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STATES ON PSEUDO–EFFECT ALGEBRAS WITH GENERAL COMPARABILITY[1]

Anatolij Dvurečenskij

Pseudo-effect algebras are partial algebras $(E;+,0,1)$ with a partially defined addition $+$ which is not necessarily commutative and therefore with two complements, left and right ones. General comparability allows to compare elements of E in some intervals with Boolean ends. Such an algebra is always a pseudo MV-algebra. We show that it admits a state, and we describe the state space from the topological point of view. We prove that every pseudo-effect algebra is in fact a pseudo MV-algebra which is a subdirect product of linearly ordered pseudo-MV-algebras. In addition, we present many illustrating examples.

Keywords: Pseudo-effect algebra, pseudo MV-algebra, general comparability, state, ideal, representable pseudo MV-algebra

AMS Subject Classification: 6D35, 03G12, 03B50

1. INTRODUCTION

Recently there appeared a whole hierarchy of non-commutative generalizations of MV-algebras: pseudo MV-algebras [17, 23] (as generalized MV-algebras) (they are always intervals in unital ℓ-groups, [8]), pseudo BL-algebras [5]. These algebras are algebraic non-commutative generalizations of non-commutative reasoning. Non-commutative reasoning becomes now a new tool of the logical investigation, see e. g. [20]. Also in the every-day life and in many psychological processes we can find non-commutative reasoning. On the other hand, nowadays there is even a programming language [1] based on a non-commutative logic.

Recently in [14] and [15], we have introduced pseudo-effect algebras which generalize both pseudo MV-algebras and quantum structures like effect algebras (for more details on quantum structures see [13]). In [14, 15], we have proved that every pseudo-effect algebra satisfying a special kind of the Riesz decomposition property is always an interval in a unital po-group (G,u) which is not necessary Abelian.

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States on MV-algebras were introduced in [3] and [22] with the intent of capturing the notion of "average degree of truth" of a proposition. In [7], we have showed that in contrast to MV-algebras, there are pseudo MV-algebras which have no state, an analogue of a probability measure. Therefore, it is of great interest to study situations when pseudo-effect algebras admit a state, and such a tool in the present paper is the study of general comparability.

Central elements of a pseudo-effect algebra E were introduced in [10] as elements $e \in E$ such that $E \cong [0,e] \times [0,e']$. Such elements form always a Boolean algebra called a center. General comparability allows roughly speaking to compare two elements $x, y \in E$ in the intervals $[0,e]$ and $[0,e']$. It implies that this pseudo-effect algebra is automatically pseudo MV-algebra.

The paper is organized as follows. In Section 2, we introduce elements of pseudo-effect algebras and pseudo MV-algebras. In Section 3, we present ideals, the hull-kernel topology of the system of maximal ideals which are also normal. In Section 4, we show that every pseudo-effect algebra satisfying general comparability has at least one state, and moreover, every extremal state on the center can be extended to a unique state on E which is also extremal. In addition, the weak topology of extremal states is homeomorphic with the topology of the center, i. e., the space is a compact, Hausdorff and totally disconnected non-void set.

In Section 5, we show that every maximal ideal of a pseudo-effect algebra satisfying general comparability is always normal. This is interesting meanwhile in every MV-algebra each maximal ideal is normal, there are MV-algebras where general comparability fail. We study here some topological properties of the weak topology of states, and we describe the faces. In addition, we extend the results also for pseudo MV-algebras which not necessarily satisfy general comparability, but every maximal ideal from the center generates a prime ideal in the pseudo MV-algebra.

In Section 6, we show that every pseudo-effect algebra satisfying general comparability is a pseudo MV-algebra which is a subdirect product of linearly ordered pseudo-effect algebras.

In Section 7, we show that any pseudo-effect algebra satisfying general comparability has a functional representation by continuous functions defined on a totally disconnected, compact, Hausdorff topological space.

Finally, in Section 8, we present examples of MV-algebras which satisfy general comparability, or do not satisfy. We study examples of MV-algebras of continuous functions on compact, Hausdorff, totally disconnected topological spaces.

2. PSEUDO–EFFECT ALGEBRAS AND PSEUDO MV–ALGEBRAS

In the present Section, we give elements of pseudo-effect algebras together with their po-group representation.

According to [14, 15], a partial algebra $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a,b,c \in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if, and only if, $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a \le b$ if, and only if, there exists an element $c \in E$ such that $a+c = b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. It is possible to show that $a \le b$ if, and only if, $b = a+c = d+a$ for some $c,d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$.

Let $E = (E;+,0,1)$ be a pseudo-effect algebra. We define $x^- := 1 \setminus x$ and $x^\sim := x \,/\, 1$ for any $x \in E$. For given an element $e \in E$, we denote by $[0,e] := \{x \in E : 0 \le x \le e\}$. Then $[0,e]$ endowed with $+$ restricted to $[0,e] \times [0,e]$ is a pseudo-effect algebra $[0,e] = ([0,e];+,0,e)$. Then, for any $x \in [0,e]$, we have $x^{-_e} := e \setminus x$ and $x^{\sim_e} := x \,/\, e$ and $e = x^{-_e} + x = x + x^{\sim_e}$. For basic properties of pseudo-effect algebras see [14] and [15].

For example, if (G,u) is a unital (not necessary Abelian) po-group with strong unit u, and

$$\Gamma(G,u) := [0,u] = \{g \in G : \ 0 \le g \le u\},$$

then $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G,u)$. In [14, 15], there was proved that the converse statement, namely if E satisfies a special kind of the Riesz decomposition property, then $E = \Gamma(G,u)$, see Theorem 2.1.

We recall that a *pseudo MV-algebra* is an algebra $(M;\oplus,^-,^\sim,0,1)$ of type $(2,1,1,\ 0,0)$ such that the following axioms hold for all $x,y,z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

In [7] it was shown that every pseudo MV-algebra is isomorphic to $\Gamma(G,u)$, where (G,u) is a unital ℓ-group with strong unit u, where $a\oplus b := (a+b)\wedge u$, $a\odot b = (a-u+b)\vee 0$ and $a^- = u-a$ and $a^\sim = -a+u$.

If M is a pseudo MV-algebra, then $(M;+,0,1)$ is a pseudo-effect algebra, where the partial operation $a+b$ is defined if, and only if, $a\le b^-$, and then $a+b := a\oplus b$.

To present the basic representations of pseudo-effect algebras, according to [14], we introduce for pseudo-effect algebras the following forms of the Riesz decomposition properties:

(a) For $a,b\in E$, we write $a \ \mathsf{com}\ b$ to mean that for all $a_1\le a$ and $b_1\le b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1,a_2,b_1,b_2\in E$ such that $a_1,a_2\le b_1,b_2$ there is a $c\in E$ such that $a_1,a_2\le c\le b_1,b_2$.

(c) We say that E fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a,b_1,b_2\in E$ such that $a\le b_1+b_2$ there are $d_1,d_2\in E$ such that $d_1\le b_1,\ d_2\le b_2$ and $a=d_1+d_2$.

(d) We say that E fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1,a_2,b_1,b_2\in E$ such that $a_1+a_2=b_1+b_2$ there are $d_1,d_2,d_3,d_4\in E$ such that $d_1+d_2=a_1,\ d_3+d_4=a_2,\ d_1+d_3=b_1,\ d_2+d_4=b_2$.

(e) We say that E fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1,a_2,b_1,b_2\in E$ such that $a_1+a_2=b_1+b_2$ there are $d_1,d_2,d_3,d_4\in E$ such that (i) $d_1+d_2=a_1,\ d_3+d_4=a_2,\ d_1+d_3=b_1,\ d_2+d_4=b_2$, and (ii) $d_2\ \mathsf{com}\ d_3$.

(f) We say that E fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1,a_2,b_1,b_2\in E$ such that $a_1+a_2=b_1+b_2$ there are $d_1,d_2,d_3,d_4\in E$ such that (i) $d_1+d_2=a_1,\ d_3+d_4=a_2,\ d_1+d_3=b_1,\ d_2+d_4=b_2$, and (ii) $d_2\wedge d_3=0$.

If G is a po-group, we say that one of the above properties hold also for G if the corresponding property holds for positive elements of G.

The following representation holds, for details see [9, 14].

Theorem 2.1. Let E be a pseudo-effect algebra satisfying (RDP_1), then there is a unique (up to isomorphism) unital po-group (G,u) satisfying (RDP_1) such that E is isomorphic with $\Gamma(G,u)$. Moreover, if ϕ^* is an isomorphism of the pseudo-effect algebra E onto $\Gamma(G,u)$ and if $\phi: E\to H$ is a mapping preserving $+$, and H a group, then there is a group homomorphism $\gamma: G\to H$ such that $\phi=\phi^*\circ\gamma$. This γ is unique.

It is possible to show that a pseudo-effect algebra is a pseudo MV-algebra iff E satisfies (RDP_2), or equivalently, iff E satisfies (RDP_1) and E is a lattice, [9, 14]. In such a case, $E\cong\Gamma(G,u)$, where (G,u) is a unital ℓ-group.

3. STATES AND IDEALS

In this Section, we present states, extremal states, ideals and normal ideal of pseudo-effect algebras.

A *state* **on** E **is any mapping** $s : E \to [0,1]$ **such that** $s(1) = 1$ **and** $s(a+b) = s(a) + s(b)$ **whenever** $a+b$ **is defined in** E**. A state** s **on** E **is said to be** *extremal* **if the equality** $s = \alpha s_1 + (1-\alpha)s_2$ **for some** $0 < \alpha < 1$**, where** s_1 **and** s_2 **are states on** E**, implies** $s = s_1 = s_2$**. We denote by** $\mathcal{S}(E)$ **and** $\mathbf{Ext}_{\mathcal{S}}(E)$ **the set of all states and the set of all extremal states on** E**, respectively. We recall that it can happen that** $\mathcal{S}(E)$ **is empty, see [7].**

A non-empty subset I **of a pseudo-effect algebra** E **is said to be an** *ideal* **of** E **if (i)** $x+y \in I$ **whenever** $x, y \in I$ **and if** $x+y$ **is defined in** E**, and (ii) if** $x \le y$ **for** $x \in E$ **and** $y \in I$**, then** $x \in I$**. Then** E **as well as** $\{0\}$ **are ideals of** E**.**

We say that a net of states, $\{s_\alpha\}$**,** *converges weakly* **to a state** s **if** $s_\alpha(a) \to s(a)$ **for any** $a \in E$**.**

An ideal I **of** E **is (i)** *normal* **if** $a+I = I+a$ **for every** $a \in E$**,[2] (ii)** *maximal* **if** I **is a proper subset of** E **and it is not included in any proper ideal of** E **as a proper subset, and (iii)** *prime* **if** $I_0(a) \cap I_0(b) \subseteq I$ **implies** $a \in I$ **or** $b \in I$ **(where** $I_0(a), I_0(b)$ **are ideals of** E **generated by the elements** a **and** b**. In [[11], there is proved that a normal ideal** I **of a pseudo-effect algebra with (RDP) is prime iff** E/I **is an antilattice[3]. If** E **is a pseudo MV-algebra, then** I **is prime iff** $a \wedge b \in I$ **implies** $a \in I$ **or** $b \in I$**.**

If s **is a state on** E**, then the** *kernel* **of the state** s**,**

$$\mathbf{Ker}(s) := \{a \in E : s(a) = 0\}$$

is a normal ideal on E**.**

The following criteria for extremal states were proved in [7].

Theorem 3.1. Let s **be a state on a pseudo MV-algebra** M**. Then the following statements are equivalent:**

(i) s **is extremal.**

(ii) $s(x \wedge y) = \min\{s(x), s(y)\}$ **for all** $x, y \in M$**.**

(iii) s **is a state-morphism.[4]**

(iv) $\mathrm{Ker}(s)$ **is a maximal ideal of** M**.**

[2] If A is a non-empty subset of E, then $a + A := \{a + x : x \in A$ and $a + x$ is defined in $E\}$. In a similar way we define $A + a$.

[3] We recall that a poset $(E; \le)$ is an *antilattice* if only comparable elements of E have a supremum or an infimum.

[4] We say that a state s on a pseudo MV-algebra M is a *state-morphism* if $s(a \oplus b) = \min\{s(a) + s(b), 1\}$, $a, b \in M$.

Moreover, every maximal ideal of M which is normal is the kernel of some extremal state. In particular, if E is a Boolean algebra, then from (ii) we see that only two-valued states on E are extremal.

Let $\mathcal{NM}(M)$ be the set of all normal maximal ideals of a pseudo MV-algebra M. We recall that according to [9], $\mathcal{NM}(M)$ can be empty.

For every $a \in M$, we put

$$M_N(a) := \{I \in \mathcal{NM}(M) : a \notin I\}.$$

Then $M_N(0) = \emptyset$, $M_N(a) \subseteq M_N(b)$ whenever $a \le b$, $M_N(a \wedge b) = M_N(a) \cap M_N(b)$, $a, b \in M$, $M_N(a \vee b) = M_N(a) \cup M_N(b)$, $a, b \in M$, and $\{M_N(a) : a \in M\}$ is the base of the so-called *hull-kernel topology* $T_{\mathcal{NM}}$ on $\mathcal{NM}(M)$.

It is possible to show, [12], that if M is a pseudo MV-algebra. The hull-kernel topology defines a Hausdorff topology such that the closed subspaces of $\mathcal{NM}(M)$ are exactly of the form

$$C = C(J) := \{I \in \mathcal{NM}(M) : I \supseteq J\}, \tag{3.1}$$

where J is an ideal of M. Similarly, every open set O is of the form

$$O = O(J) := \{I \in \mathcal{NM}(M) : I \not\supseteq J\}. \tag{3.2}$$

If each value of 1 is normal-valued, then $T_{\mathcal{NM}}$ is compact.

Moreover, the mapping $\theta : \mathbf{Ext}_{\mathcal{S}}(M) \to \mathcal{NM}(M)$ defined by

$$\theta(s) := \mathbf{Ker}(s), \ \ s \in \mathbf{Ext}_{\mathcal{S}}(M),$$

is a homeomorphism, [12, Thm 3.3].

An ideal I of a pseudo-effect algebra E is said to be the *Riesz ideal* if, for $x \in I$, $a, b \in E$ and $x \le a + b$, there exist $a_1, b_1 \in I$ such that $x = a_1 + b_1$ and $a_1 \le a$ and $b_1 \le b$.

For example, if E is a pseudo-effect algebra with (RDP), then any ideal of E is Riesz.

Let P be an ideal of a pseudo-effect algebra E. For $a, b \in E$, we write $a \sim_P b$ iff there are two elements $e, f \in P$ such that $a \setminus e = b \setminus f$. We recall that $a \sim_P b$ iff $e' / a = b \setminus f$ for some $e', f \in P$ iff $e' / a = f' / b$ for some $e', f' \in P$.

Let P be a normal Riesz ideal of a pseudo-effect algebra E. Then $\sim_P$ is an equivalence on E such that $(E/P; +, [0]_P, [1]_P)$ is a pseudo-effect algebra, where $[a]_P := \{b \in E : b \sim_P a\}$, $E/P := \{[a]_P : a \in E\}$, and $[a]_P + [b]_P = [c]_P$ if, and only if, there are $a_1 \in [a]_P$, $b_1 \in [b]_P$ and $c_1 \in [c]_P$ such that $a_1 + b_1 = c_1$.

4. CENTRAL ELEMENTS AND GENERAL COMPARABILITY

In this central Section, we show that every pseudo-effect algebra satisfying general comparability has at least one state, and we show that every extremal state on the center can be extended to a unique state on E

which is also extremal. In addition, the topology of extremal states on E makes the set a compact, Hausdorff, totally disconnected nonempty topological space. We recall that it can happen, that there is a pseudo MV-algebra without any state.

An element e of a pseudo-effect algebra E is said to be *central* (or *Boolean*) if there exists an isomorphism

$$f_e : E \to [0, e] \times [0, e^{\sim}] \tag{4.1}$$

such that $f_e(e) = (e, 0)$ and if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$ for any $x \in E$.

We denote by $C(E)$ the set of all central elements of E, and $C(E)$ is said to be the *center* of E. We recall that $0, 1 \in C(E)$, in addition, see [10], (i) if $e \in C(E)$, then $e^{\sim} = e^{-}$, we denote $e' = e^{\sim}$; (ii) $C(E) = (C(E); \vee, \wedge, ', 0, 1)$ is a Boolean algebra; (iii) if $x \in E$ and $e \in C(E)$, then $x \wedge e \in E$; (iv) if $\{e_i\}_{i=1}^n$ is a finite system of central elements of E such that $e_i \wedge e_j = 0$ for $i \neq j$ and $e_1 \vee \cdots \vee e_n = 1$, then for any $x \in E$, $x = x \wedge e_1 + \cdots + x \wedge e_n$; (v) if E satisfies (RDP), then $e \in C(E)$ iff $e \wedge e^{\sim} = 0$, or equivalently, iff $e \wedge e^{-} = 0$, and (vi) the mappings $p_e : E \to [0, e]$ and $p_{e'} : E \to [0, e']$ defined by $p_e(x) = x \wedge e$, and $p_{e'}(x) = x \wedge e'$, $x \in E$, are surjective homomorphisms such that $f_e(x) = [p_e(x), p_{e'}(x)]$ for any $x \in E$.

We say that a pseudo-effect algebra E satisfies *general comparability* if, given $x, y \in E$, there is a central element $e \in E$ such that $p_e(x) \leq p_e(y)$ and $p_{e'}(x) \geq p_{e'}(y)$. This means that the coordinates of the elements $x = (p_e(x), p_{e'}(x))$ and $y = (p_e(y), p_{e'}(y))$ can be compared in $[0, e]$ and $[0, e']$, respectively.

For example, (i) every linearly ordered pseudo-effect algebra trivially satisfies general comparability; (ii) so does any Cartesian product of linearly ordered pseudo-effect algebras, (iii) every σ-complete pseudo MV-algebra satisfies general comparability [10, Prop 4.1] (Example 8.7 below gives an MV-algebra satisfying general comparability and that is not σ-complete), and (iv) if H is a normal ideal of E and if E satisfies general comparability, so does satisfy E/H.

On the other hand, let $G = \mathbb{R}^2$ with the strict ordering and $u = (1, 1)$, then $E = \Gamma(G, u)$ is an effect algebra with (RDP) which does not satisfy general comparability, because the only central elements of E are $0, 1$.

We recall that a topological space X is said to be (i) *connected* if it cannot be expressed as a union of two nonempty clopen subsets, and (ii) *totally disconnected* if there is a base consisting of clopen sets. For example, if X is finite, or if X is a Cantor set in $[0, 1]$, then X is totally disconnected.

Let now Ω be a compact Hausdorff topological space and let $C(\Omega)$ be the set of all continuous real-valued functions on Ω. Then $C(\Omega)$ is an Abelian ℓ-group with strong unit 1_Ω under the pointwise ordering of functions. Define the MV-algebra $M(\Omega) = \Gamma(C(\Omega), 1_\Omega)$. Then $C(M(\Omega)) = \{\chi_A : A$ is clopen in $\Omega\}$. The system of all clopen subsets of Ω forms a Boolean algebra of a Stone space iff the topology of Ω is totally disconnected. Therefore, $M(\Omega)$ can satisfy general comparability only if Ω is totally disconnected.

For example, if $\Omega = [0,1]$ with the usual topology, then $M([0,1])$ is an MV-algebra which does not satisfy general comparability, while $C(M([0,1])) = \{0_\Omega, 1_\Omega\}$. The same is true for any connected compact Hausdorff space X.

Using the direct product of such algebras, we can obtain infinitely many examples of MV-algebras where general comparability fails.

In [10, Thm 4.2], there was proved that every pseudo-effect algebra satisfying general comparability is practically a pseudo MV-algebra:

Theorem 4.1. Let E be a pseudo-effect algebra satisfying general comparability. Then E is a lattice, and E can be organized into a pseudo MV-algebra such that the partial addition derived from E as the pseudo MV-algebra coincides with the original $+$ taken in the pseudo-effect algebra.

Proposition 4.2. Let e be a central element of a pseudo-effect algebra E and let s be a state on E.

(a) If $s(e) = 0$, then $s \circ p_e = 0$ and $s = s \circ p_{e'}$.

(b) If $s(e) = 1$, then $s = s \circ p_e$ and $s \circ p_{e'} = 0$.

(c) If $s(e) = \alpha$, where $0 < \alpha < 1$, then the functions $s_1 = \alpha^{-1} s \circ p_e$ and $s_2 = (1-\alpha)^{-1} s \circ p_{e'}$ are distinct states on E such that $s = \alpha s_1 + (1-\alpha) s_2$.

(d) If s is extremal, then $s(e) \in \{0,1\}$.

Proof. (a) Since $p_e(x) = x \wedge e$, and $p_{e'}(x) = x \wedge e'$, $x \in E$, we have $s(x) = s(x \wedge e) + s(x \wedge e') = s(x \wedge e') = s \circ p_{e'}(x)$ and $s \circ p_e(x) = 0$.

(b) It is similar as (a).

(c) s_1 and s_2 are states while p_e and $p_{e'}$ are homomorphisms of E onto $[0,e]$ and $[0,e']$, respectively. Then $s = \alpha s_1 + (1-\alpha)s_2$, and since $s_1(e) = \alpha^{-1} s(e) = 1$ and $s_2(e) = 0$, we have that $s_1 \neq s_2$.

(d) This follows immediately from (c). □

Proposition 4.3. Let s be a state on a pseudo-effect algebra E and let $K = C(E) \cap \mathrm{Ker}(s)$.

(i) If s is extremal state, then K is a maximal ideal of $C(E)$.

(ii) If E has the property whenever $t \in \mathcal{S}(E)$ such that $\mathrm{Ker}(t) \supseteq K$ implies $t = s$, then s is extremal.

Proof. (i) K is an ideal of $C(E)$. For any $e \in C(E)$, (d) of Proposition 4.2 shows that $s(e) \in \{0,1\}$. Hence, either $e \in K$ or $e' \in K$ which yields K is a maximal ideal of $C(E)$.

(ii) Let $s = \alpha s_1 + (1-\alpha)s_2$ for some $0 < \alpha < 1$. Then $\mathbf{Ker}(s) \subseteq \mathbf{Ker}(s_i)$, $i = 1, 2$, which implies $K \subseteq \mathbf{Ker}(s_i)$, $i = 1, 2$, i. e., $s = s_1 = s_2$. $\square$

It is known that every extremal state on a Boolean algebra is two-valued. In what follows, we show every two-valued state on $C(E)$ can be uniquely extended to an extremal state on a pseudo-effect algebra E provided E satisfies general comparability. In particular, every pseudo MV-algebra satisfying general comparability has at least one state. We recall that there are examples of pseudo MV-algebras admitting no state [7].

Theorem 4.4. Let E be a pseudo-effect algebra satisfying general comparability, and let K be a maximal ideal of $C(E)$. Then there is a unique state s on E such that $C(E) \cap \mathrm{Ker}(s) = K$. This state is extremal.

Proof. Let K be a maximal ideal of $C(E)$. We denote by $I(K)$ the ideal of E generated by K. According to [11, Prop 3.1],

$$I(K) = \{x \in E : x = x_1 + \cdots + x_n,\ x_i \le e_i \in K,\ 1 \le i \le n,\ n \ge 1\}.$$

It is possible to show that

$$I(K) = \{x \in E : x = x_1 + \cdots + x_n,\ x_i \le e \in K, 1 \le i \le n,\ n \ge 1\}.$$

Step 1. Let $x \in E$, $y \in I(K)$, $x + y \in E$ and let $x, y \le f$ for some $f \in K$. Then $x + y = y' + x$ for some $y' \in I(K)$. Indeed, since $x, y \in I(K)$, we have $x + y \in I(K)$, therefore $x + y = (x + y) \setminus x + x$ and $y' = (x + y) \setminus x \in I(K)$.

Step 2. Let $x \in E$, $y \le f \in E$ and let $x+y \in E$. Then $x+y = x\wedge f+x\wedge f'+y\wedge f = x \wedge f + y \wedge f + x \wedge f' = y' + x \wedge f + x \wedge f' = y' + x$, where $y' \in I(K)$, when we have used Step 1.

Step 3. Let $x + y_1 + \cdots + y_n \in E$, where $y_i \le f \in K$ for any i. Then $x + y_1 + y_2 + \cdots + y_n = y_1' + x + y_2 + \cdots + y_n = \cdots = y_1' + \cdots + y_n' + x$, and all $y_i' \in I(K)$.

In a similar way we prove that if $z + x \in E$, where $z \in I(K)$, then $z + x = x + z'$ for some $z' \in I(K)$. In other words, we have proved that $I(K)$ is a normal ideal of E.

Claim. *$E/I(K)$ is linearly ordered.*

Let $x, y \in E$ be given. Due to general comparability, there is $e \in C(E)$ such that $p_e(x) \le p_e(y)$ and $p_{e'}(x) \ge p_{e'}(y)$. Since K is maximal, then either $e \in K$ or $e' \in K$. In the first case, $p_e(E) \subseteq I(K)$, and we have $x/I(K) = p_e(x)/I(K)+p_{e'}(x)/I(K) = p_{e'}(x)/I(K) \ge p_{e'}(y)/I(K) = p_{e'}(y)/I(K)+ p_{e'}(y)/I(K) = y/I(K)$.

Similarly, if $e' \in K$, then $x/I(K) \le y/I(K)$.

According to [7, Thm 5.5], $E/I(K)$ admits a unique state, say t. Then $s(a) := t(a/I(K))$, $a \in E$, is a state on E such that $K \subseteq I(K) \subseteq \mathbf{Ker}(s)$. For any $e \in C(E) \setminus K$, we have $e' \in K$ and so $s(e') = 0$, whence $s(e) = 1$, i. e., $e \notin \mathbf{Ker}(s)$ which proves $C(E) \cap \mathbf{Ker}(s) = K$.

If s_1 is a state on E and $C(E) \cap \mathbf{Ker}(s_1) = K$, then $\mathbf{Ker}(s_1) \supseteq I(K) \supseteq K$. Therefore, s_1 induces a state $\hat{s}_1$ on $E/I(K)$ given by $\hat{s}_1(a/I(K)) = s_1(a)$, $a/I(K) \in E/I(K)$. Since $E/I(K)$ has a unique state, t, we have $\hat{s}_1 = t$, i. e., $s_1 = s$.

We claim s is an extremal state on E. According to Theorem 4.1, E is a pseudo MV-algebra. Using the criterion for extremal states, Theorem 3.1, we have $s(a \wedge b) = t((a \wedge b)/I(K)) = t(a/I(K) \wedge b/I(K)) = \min\{t(a/I(K)), t(b/I(K))\} = \min\{s(a), s(b)\}$, which proves that s is an extremal state on E. □

It is worth to recall that if K is a maximal ideal of $C(E)$ in Theorem 4.4, then it is not necessary that $I(K)$ is a maximal ideal of E. Indeed, let $G = \mathbb{Z} \times_{lex} \mathbb{Z}$, and let $E = \Gamma(G, (1, 0))$. Then E is linearly ordered, it satisfies general comparability, $C(E) = \{0, 1\}$, and $K = \{0\}$ is a unique maximal ideal in $C(E)$, therefore, $I(K) = \{0\}$ and it is contained in a unique maximal ideal $I = \{(0, n) : n \geq 0\}$ of E.

Corollary 4.5. Every pseudo-effect algebra E satisfying general comparability admits at least one state. Moreover, every two-valued state s_0 on $C(E)$ can be extended to a unique extremal state s on E such that $s|C(E) = s_0$.

P r o o f. According to Theorem 4.4, E has at least one state. Let now s_0 be any two-valued state on $C(E)$. Then $K := \mathbf{Ker}(s_0)$ is a maximal ideal of $C(E)$, and by Theorem 4.4, there is a unique extremal state s on E such that $C(E) \cap \mathbf{Ker}(s) = \mathbf{Ker}(s_0)$. Due to (d) of Proposition 4.2, we see that $s|C(E) = s_0$. □

We denote by $\mathcal{M}(C(E))$ the set of all maximal ideals of the Boolean algebra $C(E)$. Its hull-kernel topology is totally disconnected.

Theorem 4.6. Let E be a pseudo-effect algebra satisfying general comparability. Then the mapping

$$\phi(s) := C(E) \cap \mathrm{Ker}(s),\ \ s \in \mathrm{Ext}_{\mathcal{S}}(E), \tag{4.2}$$

defines a homeomorphism ϕ of $\mathrm{Ext}_{\mathcal{S}}(E)$ onto $\mathcal{M}(C(E))$.

P r o o f. In view of Theorem 4.4, we see that ϕ is a bijection of $\mathrm{Ext}_{\mathcal{S}}(E)$ onto $\mathcal{M}(C(E))$. The space $\mathcal{M}(C(E))$ is totally disconnected, therefore, it has a basis consisting of clopen sets of the form

$$U = \{K \in \mathcal{M}(C(E)) : e \notin K\}$$

for $e \in C(E)$. We observe that

$$\phi^{-1}(U) = \{s \in \mathrm{Ext}_{\mathcal{S}}(E) : s(e) \neq 0\},$$

which is an open subset of $\mathrm{Ext}_{\mathcal{S}}(E)$. This proves that ϕ is continuous.

Claim. *If X is a nonempty compact subset of $\mathrm{Ext}_{\mathcal{S}}(E)$, then*

$$X = \{s \in \mathrm{Ext}_{\mathcal{S}}(E) : \mathrm{Ker}(X) \subseteq \mathrm{Ker}(s)\}, \tag{4.3}$$

where $Ker(X) = \bigcap_{s\in X} Ker(s)$.

This claim was proved in [12, Lem 3.2].

Therefore, $\phi(X) = \{C(E) \cap \mathbf{Ker}(s) : s \in X\} = \{C(E) \cap \mathbf{Ker}(s) : \mathbf{Ker}(X) \subseteq \mathbf{Ker}(s)\} = \{C(E) \cap \mathbf{Ker}(s) : \mathbf{Ker}(X) \cap C(E) \subseteq \mathbf{Ker}(s) \cap C(E)\}$, which by (3.1) proves that $\phi(X)$ is a closed subset of $\mathcal{M}(C(E))$.

Since $\mathbf{Ext}_{\mathcal{S}}(C(E))$ is a compact space, any closed subset X of $\mathbf{Ext}_{\mathcal{S}}(C(E))$ is compact, we see that ϕ is a closed mapping, whence ϕ is a homeomorphism. □

As a direct consequence of Theorem 4.6 we have the following statement.

Corollary 4.7. If E is pseudo-effect algebra satisfying general comparability, then $\mathrm{Ext}_{\mathcal{S}}(E)$ is a nonempty, compact and totally disconnected.

5. STATES WHEN EVERY MAXIMAL IDEAL IS NORMAL

In this Section, we study the state space of pseudo MV-algebras. We show that in every such an algebra, every maximal ideal is normal, and we study the weak topology of states to describe the closed faces. We note that we generalize the results known for Abelian unital po-groups with interpolation, see [19, Section 8]. In addition, we extend the results also for pseudo MV-algebras E which not necessarily satisfy general comparability, but in which every maximal ideal K of the center generated is a prime ideal $I(K)$ in E.

Proposition 5.1. Let I be a maximal ideal of a pseudo-effect algebra E with (RDP_0). Then $I \cap C(E)$ is a maximal ideal of $C(E)$.

If K is a maximal ideal of $C(E)$, then $I(K)$, the ideal of E generated by K is the set

$$I(K) = \bigcup_{e\in K} [0, e],$$

and $I(K)$ is normal.

Proof. It is clear that $I \cap C(E)$ is an ideal of $C(E)$. To show that it is maximal in $C(E)$, assume $e \in C(E)$ and $e \notin I$. The ideal generated by I and

e contains 1. Hence, [11, Prop 3.1], $1 = x + e_1 + \cdots + e_n$, where $x \in I$ and $e_1, \ldots, e_n \le e$, $e_i \in E$ for any i. Therefore, $e' = x\wedge e' + e_1\wedge e' + \cdots + e_n \wedge e' = x\wedge e'$ which implies $e' \le x \in I$, i. e., $e' \in I$.

The formula for $I(K)$ follows from [11, Prop 3.3] and from reasonings from the proof of Theorem 4.4. □

Theorem 5.2. Every maximal ideal of a pseudo-effect algebra satisfying general comparability is normal.

Proof. Let now I be any maximal ideal of E, and let $K = I \cap C(E)$. According to Proposition 5.1, K is a maximal ideal of $C(E)$. Let $I(K)$ be the ideal of E generated by K. According to Proposition 5.1, $I(K)$ is a normal ideal of E, and $E/I(K)$ is a linearly ordered pseudo-effect algebra having a unique state, say t. Then the mapping $s(a) := t(a/I(K))$, $a \in E$, is an extremal state on E, and it contains $I(K)$, Theorem 4.4.

Let $I/I(K) = \{x/I(K) : x \in I\}$. Then it is easy to verify that $I/I(K)$ is a proper ideal of $E/I(K)$. Since Ker(t) is a unique maximal ideal of $I/I(K)$, [7, Thm 5.5], it contains $I/I(K)$. Therefore, $I \subseteq$ Ker(s). The maximality of I implies $I =$ Ker(s) which proves that I is normal. □

We have note that we have a stateless pseudo MV-algebra, see [7]. Thus it gives an example of a pseudo-effect algebra do not satisfying general comparability. The following example is from [4].

We apply similar notations as in [18]. Let $\mathbb{R}$ be the set of all real numbers with the natural linear order. We denote by $A(\mathbb{R})$ the set of all order-preserving permutations of $\mathbb{R}$. Then $A(\mathbb{R})$ is a group under composition. For $f, g \in A(\mathbb{R})$ we put $f \le g$ if $f(t) \leqq g(t)$ for each $t \in \mathbb{R}$. The relation $\le$ is a partial order on $A(\mathbb{R})$ and under this partial order, $A(\mathbb{R})$ turns out to be a lattice ordered group.

Example 5.3. Let $a \in A(\mathbb{R})$, $a(t) \ge t$ for any $t \in \mathbb{R}$, and $a(t_0) > t_0$ for some $t_0 \in \mathbb{R}$. Then $M = \Gamma(G_a, a)$ is a stateless pseudo MV-algebra, where G_a denotes the convex ℓ-subgroup of $A(\mathbb{R})$ generated by the element a, and any maximal ideal of M is not normal. In addition, general comparability fails to hold in M.

The above results can be generalized as follows.

Theorem 5.4. If, for every maximal ideal K in $C(E)$ of a pseudo MV-algebra E, the ideal $I(K)$ generated by K is prime, then every extremal state on $C(E)$ can be uniquely extended to an extremal state on E, and every maximal ideal of E is normal. In addition, the mapping ϕ defined by (4.2) defines a homeomorphism of $\mathrm{Ext}_{\mathcal{S}}(E)$ onto $\mathcal{M}(C(E))$.

Proof. According to Proposition 5.1, the ideal $I(K)$ is normal. Since $I(K)$ is prime, this implies $E/I(K)$ is linear, [11, Prop 4.6], and it contains

a unique state, t. It induces a state s on E by $s(a) := t(a/I(K))$. It is evident that s is extremal, Theorem 3.1. Moreover, Ker$(s) \supseteq I(K) \supseteq K$. If now $e \in C(E) \setminus K$, then $e' \in K$ and $s(e') = 0$ which gives $e \notin$ Ker(s).

As in the proof of Theorem 4.4, if s_1 is any state such that Ker$(s_1) \cap C(E) = K$, then $s_1 = s$.

Repeating the proof of Theorem 5.2, we see that every maximal ideal of E is normal.

The rest follows the same ideas as the proof of Theorem 4.6. □

We recall that every pseudo-effect algebra satisfying general comparability satisfies the conditions of Theorem 5.4.

Suppose that $E = \Gamma(G, u)$, where G satisfies (RDP$_1$) and let $e \in C(E)$. The mapping (4.1) can be extended also for the whole group G as follows.

We recall that an *o-ideal* of a po-group G is any normal convex directed subgroup H of G. A subgroup H of an ℓ-group G is an o-ideal of G iff H is an ℓ-ideal of G. If (G, u) is a unital po-group, we denote by $\mathcal{OI}(G, u)$ the set of all o-ideals of (G, u). According to [11, Thm 4.2], we have the following result.

Theorem 5.5. **Let $E = \Gamma(G, u)$, where (G, u) is a unital po-group satisfying (RDP$_1$). For any ideal I of E we set**

$$\phi(I) = \{x \in G : \exists\ x_i, y_j \in I, \quad x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \tag{5.1}$$

Then $\phi(I)$ is an o-ideal of (G, u) if, and only if, I is a normal ideal of E. In such the case

$$(E/I, u/I) = \Gamma(G/\phi(I), u/\phi(I)).$$

In addition, if K is an o-ideal of (G, u), then its restriction to E, denoted by $\psi(K)$, gives a normal ideal of E, i. e.,

$$\psi(K) := K \cap E \in \mathcal{I}(E), \quad K \in \mathcal{I}(G, u).$$

Moreover, both mappings, ϕ and ψ, are mutually bijective and preserving the set-theoretical inclusion.

Let now $e \in C(E)$ and let $I_0(e)$ and $I_0(e')$ be the ideal of E generated by e and e', respectively. It is easy to verify that $I_0(e)$ and $I_0(e')$ are normal ideals. In view of Theorem 5.5, $G(e) := \phi(I_0(e))$ and $G(e') = \phi(I_0(e'))$ are o-ideals of G such that $G(e) \cap G(e') = \{0\}$. Indeed, first let $x \in I_0(e) \cap I_0(e')$. Then $x = x_1 + \cdots + x_n = y_1 + \cdots + y_m$ where $x_i \le e$ and $y_j \le e'$. The Riesz decomposition property implies that there is a system of elements $\{c_{ij} : 1 \le i \le n,\ 1 \le j \le m\}$ such that $x_i = \sum_j c_{ij}$ and $y_j = \sum_i c_{ij}$. Therefore, $c_{ij} \le e \wedge e' = 0$, i. e., $x = 0$. The general case follows from Theorem 5.5.

Therefore,

$$G = G(e) \oplus G(e').$$

Define now two mappings $f_e : G \to G(e)$ and $f_{e'} : G \to G(e')$ by $f_e(g) = g_1$ and $f_{e'}(g) = g_2$ whenever $g = (g_1, g_2)$. Then f_e is a positive group-homomorphism from G onto $G(e)$ and similarly, $f_{e'}$ is a positive group-homomorphism from G onto $G(e')$. Moreover, $p_e(x) = f_e(x)$ for any $x \in E$.

Proposition 5.6. Let $e \in C(E)$, where $E = \Gamma(G, u)$ for some unital po-group satisfying (RDP$_1$). If $x \in G^+$ and $x \le nu$, then $f_e(x) = x \wedge ne$.

Proof. As $x \le nu$, then $f_e(x) \le f_e(nu) = ne$. In addition, $0 = f_{e'}(x) = x - f_e(x)$ whence $f_e(x) \le x$. Assume $y \le x$ and $y \le ne$ for some $y \in G$. Then $f_e(y) \le f_e(x)$ and $y - f_e(y) = f_{e'}(y) \le f_{e'}(ne) = 0$. Whence $y \le f_e(y) \le f_e(x)$ which gives $f_e(x) = x \wedge ne$. □

We recall every state on $E = \Gamma(G, u)$, E with (RDP$_1$), can be uniquely extended to a state on (G, u)[5], and vice-versa, the restriction of any state on (G, u) gives a state on E.

According to [9], we introduce two functions f_* and f^* on G as follows. For any $x \in G$, we set

$$\begin{aligned} f^*(x) &= \inf\{l/n : l \in \mathbb{Z}, n \ge 1,\ nx \le lu\}, \\ f_*(x) &= \sup\{k/i : k \in \mathbb{Z}, i \ge 1,\ ku \le ix\}. \end{aligned}$$

These functions have a very close connection with the existence of states on (G, u) while if s is a state on (G, u), then $f_*(x) \le s(x) \le f^*(x)$ for any $x \in G$. Moreover, if G is linearly ordered, then (G, u) has a unique state, say s, and we have

$$s(x) = \inf\{l/n : l \in \mathbb{Z}, n \ge 1,\ nx \le lu\} = \sup\{k/i : k \in \mathbb{Z}, i \ge 1,\ ku \le ix\}$$

for any $x \in G$. Moreover, if $x \in G^+$, then

$$s(x) = \inf\{l/n : l, n \ge 1,\ nx \le lu\} = \sup\{k/i : k \ge 0, i \ge 1,\ ku \le ix\}. \tag{5.3}$$

We recall that a *face* of a convex set $\mathcal{S}$ is a convex subset F of $\mathcal{S}$ such that if $x = \alpha x_1 + (1-\alpha)x_2 \in F$ for $x_1, x_2 \in \mathcal{S}$, then $x_1, x_2 \in F$. Every face is roughly speaking (i) the empty set, or (ii) the whole $\mathcal{S}$, or (iii) an extremal point or line segment connecting pairs of adjacent extremal points of $\mathcal{S}$. Moreover, given any subset $X \subseteq \mathcal{S}$, there is a smallest face of $\mathcal{S}$ that contains X.

Proposition 5.7. Let E be a pseudo-effect algebra and let X be a subset of E. Then the set

$$F = \{s \in \mathcal{S}(E) : X \subseteq \mathrm{Ker}(s)\}$$

is a closed face of $\mathcal{S}(E)$.

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[5] A *state* on a unital po-group (G, u) is any mapping $\hat{s} : G \to \mathbb{R}$ such that (i) $\hat{s}(g) \ge$ for any $g \in G^+$, (ii) $\hat{s}(g + h) = \hat{s}(g) + \hat{s}(h)$ for all $g, h \in G$, and (iii) $\hat{s}(u) = 1$. We denote by $\mathcal{S}(G, u)$ and $\mathrm{Ext}_{\mathcal{S}}(G, u)$ the sets of all states and all extremal states, respectively, on (G, u).

Proof. If $\mathcal{S}(E)$, the statement is evident. So let the state space $\mathcal{S}(E)$ be nonempty. Then it is clear that F is a closed convex subset of $\mathcal{S}(E)$. If $s = \alpha s_1 + (1-\alpha)s_2 \in F$ with $s_1, s_2 \in \mathcal{S}(E)$ and $0 < \alpha < 1$, then for any $x \in X$, we have $s(x) = \alpha s_1(x) + (1-\alpha)s_2(x) = 0$. Therefore, s_1 and s_2 vanish on X, so that $s_1, s_2 \in F$. □

Example 8.7 gives an MV-algebra such that every maximal ideal K_i generates a prime ideal $I(K_i)$ $(i \ge 1)$, but $I(K_\infty)$ is not prime. For such algebras we have the following result.

Proposition 5.8. Let $E = \Gamma(G,u)$ be a pseudo MV-algebra and let K be a maximal ideal of $C(E)$ such that $I(K)$ is prime. Then there is a unique state s on E such that $\mathrm{Ker}(s) \cap C(E) = K$, this state is extremal, and for its extension $\hat{s}$ on (G,u) we have

$$\begin{aligned}\hat{s}(x) &= \inf\{l/n : l,n \ge 1,\ nf_e(x) \le lu,\ e \in C(E)\setminus K\}\\ &= \sup\{k/i : k \ge 0, i \ge 1,\ ke \le if_e(x),\ e \in C(E)\setminus K\}.\end{aligned}$$

Proof. According to Proposition 5.1, $I(K)$ is normal and, by the hypothesis, is prime, so that $E/I(K)$ is linear, and according to Theorem 5.5, $E/\phi(I(K))$ is linear. The group $(G/\phi(I(K)), u/\phi(I(K)))$ has a unique state, [9, Prop 3.4], as well as $E/I(K)$. Applying again Theorem 5.5, if t is a state on $E/I(K)$, then $\hat{t}$ is a unique state on $(G/\phi(I(K)), u/\phi(I(K)))$. Moreover, $s(a) := t(a/I(K)$, $a \in E$, is a unique state on E such that $\mathrm{Ker}(s) \cap C(E) = K$ Therefore, for the extension $\hat{t}$ of t onto G, we have (5.3), and our formulas follow directly from (5.3), that is

$$\begin{aligned}\hat{s}(x) &= \hat{t}(x/\phi(I(K))) = \inf\{l/n : l,n \ge 1,\ nx/\phi(I(K)) \le lu/\phi(I(K))\}\\ &= \sup\{k/i : k \ge 0, i \ge 1,\ ku/\phi(I(K)) \le ix/\phi(I(K)\}.\end{aligned}$$

Let now $nx/\phi(I(K)) \le lu/\phi(I(K))$. There is an $a \in \phi(I(K))$ such that $nx - a \le lu$. But then $a = a_1 + \cdots + a_n - b_1 - \cdots - b_m$, where $a_i, b_j \le e' \in K$. Then $f_e(nx-a) \le f_e(lu)$, i.e., $nf_e(x) - f_e(a) = nf_e(x) \le le$. Conversely, if $nf_e(x) \le le$ for some $e' \in K$, then $nx/\phi(I(K)) \le lu/\phi(I(K))$. In a similar way we prove the second property for the supremum. □

We recall that according to [12, Thm 4.4], if $E = \Gamma(G,u)$ for some unital ℓ-group (G,u), then $\mathrm{Ker}_{\mathcal{S}}(E)$ and $\mathrm{Ker}_{\mathcal{S}}(G,u)$ are homeomorphic compact sets which are simultaneously non-void or void.

Theorem 5.9. Let E be a pseudo MV-algebra such that $I(K)$ is prime for any maximal ideal K of $C(E)$. Let X be a subset of states on E, and set

$$V = \{s \in \mathcal{S}(E) : \mathrm{Ker}(X) \subseteq \mathrm{Ker}(s)\},$$
$$W = \{s \in \mathcal{S}(E) : C(E) \cap \mathrm{Ker}(X) \subseteq \mathrm{Ker}(s)\}.$$

Then $V = W$ and V is a closed face of $\mathcal{S}(E)$. Moreover, V equals the closure of the face generated by X in $\mathcal{S}(E)$.

P r o o f. According to Proposition 5.7, V and W are closed faces of $\mathcal{S}(E)$. It is clear that $V \subseteq W$. Now let Y be the closure of the face generated by X in $\mathcal{S}(E)$. Since Y is a closed convex set and W equals the closure of the convex hull of extremal points, Ker(W), it suffices to verify that Ker$(W) \subseteq Y$. Thus consider any state $t \in$ Ker(W). Because W is a face of $\mathcal{S}(E)$, we have that t is an extremal state of E. By Proposition 5.7, the set $K = C(E) \cap$ Ker(t) is a maximal ideal of $C(E)$.

Set $A = C(E) \setminus K$, which is nonempty. As $t \in W$, we have $K = C(E) \cap$ Ker$(t) \supseteq C(E) \cap$ Ker(X), which implies that A is disjoint from Ker(X). Hence, given any $e \in A$, we may choose a state $s_e \in X$ such that $s_e(e) > 0$. The function $t_e = s_e(e)^{-1} s_e \circ p_e$ is a state on E such that $t_e(e) = 1$. By Proposition 4.2, $t_e = t_e \circ p_e$. Hence, applying [19, Prop 6.15, Prop 5.7], we have that t_e lies in the face generated by X, and hence $t_e \in Y$.

Consider the downward-direct set $\{t_e : e \in A\}$ in Y, and we assert that $t_e \to t$ weakly on E.

Using (5.3), we have that for any $x \in E$ and any $\epsilon > 0$ there exist $k \in \mathbb{Z}^+$ and $i, l, n \in \mathbb{N}$ such that $k/i > t(x) - \epsilon$ and $l/n < t(x) + \epsilon$, while also $kf \leq i f_e(x)$ and $n f_g(x) \leq lg$ for some $f, g \in A$. Note that $f \wedge g \in A$. For any $e \in A$ with $e \leq f \wedge g$, we have $ke = f_e(kf) \leq i f_e f_f(x) = i f_e(x)$ and $n f_e(x) = n p_e p_g(x) \leq p_e(lg) = le$, whence $k = k t_e(e) \leq i t_e f_e(x) = i t_e(x)$ and similarly $n t_e(x) \leq l$. Consequently, we have

$$t(x) - \epsilon < k/i \leq t_e(x) \leq l/n < t(x) + \epsilon,$$

which implies $vert t_e(x) - t(x)| < \epsilon$ for all $e \in A$ such that $e \leq f \wedge g$.

Therefore, $t_e(x) \to t(x)$ for any $x \in E$. Since Y is closed in $\mathcal{S}(E)$, we conclude that $t \in Y$.

Thus Ext$(W) \subseteq Y$, and hence $W \subseteq Y$, by the Krein–Mil'man theorem. Therefore, $Y = V = W$. □

Corollary 5.10. Let E be a pseudo MV-algebra such that $I(K)$ is prime for any maximal ideal K of $C(E)$ and let $X \subseteq \mathcal{S}(E)$. If $\mathrm{Ker}(X) = \{0\}$, then $\mathcal{S}(E)$ equals the closure of the face generated by X in $\mathcal{S}(E)$.

Corollary 5.11. Let E be a pseudo MV-algebra such that $I(K)$ is prime for any maximal ideal K of $C(E)$. Then the closure of any face of $\mathcal{S}(E)$ is a face of $\mathcal{S}(E)$. Moreover, the closed faces of $\mathcal{S}(E)$ are exactly the sets

$$F_H = \{s \in \mathcal{S}(S) : H \subseteq \mathrm{Ker}(s)\},$$

where H is any normal ideal of E.

P r o o f. It immediately follows from Theorem 5.9 that the closures of faces are faces in $\mathcal{S}(E)$. On the other hand, by Proposition 5.7, F_H is a closed face of $\mathcal{S}(E)$. Conversely, let F be any closed face of $\mathcal{S}(E)$ and set $H = \mathbf{Ker}(F)$. Then H is a normal ideal of E, and, by Theorem 5.9,

$$F = \{s \in \mathcal{S}(E) : \mathbf{Ker}(F) \subseteq \mathbf{Ker}(s)\} = F_H. \qquad \square$$

6. REPRESENTABILITY OF PSEUDO–EFFECT ALGEBRAS WITH GENERAL COMPARABILITY

We have seen that every pseudo-effect algebra satisfying general comparability is automatically a pseudo MV-algebra such that every maximal ideal is normal. In the present Section, we show a more stronger result saying that every pseudo MV-algebra satisfying general comparability can be represented as a subdirect product of linearly ordered pseudo MV-algebras.

Let $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ be a family of pseudo MV-algebras. The Cartesian product $M := \prod_{t\in T} M_t$, where $\oplus, ^-, ^\sim, 0, 1$ are defined in a usual way by coordinates, is said to be a *direct product* of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$. Then M is a pseudo MV-algebra. A pseudo MV-algebra M is a *subdirect product* of a family of $\{(M_t; \oplus_t, ^{-_t}, ^{\sim_t}, 0_t, 1_t)\}_{t\in T}$ of pseudo MV-algebras iff there exists a one-to-one homomorphism $h: M \to \prod_{t\in T} M_t$ of pseudo MV-algebras [6] such that, for each $t \in T$, $\pi_t \circ h$ is a homomorphism from M onto M_t, where π_t is the tth projection $\prod_{t\in T} M_t$ onto M_t.

According to [17], we say that a pseudo MV-algebra M is *representable* if it can be represented as a subdirect product of linear pseudo MV-algebras. It is well-known that every MV-algebra is representable (see e. g., [2]).

In [7], we have proved that the family of all representable pseudo MV-algebras form a variety, and every such MV-algebra has at least one state.

Proposition 6.1. Let E be a pseudo-effect algebra satisfying (RDP_0). Let K be a maximal ideal of $C(E)$ and $I(K)$ the ideal of E generated by K. Then

$$\bigcap_K I(K) = \{0\}. \tag{6.1}$$

P r o o f. Let the Boolean algebra $C(E)$ be represented as a system of all clopen subsets of the compact, Hausdorff, totally disconnected topological space $\Omega = C(E)$. For every $\omega \in \Omega$, the set $K_\omega = \{e \in C(E) : \omega \notin e\}$ is a maximal ideal of $C(E)$, and conversely, any $K = K_\omega$ for a unique $\omega \in \Omega$. For elements of $C(E)$, we can identified finite joins and meets in it with the set-theoretical unions and intersections, respectively.

Take $x \in \bigcap_K I(K)$, then $x \in I(K_\omega)$ for any $\omega \in \Omega$. Fix ω_0 and by Proposition 5.1, there is an $e_0 \in K_{\omega_0}$ such that $x \le e_0$. For any $\omega \in e_0$, by

[6] We recall that a mapping $h: M_1 \to M_2$ of two pseudo MV-algebras M_1 and M_2 is said to be a *homomorphism* if h preserves $\oplus$, $^-$, $^\sim$ and 0 and 1.

Proposition 5.1, there is an $e_\omega \in K_\omega$ such that $x \le e_\omega$ and $\omega \in e'_\omega$. Since $e_0 \subseteq \bigcup_{\omega\in e_0} e'_\omega$, and e_0 is closed, the compactness of Ω implies $e_0 \subseteq \bigcup_{i=1}^n e'_{\omega_i}$, so that $e_0 \le \bigvee_{i=1}^n e'_{\omega_i}$. Then $e_0 = \bigvee_{i=1}^n (e'_{\omega_i} \wedge e_0)$ and $e'_0 = \bigwedge_{i=1}^n (e_{\omega_i} \vee e'_0) \ge x$. Therefore, $x \le e_0 \wedge e'_0 = 0$. □

Theorem 6.2. Every pseudo MV-algebra M satisfying general comparability is representable.

P r o o f. Let K be any maximal ideal of $C(M)$. According the Claim of the proof of Theorem 4.4, the ideal $I(K)$ of M generated by K is normal and $M/I(K)$ is a linearly ordered pseudo MV-algebra. Since (6.1) holds, then it is easy to verify that M is a subdirect product of $\prod_K M/I(K)$. □

Theorem 6.2 can be generalized also for pseudo MV-algebras M not necessarily satisfying general comparability, but in which every maximal ideal of the center gives a prime ideal in M.

Theorem 6.3. Let every maximal ideal K of the center $C(M)$ of a pseudo MV-algebra M generate a prime ideal $I(K)$ in M. Then E is representable.

P r o o f. It follows the same ideas as the proof of Theorem 6.2. □

It is worthy to recall that the family of all pseudo-effect algebras satisfying general comparability is not a variety: It is closed under direct products, every quotient does again satisfy, but there is an MV-algebra satisfying general comparability having an MV-subalgebra where general comparability fails. Indeed, take $M(C[0,1])$ from the Section 4. Because it is commutative, it is an MV-subalgebra of a subdirect product of linearly ordered MV-algebras, and each of them satisfies general comparability.

7. FUNCTIONAL REPRESENTATIONS OF PSEUDO–EFFECT ALGEBRAS

Pseudo-effect algebras satisfying general comparability are not necessary commutative. However, we show that there is a homomorphism of E onto an MV-algebra M of continuous functions from $M(\Omega)$ such that M is dense in the sup-norm of $M(\Omega)$ for some compact, Hausdorff, totally disconnected topological space homeomorphic with $C(E)$.

Thus, let Ω be a compact, Hausdorff, totally disconnected topological space, and let $M(\Omega)$ be the system of all continuous functions from $[0,1]^\Omega$. For any $\omega \in \Omega$, let $I_\omega = \{f \in M(\Omega) : f(\omega) = 0\}$. Then I_ω is a maximal ideal on $M(\Omega)$, and conversely, every maximal ideal $I = I_\omega$ for a unique $\omega \in \Omega$, [2, Thm 3.4.3]. Because there is a one-to-one correspondence among extremal states and maximal ideals given by $s \leftrightarrow \mathrm{Ker}(s)$, according to the Riesz–Markov theorem, for any I_ω, there is a unique Baire probability measure μ_ω on $\mathcal{B}(\Omega)$, the Baire σ-algebra generated by all compact G_δ sets

on Ω, or equivalently, generated by $\{f^{-1}([a,\infty)) : f \in C(\Omega),\ a \in \mathbb{R}\}$. It can happen that $\mu_\omega = \delta_\omega$, where δ_ω is the Dirac measure concentrated on ω; such a situation is, for example, when the Baire σ-algebra $\mathcal{B}(\Omega)$ coincide with the Borel σ-algebra over Ω, i. e. the σ-algebra generated by open subsets of Ω.

We have $C(M(\Omega)) = \{\chi_A : A \text{ is clopen in } \Omega\}$. Each two-valued state on $C(M(\Omega))$ is concentrated on a unique point ω. If e. g., Baire and Borel σ-algebras coincide, every two-valued state on $C(M(\Omega))$ can be uniquely extended to a unique state on $M(\Omega)$, s_ω, which is defined by $s_\omega(f) := f(\omega)$, $f \in M(\Omega)$, which is also extremal.

However it can happen that $M(\Omega)$ does not satisfy general comparability, see Examples 8.4 − 8.6.

In what follows, we show that if E satisfies e. g. general comparability, then E can be homomorphically embedded into $M(\Omega)$, where $\Omega = \mathbf{Ext}_{\mathcal{S}}(E)$.

Let $E = \Gamma(G,u)$ be a pseudo-effect algebra satisfying ($\mathbf{RDP}_1$), and let s be a state on E and $\hat{s}$ its unique extension on (G,u). We set $s(E) := \{s(a) : a \in E\}$.

By [19, Lemma 4.21], $\hat{s}(G) = \{\hat{s}(g) : g \in G\}$ is a subgroup of the group $\mathbb{R}$ of all real numbers which is either cyclic or dense in $\mathbb{R}$. In the first case $\hat{s}$ is said to be *discrete*. In such a case $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$.

A state s on pseudo-effect algebras E is said to be *discrete* if $s(E) = \{s(a) : a \in E\} \subseteq \{0, 1/n, 2/n, \dots, n/n\}$ for some integer $n \geq 1$. It can happen that $s(E)$ is a proper subset of $\{0, 1/n, 2/n, \dots, n/n\}$. Indeed, let $E = \{0, a, a', 1\}$, and let $s(a) = 0.3$ and $s(a') = 0.7$.

We now show that there is a one-to-one correspondence among discrete states on E and (G,u), respectively.

Proposition 7.1. **Let $E = \Gamma(G,u)$ be a pseudo-effect algebra with (RDP_1). Then a state s on E is discrete if, and only if, its extension $\hat{s}$ on (G,u) is discrete.**

Proof. If $\hat{s}$ is discrete, it can be easily seen that s is discrete. Conversely, let s be discrete. It means $s(E) \subseteq \{0, 1/n, 2/n, \dots, n/n\}$ for some integer $n \geq 1$; let n be the smallest one. We suppose that $s(E) = \{0, k_1/n, \dots, k_m/n, 1\}$, where $1 \leq k_1 < \dots < k_m \leq n$. Since n is minimal, this implies that the greatest common divisor of $n, k_1, \dots, k_m$ is 1. From the elementary arithmetic this yields that there are integers $a_0, a_1, \dots, a_m \in \mathbb{Z}$ such that $a_0 n + a_1 k_1 + \dots + a_m k_m = 1$. Therefore, $1/n \in \hat{s}(G)$, i. e., $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$.
□

A pseudo-effect algebra E is said to be *weakly divisible* if, for any integer $n \geq 1$, there is an element $v \in E$ such that $nv = 1$. If E is weakly divisible, then E has no discrete state. Indeed, for any state s of E we have $1/n \in s(E)$ for any integer $n \geq 1$. For example, $M(\Omega)$ is weakly divisible.

Suppose that $\Omega = \mathbf{Ext}_{\mathcal{S}}(E)$. If E satisfies general comparability, then Ω is a nonempty compact, Hausdorff, totally disconnected topological space, Corollary 4.7.

Define a mapping $\psi : E \to C(\mathbf{Ext}_{\mathcal{S}}(E))$ defined by

$$\psi(a) := s(a), \quad a \in E, \quad s \in \mathbf{Ext}_{\mathcal{S}}(E),$$

supposing $\mathbf{Ext}_{\mathcal{S}}(E) \neq \emptyset$.

If E is a pseudo MV-algebra with the nonempty state space, then in view of Theorem 3.1, $\psi(E)$ is an MV-algebra which is an MV-subalgebra of the MV-algebra $M(\mathbf{Ext}_{\mathcal{S}}(E))$, and ψ is a homomorphism of pseudo MV-algebras.

Proposition 7.2. **Let E be a pseudo MV-algebra with the nonempty state space $\mathcal{S}(E)$. Set $\mathrm{Ker} = \bigcap\{\mathrm{Ker}(s) : s \in \mathrm{Ext}_{\mathcal{S}}(E)\}$. Then Ker is a normal ideal of E, and E/Ker is isomorphic with $M(\mathrm{Ext}_{\mathcal{S}}(E))$.**

P r o o f. From the above it is easy to see that Ker is a normal ideal of E. Then $E/$Ker is a pseudo MV-algebra. We show that it is commutative. Assume that $n(a/$Ker$)$ is defined in $E/$Ker for any integer $n \geq 1$. Since every extremal state on E defines an extremal state on $E/$Ker, and vice-versa, we have that for any extremal state $n(s(a)) \leq 1$ for any integer. Therefore, $a \in$ Ker, so that $a/$Ker $= 0/$Ker. This implies that $E/$Ker is Archimedean[7]. By [8] this implies that $E/$Ker is commutative, i. e., an MV-algebra.

This MV-algebra is therefore isomorphically representable by $[0,1]$-valued continuous functions defined on $\mathbf{Ext}_{\mathcal{S}}(E/\mathbf{Ker})$ which is homeomorphic with $\mathbf{Ext}_{\mathcal{S}}(E)$. The isomorphism between $E/$Ker and $\psi(E)$ is given by $a \mapsto \psi(a)$. □

Theorem 7.3. **Let E be a pseudo-effect algebra satisfying general comparability. Set**

$$M = \{f \in M(\mathrm{Ext}_{\mathcal{S}}(E)) : f(s) \in s(E) \text{ for all discrete } s \in \mathrm{Ext}_{\mathcal{S}}(E)\}.$$

Then $\psi(E)$ is an MV-subalegbra of M which is dense in the sup-norm in M.

P r o o f. It is clear that $\psi(E) \subseteq M$. Because E satisfies general comparability, so satisfies also $E/$Ker. By Proposition 7.2, $E/$Ker is an MV-algebra satisfying general comparability theorem. By [19, Thm 8.20], we obtain the result in question. □

As a direct corollary of Theorem 7.3, we have that if E has no discrete extremal state (this can happen e. g. if E is weakly divisible) we have the following situation.

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[7]A pseudo MV-algebra E is *Archimedean* if the existence of $na \in E$ for any $n \geq 1$ entails $a = 0$.

Corollary 7.4. Let the conditions of Theorem 7.3 be satisfied. If there is no discrete state, then $\psi(E)$ is dense in M.

8. EXAMPLES

We present here examples of MV-algebras which do or do not satisfy general comparability. We study examples of MV-algebras of continuous functions on compact, Hausdorff, totally disconnected topological spaces.

Example 8.1. Let $\Omega_n = \{\omega_1, \dots, \omega_n\}$, $n \geq 1$. It is compact, Hausdorff and totally disconnected in the discrete topology. Then $M(\Omega_1) = [0,1]$ and $M(\Omega_n) = [0,1]^n$, and they satisfy general comparability.

We say that a topological space Ω is *basically disconnected* provided the closure of every open F_σ subset of Ω is open.

Example 8.2. Let Ω be a basically disconnected topological space. Then $M(\Omega)$ satisfies general comparability. This follows from [19, Cor. 9.3], because $C(\Omega)$ is Dedekind σ-complete iff Ω is basically disconnected. Then $M(\Omega)$ is a σ-complete MV-algebra, and every σ-complete MV-algebra satisfies general comparability [10, Prop 4.1].

In what follows we show that if Ω is a Hausdorff, compact, totally disconnected topological space, then $M(\Omega)$ does not necessarily satisfy general comparability.

Example 8.3. Let $X = [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers. Since it does not contains any interval of nonzero length, it is Hausdorff, totally disconnected and regular. Any set of the form $[0,a), (a,b), (b,1]$, where a and b are arbitrary irrational numbers from the real interval $(0,1)$, are clopen. Its Čech–Stone compactification, $\Omega = \beta X$, is compact, Hausdorff and totally disconnected [16, Thm 6.2.12, pp. 447–487]. Let $f(x) = x$ and $g(x) = 1 - x$, $x \in X$. Then f and g are continuous in X, and let $\hat{f}$ and $\hat{g}$ be their continuous extension to βX. We assert that there is no clopen set A in Ω such that $\hat{f}\chi_A \leq \hat{g}\chi_A$ and $\hat{f}\chi_{A^c} \geq \hat{g}\chi_{A^c}$. Indeed, we have $f(1/2) = 1/2 = g(1/2)$ and either $1/2 \in A$ or $1/2 \in A^c$. In the first case, A contains points x_1 and x_2 such that $f(x_1) < 1/2 < g(x_1)$ and $f(x_2) > 1/2 > g(x_2)$.

This implies that general comparability fails to hold in this $M(\Omega)$.

Example 8.4. Let Ω be the Čech–Stone compactification of $X = [0,1] \cap \mathbb{Q}$. We assert that for any $x_0 \in (0,1) \cap X$, the ideal $I(K_{x_0})$ of $M(\Omega)$ generated by the maximal ideal $K_{x_0} = \{\chi_A : x_0 \notin A\}$ is not prime.

Let f and g be piecewise linear functions connecting the points $(0,1), (x_0,0)$ and $(0,0)$, and $(0,0), (x_0,0)$ and $(1,1)$, respectively. Then f and g are

nonzero and let $\hat{f}$ and $\hat{g}$ be their continuous extension to Ω. Then $\hat{f} \wedge \hat{g} = 0 \in I(K_{x_0})$, but there is no clopen set A not containing point x_0 such that $\hat{f} \leq \chi_A$ or $\hat{g} \leq \chi_A$.

Example 8.4. **Let $\mathcal{C}$ be the Cantor set in $[0,1]$, that is, the set of all real numbers x of the form $x = \sum_{n=1}^\infty \frac{2\alpha_n}{3^n}$, where $\alpha_n \in \{0,1\}$ for any $n \geq 1$. This space is Hausdorff, compact and totally disconnected, but $M(\mathcal{C})$ does not satisfy general comparability. Indeed, let $x \in (0,1) \cap \mathcal{C}$, and let f and g be functions which are piecewise linear connecting the points $(0,1), (x,0)$ and $(1,0)$, and $(0,0), (x,0)$ and $(1,1)$, respectively. Then there is no clopen subset A of $\mathcal{C}$ such that $f\chi_A \leq g\chi_A$ and $f\chi_{A^c} \geq g\chi_{A^c}$.**

Similarly, $I(K)$ is not a prime ideal.

A more general case than later is the following example

Example 8.6. **Let Ω be any compact, Hausdorff, totally disconnected space, $\Omega \subset [0,1]$ and let Ω does not contain any isolated point. Then $M(\Omega)$ does not satisfy general comparability, and similarly the ideal $I(K)$ is not prime.**

Example 8.5 is interesting also from another point of view. The system of all clopen sets of $\mathcal{C} \subset [0,1]$ is an open basis of $\mathcal{C}$. It has a countable subbase. Consequently, the Baire σ-algebra, $\mathcal{B}(\mathcal{C})$, coincides with the Borel σ-algebra generated by all open subsets of $\mathcal{C}$. By [6, Cor III.5.9], every state on $C(M(\mathcal{C}))$ can be extended to a unique probability measure μ on $\mathcal{B}(\mathcal{C})$, and according to the Riesz–Markov theorem, it defines a unique state s on $M(\mathcal{C})$ such that $s(f) = \int f(\omega) d\mu(\omega)$. In this case, in spite of the fact that $M(\mathcal{C})$ is not satisfying general comparability as well as $I(K)$ is not any prime ideal of $M(\mathcal{C})$, every state on $C(M(\mathcal{C}))$, not only any extremal state, can be extended to a unique state on $M(\mathcal{C})$. In particular, every two-valued state on $C(M(\mathcal{C}))$ (it is concentrated in some point ω) can be uniquely extended to a unique (extremal) state on $M(\Omega)$, s_ω, which is defined by $s_\omega(f) := f(\omega)$, $f \in M(\mathcal{C})$.

Example 8.7. **Let S be the set of all real sequences $\{a_n\}$ such that $\lim_n a_n$ exists in $\mathbb{R}$. Then S is an Abelian ℓ-group with strong unit $\{1\}$, and $S_0 = \Gamma(S, \{1\})$ is an MV-algebra. A sequence $\{a_n\}$ is a central element iff $a_n \in \{0,1\}$ for any $n \geq 1$ and for all but finitely many n either $a_n = 0$ or $a_n = 1$. S_0 does not satisfies general comparability; take e. g., $a = \{a_n\}$ and $b = \{b_n\}$, where $a_n = 1/2$ and $b_n = 1/2 + (-1)^n/2^n$ $(n \geq 1)$.**

Any maximal ideal of $C(S_0)$ is of the form $K_i = \{\{a_n\} : a_i = 0\}$ for $i \geq 1$, or $K_\infty = \{\{a_n\} : \lim_n a_n = 0\}$. Then $I(K_i)$ is a prime ideal of S_0 for any integer $i \geq 1$ but $I(K_\infty)$ is not prime. In addition $\bigcap_{i=1}^\infty I(K_i) = \{0\}$.

In what follows, we show that in some cases $M(\Omega)$ satisfies general comparability iff $I(K)$ is a prime ideal of $M(\Omega)$ for any maximal ideal K of $C(M(\Omega))$.

We recall that a topological space Ω is *Fréchet* provided, for every $A \subseteq \Omega$ and every $\omega \in \overline{A}$, there exists a sequence $\{\omega_n\}$ of points of A converging to ω.

Proposition 8.8. **If Ω is a Fréchet, Hausdorff, compact and totally disconnected topological space, then $M(\Omega)$ satisfies general comparability if, and only if, $I(K)$ is a prime ideal of $M(\Omega)$ for any maximal ideal K of $C(M(\Omega))$.**

P r o o f. **By Claim of Theorem 4.4, general comparability of $M(\Omega)$ implies that $I(K)$ is a prime ideal of $M(\Omega)$ for any maximal ideal K of $C(M(\Omega))$.**

Suppose now that $M(\Omega)$ does not satisfy general comparability. That is, there are two continuous functions f and g in $M(\Omega)$ such that $f\chi_A \not\leq g\chi_A$ or $f\chi_{A^c} \not\geq g\chi_{A^c}$ for any clopen set A. Set $U = \{\omega \in \Omega : f(\omega) < g(\omega)\}$. Then U is an open F_σ set, and therefore, the closure $\overline{U}$ of U is not open. Hence there exists a point $\omega \in \overline{U} \setminus U$ such that, for every neighborhood O_ω of the point ω, we have $O_\omega \cap (\Omega \setminus \overline{U}) \neq \emptyset$. Therefore, $\omega \in \overline{\Omega \setminus \overline{U}}$, and there exists a sequence $\{\omega_n\}$ of points in $\Omega \setminus \overline{U}$ converging to ω. Let $K = \{\omega_n : n \geq 1\} \cup \{\omega\}$, and let us define a continuous function $h : \ \overline{U} \cup K \to [0,1]$ by $h(\omega) = 0$ for $\omega \in \overline{U}$ and $h(\omega_n) = 1/n$ $(n \geq 1)$. Let $\tilde{h}$ be the continuous extension of h onto Ω.

Since $\omega \in \overline{U}$, there exists a sequence $\{y_n\}$ in U such that $\{y_n\}$ converges to ω. Let $H = \{y_n : \ n \geq 1\} \cup \{\omega\}$ and let us define a continuous function $k : (\Omega \setminus U) \cup H \to [0,1]$ by $k(\omega) = 0$ if $\omega \in \Omega \setminus U$ and $k(y_n) = 1/n$. Let $\tilde{k}$ be the continuous extension of k onto Ω.

We have $h \wedge k = 0$, i. e., $\tilde{h} \wedge \tilde{k} = 0$. Assume that $K = I_\omega = \{\chi_B : B$ is clopen in $\Omega, \ \omega \notin B\}$, and let $B \in I_\omega$. Then $\omega \notin B$ and it cannot happen that $h \leq \chi_B$ or $g \leq \chi_B$ while χ_B takes the value 0 on some neighborhood of the point ω. □

Finally we recall that the author does not know whether any pseudo MV-algebra E such that every maximal ideal K of $C(E)$ induces the prime ideal $I(K)$ of E does satisfy general comparability.

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