geqslant q\}^{\ell u} \cap \mu(X).$$

Now, since for every $\mu(x)$ such that $\nu(x) \geqslant q$, we have that $\mu(x) \geqslant p$ (by Lemma 1), it follows that $\bigwedge_{DM(P)} \{\mu(x) \,|\, \nu(x) \geqslant q\} \geqslant p$, and hence $\uparrow \bigwedge_{DM(P)} \{\mu(x) \,|\, \nu(x) \geqslant q\} \cap \mu(X) \subseteq \uparrow p \cap \mu(X)$.

To prove the other inclusion, let $\mu(y) \geqslant p$ (i. e., $\mu(y) \in \uparrow p \cap \mu(X)$). Then, by Lemma 1, $\nu(y) \geqslant q$. Since $\{\mu(x) \,|\, \nu(x) \geqslant q\} \subseteq \{\mu(x) \,|\, \nu(x) \geqslant q\}^{\ell u}$, we have that $\mu(y) \in \{\mu(x) \,|\, \nu(x) \geqslant q\}^{\ell u} \cap \mu(X)$, i. e., $\uparrow p \cap \mu(X) \subseteq \{\mu(x) \,|\, \nu(x) \geqslant q\}^{\ell u} \cap \mu(X)$, which proves that $\sim$ is symmetric.

Finally, suppose $\mu \sim \nu$ and $\nu \sim \rho$, and F and G are the corresponding isomorphisms from P_μ to P_ν and from P_ν to P_ρ, respectively. Directly by the definition of F and G, it follows that $F \circ G$ is the isomorphism by which $\mu \sim \rho$. Therefore, $\sim$ is transitive. □

We say that fuzzy sets $\mu, \nu \in \mathcal{F}_P(X)$ are *equivalent* if $\mu \sim \nu$.

If the poset P is replaced by a complete lattice L, then the above definition of equivalent fuzzy sets coincides with the definition of equivalent lattice valued fuzzy sets from [15]:

$$F(\uparrow p \cap \mu(X)) = \uparrow \bigwedge \{\nu(x) \,|\, \mu(x) \geqslant p\} \cap \nu(X), \quad p \in L. \tag{$*'$}$$

Indeed, in any lattice we have

$$\{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} = \bigwedge \{\nu(x) \,|\, \mu(x) \geqslant p\},$$

hence in this case the last formula is equivalent with the formula $(*)$.

In particular, if we consider classical fuzzy sets with co-domain $[0,1]$, then the isomorphism $(*)$ has the form

$$F([p,1] \cap \mu(X)) = [\inf\{\nu(x) \,|\, \mu(x) \geqslant p\}, 1] \cap \nu(X), \quad p \in [0,1]. \tag{$*''$}$$

Obviously, definition $(*)$ is the most general.

Recall that for equivalent fuzzy sets μ and ν, the correspondence $f : \mu(x) \mapsto \nu(x)$ for $x \in X$, is by definition a bijection from $\mu(X)$ to $\nu(X)$. These sets of images are ordered subsets of P and we prove that they are order isomorphic.

Theorem 2. Let $\mu, \nu \in \mathcal{F}_P(X)$ and $\mu \sim \nu$. Then for all $x, y \in X$

$$\mu(x) \leqslant \mu(y) \quad \text{if and only if} \quad \nu(x) \leqslant \nu(y). \tag{$**$}$$

P r o o f. Let $\mu(x) \leqslant \mu(y)$. Then $\uparrow\mu(y) \subseteq \uparrow\mu(x)$, hence $\uparrow\mu(y) \cap \mu(X) \subseteq \uparrow\mu(x) \cap \mu(X)$ and finally, since $\mu \sim \nu$, we have $\uparrow\nu(y) \cap \nu(X) \subseteq \uparrow\nu(x) \cap \nu(X)$. Therefore $\nu(x) \leqslant \nu(y)$. By the symmetry of $\sim$, the opposite implication is also satisfied, and we are done. □

Remark. In the paper [8] condition $(**)$ and the notion of equivalent fuzzy sets were used to characterize fuzzy sets with equal supports and only finite number of values in the interval $[0,1]$.

Now we are able to prove our main result, namely that equivalent fuzzy sets have equal families of cuts, which justifies the used terminology.

Theorem 3. Let $\mu, \nu : X \to L$. Then $\mu \sim \nu$ if and only if fuzzy sets μ and ν have equal families of cuts.

P r o o f. Let $\mu \sim \nu$, and let $p \in P$. We prove that for every $p \in L$ there is $q \in L$ such that $\mu_p = \nu_q$.

Since $F(\uparrow p \cap \mu(X)) := \{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X)$, is a bijection from P_μ to P_ν, there is an element $q \in P$, such that $F(\uparrow p \cap \mu(X)) = \uparrow q \cap \nu(X)$. By Lemma 1, for p and q, we have that for all $y \in X$, $\mu(y) \geqslant p$ if and only if $\nu(y) \geqslant q$. Hence, $\mu_p = \nu_q$.

By symmetry of $\sim$, we get the same result if we start from q.

To prove the converse, suppose that the families of cuts of μ and ν are equal. Then, for every $p \in P$ there is $q \in P$, such that $\mu_p = \nu_q$ and vice-versa.

Now, we consider posets P_μ and P_ν. They consist of the sets $\uparrow p \cap \mu(X)$ and $\uparrow p \cap \nu(X)$ $(p \in P)$, respectively and they are order isomorphic with the lattices μ_P of cuts of μ and ν_P of cuts of ν (by Proposition 6). By the assumption, $\mu_P = \nu_P$, therefore, P_μ is isomorphic with the P_ν. If $\mu_p = \nu_q$, then $\uparrow p \cap \mu(X) \mapsto \uparrow q \cap \mu(X)$ is the required isomorphism by the proof of Proposition 6. Now, we have to prove that the isomorphism is indeed the F from definition (*). By Proposition 5, families of cuts of lattice valued fuzzy sets $\mu^{DM(P)}$ and $\nu^{DM(P)}$ are the same. By Theorem 1 from [15], mapping $F(\uparrow p \cap \mu(X)) := \uparrow \bigwedge_{DM(P)} \{\nu(x) \,|\, \mu(x) \geqslant p\} \cap \nu(X)$, $p \in DM(P)$ is an isomorphism from the lattice $DM(P)_\mu$ onto the lattice $DM(P)_\nu$. This mapping is the extension of the mapping F from P_μ to P_ν, since images of fuzzy sets in both cases belong to P, and mapping is given by $F(\uparrow p \cap \mu(X)) = \uparrow q \cap \nu(X)$, for q for which $\mu_p = \nu_q$. The theorem is now proved by $\uparrow \bigwedge_{DM(P)} \{\nu(x) \,|\, \mu(x) \geqslant p\} \cap \nu(X) = \{\nu(x) \,|\, \mu(x) \geqslant p\}^{\ell u} \cap \nu(X)$. □

In the case of fuzzy sets with finite number of values in the interval $[0, 1]$, condition $(**)$ from Theorem 2 is also sufficient in order that μ and ν are equivalent (see [8]), i. e., that they have equal collections of cut sets. The counter-example for the general case is given in Example 2.

Example 2. Let p be a positive real number less then 1, and X the set of numbers from p to 1:

$$p \in \mathbb{R},\ 0 < p < 1, \quad X = [p, 1].$$

Consider fuzzy sets $\mu, \nu : X \to [0, 1]$, defined as follows: for every $x \in X$,

$$\begin{aligned} \mu(x) &:= \begin{cases} x, & x > p \\ 0, & x = p, \end{cases} \\ \nu(x) &:= x. \end{aligned}$$

Now it is easy to check that the condition $(**)$ is fulfilled: for all $x, y \in X$,

$$\mu(x) \leqslant \mu(y) \quad \text{if and only if} \quad \nu(x) \leqslant \nu(y).$$

However, μ and ν do not have equal collections of cut sets. Indeed, observe the p-cut of μ (recall that p is the smallest real number in the set X):

$$\mu_p = \{x \in X \,|\, \mu(x) \geqslant p\} = (p, 1],$$

but there is no $q \in [0, 1]$ such that $\nu_q = \mu_p$.

The reason for the difference of cuts is that μ and ν are not equivalent in the sense of our definition. Indeed, we have

$$\mu(X) = \{0\} \cup (p, 1], \quad \nu(X) = [p, 1].$$

Hence

$$F([0,1]\cap\mu(X)) = F(\{0\}\cup(p,1]) = [\inf\{\nu(x)\,|\,\mu(x)\geqslant 0\},1]\cap\nu(X) = [p,1],$$

and similarly

$$F([p,1]\cap\mu(X)) = F((p,1]) = [p,1].$$

Therefore, F is not injective, hence it is not an isomorphism. □

Our last example illustrates the definition and properties of equivalent fuzzy sets whose co-domain is a poset which is not a lattice.

Example 3. Consider a three-element set $X = \{x, y, z\}$ and a seven element poset P given by its diagram in Figure 3.

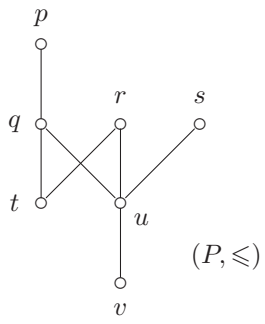


Fig. 3.

Let μ, ν and ρ be fuzzy sets as functions from X to P, defined as follows:

$$\mu = \begin{pmatrix} x & y & z \\ q & t & s \end{pmatrix}, \qquad \nu = \begin{pmatrix} x & y & z \\ r & t & s \end{pmatrix}, \qquad \rho = \begin{pmatrix} x & y & z \\ t & s & v \end{pmatrix}$$

The sets of images for these fuzzy sets are isomorphic order subsets of P (Figure 4), i. e., each pair of these fulfils the condition $(**)$.

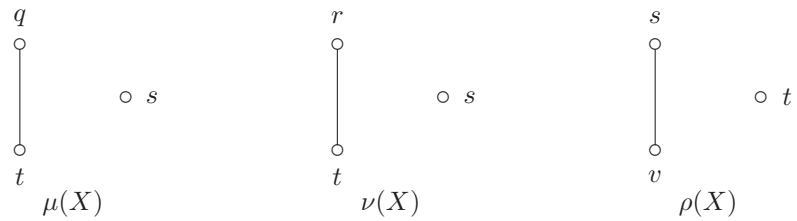


Fig. 4.

Still, only fuzzy sets μ and ν are equivalent in the sense of our definition. It means that posets P_μ and P_ν are isomorphic, and consequently collections of cut sets coincide (Figure 5).

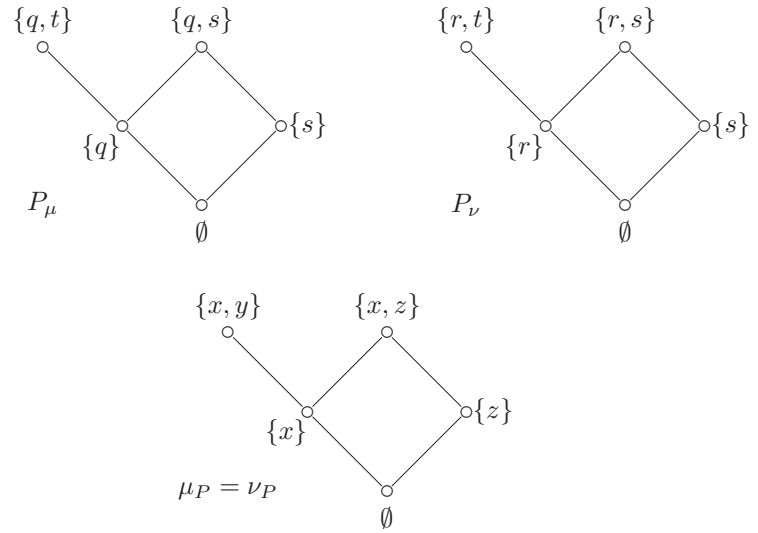


Fig. 5.

Fuzzy set ρ is not equivalent with the previous two, since its poset P_ρ is not isomorphic with the above ones; hence also its poset of cut set does not coincide with theirs (Figure 6).

Fig. 6.

4. CUTS OF LR FUZZY QUANTITIES

In this part the results of the previous sections are applied to a special type of fuzzy sets – LR fuzzy quantities.

We consider fuzzy quantities (fuzzy real numbers) as mappings $\mu : \mathbb{R} \to [0, 1]$, from set of reals to [0,1], such that there is exactly one $a \in \mathbb{R}$, such that $\mu(a) = 1$ and all the cut sets are intervals.

LR-representation of a fuzzy number $\mu : \mathbb{R} \to [0, 1]$ (where $\mu(a) = 1$ for $a \in \mathbb{R}$) is an ordered pair of functions (μ_L, μ_R), where μ_L is a monotonously nondecreasing function from $(-\infty, a)$ to [0,1] and μ_R is a monotonously nonincreasing function

from (a, ∞) to [0,1]. We consider functions μ_L and μ_R as fuzzy sets on sets $(-\infty, a)$ and (a, ∞), respectively.

Obviously, Theorem 3 is true for the case of fuzzy quantities and it can be easily re-formulated in this setting.

However, one can raise a question whether it is possible to simplify conditions from Theorem 3 and re-formulate the conditions using the fuzzy sets μ_L and μ_R instead of μ. The following proposition is easy to verify:

Proposition 7. Let μ and ν be two fuzzy real numbers in LR representation. If μ and ν are equivalent, then $\mu(a) = 1$ if and only if $\nu(a) = 1$ and the related fuzzy sets μ_L and ν_L and μ_R and ν_R are (in pairs) equivalent.

The converse of this theorem is not true, which is illustrated in the following example.

Example 4. Let μ and ν be fuzzy numbers, defined by:

$$\mu(x) := \begin{cases} 0, & x \leq 0 \\ 0.5x, & x \in (0,2) \\ 1, & x = 2 \\ -0.5x + 2, & x \in (2,4) \\ 0, & x \geq 4, \end{cases}$$

$$\nu(x) := \begin{cases} 0, & x \leq 0 \\ x, & x \in (0,0.5) \\ \frac{1}{3}x + \frac{1}{3}, & x \in (0.5,2) \\ 1, & x = 2 \\ -0.5x + 2, & x \in (2,4) \\ 0, & x \geq 4. \end{cases}$$

Condition $\mu(a) = 1$ if and only if $\nu(a) = 1$ for all $a \in R$ is satisfied.

Moreover, fuzzy set μ_L is equivalent with ν_L and fuzzy set μ_R is equivalent with ν_R.

However, the conditions of Theorem 3 are not satisfied and fuzzy sets μ and ν are not equivalent.

5. CONCLUSION

The above results completely characterize fuzzy sets according to the equality of collections of cuts. As it is pointed out, this characterization is essentially set and order theoretic. Therefore it can be used to classify e. g., all fuzzy algebraic and ordered structures, fuzzy topologies, fuzzy relations, graphs etc.

According to some general approach to cut sets (see e. g., Liu and Luo [6]), some further investigation could lead to analogue result concerning different generalizations of cut sets, or to some classification of cutworthy properties of fuzzy systems.

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