Kybernetika

VOLUME *41* (2005), NUMBER 5

The Journal of the Czech Society for Cybernetics and Information Sciences

Published by:

Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic

Editor-in-Chief:

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Pod Vodárenskou věží 4, 182 08 Praha 8

Kybernetika is a bi-monthly international journal dedicated for rapid publication of high-quality, peer-reviewed research articles in fields covered by its title.

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The Journal has been monitored in the Science Citation Index since 1977 and it is abstracted/indexed in databases of Mathematical Reviews, Current Mathematical Publications, Current Contents ISI Engineering and Computing Technology.

K y b e r n e t i k a. Volume *41* (2005) ISSN 0023-5954, MK ČR E 4902.

Published bi-monthly by the Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. — Address of the Editor: P. O. Box 18, 182 08 Prague 8, e-mail: kybernetika@utia.cas.cz. — Printed by PV Press, Pod vrstevnicí 5, 140 00 Prague 4. — Orders and subscriptions should be placed with: MYRIS TRADE Ltd., P. O. Box 2, V Štíhlách 1311, 142 01 Prague 4, Czech Republic, e-mail: myris@myris.cz. — Sole agent for all "western" countries: Kubon & Sagner, P. O. Box 34 01 08, D-8 000 München 34, F.R.G.

Published in October 2005.

KYBERNETIKA — VOLUME *41* (2005), NUMBER 5, PAGES 637 – 660

DIRECT ALGORITHM FOR POLE PLACEMENT BY STATE–DERIVATIVE FEEDBACK FOR MULTI–INPUT LINEAR SYSTEMS – NONSINGULAR CASE

Taha H. S. Abdelaziz and Michael Valášek

This paper deals with the direct solution of the pole placement problem by state-derivative feedback for multi-input linear systems. The paper describes the solution of this pole placement problem for any controllable system with nonsingular system matrix and nonzero desired poles. Then closed-loop poles can be placed in order to achieve the desired system performance. The solving procedure results into a formula similar to Ackermann one. Its derivation is based on the transformation of linear multi-input systems into Frobenius canonical form by coordinate transformation, then solving the pole placement problem by state derivative feedback and transforming the solution into original coordinates. The procedure is demonstrated on examples. In the present work, both time-invariant and time-varying systems are treated.

Keywords: pole placement, state-derivative feedback, linear MIMO systems, feedback stabilization

AMS Subject Classification:

1. INTRODUCTION

Pole placement technique is one of the most important approaches for linear control systems design. The state feedback control problem has been investigated in control community during the last four decades. There have been developed the design methods for a wide class of linear systems under full-state feedback with the objective of stabilizing control systems (e. g. [8, 18, 19, 20, 21]).

However, this paper focuses on a special feedback using only state derivatives instead of full-state feedback. Therefore this feedback is called state derivative feedback. The problem of arbitrary pole placement using state-derivative feedback naturally arises. To the best knowledge of the authors there have been yet no general study solving this feedback for pole placement based on traditional approaches to pole placement by state feedback. The problem of state derivative feedback has been investigated within the treatment of generalized class of singular linear dynamic systems using geometric approach in [12] and [10]. Only recently, the authors have derived [1, 2] a pole placement technique by state-derivative feedback for single-input time-invariant and time-varying linear systems. However, the generalization of these results for multi-input systems is not an easy task. This paper is the first attempt to solve the aforementioned problem with a simple direct way.

In general, it is well known from classical control theory that derivative feedback is sometimes essential for achieving desired control objectives [12]. However, the motivation for the state derivative feedback in this paper comes from controlled vibration suppression of mechanical systems. The main sensors of vibration are accelerometers. From accelerations it is possible to reconstruct velocities with reasonable accuracy but not any longer the displacements. Therefore the available signals for feedback are accelerations and velocities only and these are exactly the derivatives of states of the mechanical systems that are the velocities and displacements. There have been published many papers (e. g. [3, 4, 9, 14, 15, 16]) describing the acceleration feedback for controlled vibration suppression. However, the pole placement approach for feedback gain determination has not been used at all or has not been solved generally. The approach in [3, 4, 15, 16] is based on dynamic derivative output feedback. The feedback uses acceleration only (the velocity is not used, therefore it is not full-state derivative feedback, but only output derivative feedback) and the acceleration is processed by dynamic filter (dynamic feedback). The feedback gains are determined using root locus analysis [3, 4, 14, 15, 16], optimization of H_2 norm of the closed loop transfer function [4], or using just numerical parameter optimization of performance indexes [9]. Another papers dealing with acceleration feedback for mechanical systems are [5, 6] but there the feedback uses all states (positions, velocities) and accelerations additionally.

In this paper a generalization of eigenvalue assignment by state-derivative feedback for multi-input time-invariant and time-varying linear systems is presented. However, this paper deals only with the case of nonsingular system matrix of the original system. The whole procedure is unique and provides more insight into the eigenvalue assignment. The proposed controller is based on the measurement and feedback of the state derivatives of the system. In this study, particular attention is directed toward the Frobenius canonical form, because of its unparallel position in arriving at the desired pole placement for linear systems. This work has successfully extended previous techniques by state feedback and modified to state-derivative feedback. The new formulations are derived through the following three steps design. The first step is an implementation of a state coordinate transformation to the Frobenius canonical form. The second step involves the subsequent employment of pole placement technique for the transformed linear systems. The third step is the transformation of the state-derivative feedback into the original coordinates. This provides a new systematic way of solving the aforementioned problem with a simple direct way. Finally, the derived technique is demonstrated on examples.

In summary, the rest of this paper is organized as follows. In Section 2, we begin with a transformation to Frobenius canonical form for multi-input systems and introduce the solution of the pole placement problem by state-derivative feedback for time-invariant systems. Section 3 deals with the extension of pole placement for multi-input time-varying systems. In Section 4, the illustrative examples and simulation results are presented. Finally, conclusion is in Section 5.

2. POLE PLACEMENT BY STATE–DERIVATIVE FEEDBACK FOR MULTI–INPUT TIME–INVARIANT SYSTEMS

In this section, we provide a detailed description of the algorithm for the pole placement problem by state-derivative feedback for linear time-invariant systems.

2.1. Pole placement problem formulation

Consider a multi-input, time-invariant, linear system with the following state-space representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^m$ are the state and the control vectors, respectively, $(m \leq n)$, while $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ are the system and control gain matrices, respectively. The fundamental assumptions imposed on the system is that, the system is completely controllable and the m columns of the matrix $\boldsymbol{B}$, $\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m]$, are linearly independent ($\boldsymbol{B}$ has a full column rank m). Further it is assumed that the system matrix $\boldsymbol{A}$ is nonsingular.

The objective is to stabilize the system by means of a linear feedback that enforces a desired characteristic behavior for the states. The design problem is to find the state-derivative feedback control law

$$\boldsymbol{u}(t) = -\boldsymbol{K}\dot{\boldsymbol{x}}(t) \tag{2}$$

that assigns prescribed closed-loop eigenvalues, that stabilizes the system and achieves the desired performance. Substituting (2) into (1) the closed-loop system dynamics becomes

$$\begin{aligned} (\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K})\,\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) &= (\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{A}\boldsymbol{x}(t) \end{aligned} \tag{3}$$

where $\boldsymbol{I}_n$ is the $n\times n$ identity matrix. In the following, matrix $(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K})$ is assumed to have a full rank in order that the closed-loop system is well defined.

The problem is to find such feedback gain matrix $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ that the self-conjugate closed-loop eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are assigned at the desired values. It will be shown that the desired eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ must be nonzero. The major difficulty is that the system matrix $\boldsymbol{A}$ is manipulated by the feedback gain $\boldsymbol{K}$ in (3) by indirect way that is not similar to the traditional state feedback modification of system matrix.

In order to overcome this difficulty, the system can be manipulated based on a transformation of coordinates. In other words, the pole placement problem is easily solved if the system is preliminarily reduced to a simple structure of the transformed matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Consequently, the pole placement methodology can be applied. A preliminary step for solving the above problem is to transform this system to the Frobenius canonical form, and the next step is to employ pole placement technique in order to arbitrarily assign the poles of the closed-loop system and achieve the above objective.

2.2. Transformation into Frobenius canonical form for time-invariant systems

Frobenius canonical form is constructed by transforming the state vector to a new coordinate system in which the system equations take a particular form. Let us take the following time-invariant linear coordinate transformation

$$\boldsymbol{z}(t) = \boldsymbol{Q}^{-1}\boldsymbol{x}(t), \quad \boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{z}(t) \tag{4}$$

where $\boldsymbol{z}(t) \in \mathbb{R}^n$ is the transformed state variable vector and the transformation matrix is $\boldsymbol{Q}^{-1} \in \mathbb{R}^{n\times n}$. Then, the Frobenius canonical form is

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_F\boldsymbol{z}(t) + \boldsymbol{B}_F\boldsymbol{u}(t) \tag{5}$$

where $\boldsymbol{A}_F \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}_F \in \mathbb{R}^{n\times m}$ are the transformed system and control gain matrices, respectively, and given by [13],

$$\boldsymbol{A}_F = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}, \quad \boldsymbol{B}_F = \boldsymbol{Q}^{-1}\boldsymbol{B} \tag{6}$$

where

$$\boldsymbol{A}_F = \begin{pmatrix}
\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \times & \times & \cdots & \times \end{pmatrix} &
\begin{matrix} 0 & & \cdots & \\ \vdots & & & \\ 0 & & \cdots & \\ \times & \times & \cdots & \times \end{matrix} &
\begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} &
\begin{matrix} & & \cdots & 0 \\ & & & \vdots \\ & & \cdots & 0 \\ \times & \times & \cdots & \times \end{matrix} \\
\begin{matrix} 0 & & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & & \cdots & 0 \\ \times & \times & \cdots & \times \end{matrix} &
\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \times & \times & \cdots & \times \end{pmatrix} &
\begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} &
\begin{matrix} 0 & & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & & \cdots & 0 \\ \times & \times & \cdots & \times \end{matrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{matrix} 0 & & & \cdots \\ \vdots & & & \\ 0 & & & \cdots \\ \times & \times & \cdots & \times \end{matrix} &
\begin{matrix} \cdots & & \cdots & 0 \\ & & & \vdots \\ \cdots & & \cdots & 0 \\ \times & \times & \cdots & \times \end{matrix} &
\begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} &
\begin{pmatrix} 0 & 1 & \cdots 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \times & \times & \cdots & \times \end{pmatrix}
\end{pmatrix}$$

$$\boldsymbol{B}_F = \begin{pmatrix}
\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \times \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \times \end{matrix} \\
\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} & \begin{matrix} \cdots \\ \\ \cdots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \times \end{matrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix} & \begin{matrix} \cdots \\ \\ \cdots \\ \cdots \end{matrix} & \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\end{pmatrix}. \tag{7}$$

It is shown that, this system is composed of m fundamental companion matrices located in blocks along the diagonal. Each of the companion matrices can be considered to represent a subsystem coupled to other subsystems. The block size is μ_j, the controllability index corresponding to $\boldsymbol{b}_j$ of matrix $\boldsymbol{B}$, and $\mu_1 + \cdots + \mu_m = n$, $j = 1, \ldots, m$. Then, the multi-input system is reduced to a coupled set of m single-input subsystems that can be easily manipulated and, consequently, solve the pole placement problem. The $\times$'s in the matrices represent generally nonzero elements.

The constant transformation matrix $\boldsymbol{Q}^{-1} \in \mathbb{R}^{n\times n}$ is constructed as follows

$$\boldsymbol{Q}^{-1} = \text{rows}\left(\boldsymbol{q}_1\ \boldsymbol{q}_1\boldsymbol{A}\cdots\boldsymbol{q}_1\boldsymbol{A}^{\mu_1-1}\ \boldsymbol{q}_2\ \boldsymbol{q}_2\boldsymbol{A}\cdots\boldsymbol{q}_2\boldsymbol{A}^{\mu_2-1}\cdots\boldsymbol{q}_m\ \boldsymbol{q}_m\boldsymbol{A}\cdots\boldsymbol{q}_m\boldsymbol{A}^{\mu_m-1}\right) \tag{8}$$

where $\boldsymbol{q}_j \in \mathbb{R}^{1\times n}$ denotes the row vector computed as follows:

$$\boldsymbol{q}_j = \boldsymbol{e}_{r_j}^{\mathrm{T}}\boldsymbol{R}^{-1}, \quad \boldsymbol{r}_j = \sum_{k=1}^{j}\mu_k, \quad j = 1, \ldots, m, \tag{9}$$

where $\boldsymbol{e}_{r_j} \in \mathbb{R}^n$ is unit vector with 1 at position $\boldsymbol{r}_j$.

The controllability matrix of system (1), $\boldsymbol{R} \in \mathbb{R}^{n\times n}$, is

$$\boldsymbol{R} = \left(\boldsymbol{b}_1\ \boldsymbol{A}\boldsymbol{b}_1\cdots\boldsymbol{A}^{\mu_1-1}\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{A}\boldsymbol{b}_2\cdots\boldsymbol{A}^{\mu_2-1}\boldsymbol{b}_2\cdots\boldsymbol{b}_m\ \boldsymbol{A}\boldsymbol{b}_m\cdots\boldsymbol{A}^{\mu_m-1}\boldsymbol{b}_m\right). \tag{10}$$

The selection of the vectors comprising the $\boldsymbol{R}$ matrix is done according to the following procedure. The process starts with all columns $\boldsymbol{b}_j$ of matrix $\boldsymbol{B}$. At step i, the columns $\boldsymbol{A}^{i-1}\boldsymbol{b}_j$ are studied for their dependence on all previous ones on the order $j = 1, \ldots, m$ from left to right. If the selected vector is linearly independent of the previously selected vectors, retain it, otherwise omit it from the selection. The selection process terminates when n linearly independent vectors are found. Arrange the n vectors in their proper order to form the matrix $\boldsymbol{R}$. It has been proven [19] that the transformation matrix $\boldsymbol{Q}^{-1}$ obtained by this procedure is nonsingular and the transformation to the generalized canonical form can be made. The above steps complete the transformation into canonical form. These results substantially simplified the manipulation of the pole placement problem. The next step is to develop the feedback gain matrix and solve the pole placement problem.

2.3. Solution of the pole placement problem for time-invariant systems

In this section, we shall show how to derive an explicit formula for the state-derivative feedback gain matrix $\boldsymbol{K}$ that assigns the desired closed-loop poles system in a computational efficient and simple direct manner. Utilizing the above transformation into canonical form, the system can be manipulated by a linear feedback for a desired behavior (i. e., the pole placement problem). By differentiating the transformation equation (4), the resulting closed-loop system in the z-coordinates is

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{Q}^{-1}\dot{\boldsymbol{x}}(t). \tag{11}$$

Hence, after the substitution of (3) and (4) in the above equation we obtain

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{Q}^{-1}(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K}^{-1})\boldsymbol{A}\boldsymbol{Q}\boldsymbol{z}(t) = \boldsymbol{A}_Z\boldsymbol{z}(t) \tag{12}$$

where $\boldsymbol{A}_Z \in \mathbb{R}^{n\times n}$ is the closed-loop system matrix in the z-coordinates and given by

$$\boldsymbol{A}_Z = \boldsymbol{Q}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK})^{-1}\boldsymbol{AQ}. \tag{13}$$

Postmultiply the above equation by $\boldsymbol{Q}^{-1}\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK})$ the above equation can be rewritten as

$$\boldsymbol{A}_Z\boldsymbol{Q}^{-1}\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK}) = \boldsymbol{Q}^{-1}. \tag{14}$$

To solve the pole placement problem, we first divide the desired poles into a self-conjugate m groups $\{\lambda^1\},\dots,\{\lambda^m\}$, with μ_j poles in each block, $j = 1,\dots,m$, where $\lambda^j \equiv (\lambda_1^j,\dots,\lambda_{\mu_j}^j)$. It is also advantageous that the desired poles are distributed among all blocks and the largest eigenvalues lies within the smallest block. The benefit of this is to smoothing and minimizing undesirable transient variations [19]. The corresponding real vectors $\{\boldsymbol{d}^1\},\dots,\{\boldsymbol{d}^m\}$, with $\boldsymbol{d}^j \equiv (d_0^j,\dots,d_{\mu_j-1}^j)$ that are the coefficients of desired characteristic equations for groups j are computed

$$\begin{aligned} \boldsymbol{D}_j(s) &= (s-\lambda_1^j)\,(s-\lambda_2^j)\cdots(s-\lambda_{\mu_j}-1) \\ &= s^{\mu_j} + d_{\mu_j-1}^j s^{\mu_j-1} + \dots + d_1^j s + d_0^j, \quad j = 1,\dots,m \end{aligned} \tag{15}$$

Then the structure of the desired closed-loop matrix can be formed as a block diagonal matrix as

$$\boldsymbol{A}_Z = \begin{pmatrix} \begin{pmatrix} \boldsymbol{0}_{\mu_1-1,1} & \boldsymbol{I}_{\mu_1-1} \\ \multicolumn{2}{c}{-\boldsymbol{d}^1} \end{pmatrix} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \begin{pmatrix} \boldsymbol{0}_{\mu_2-1,1} & \boldsymbol{I}_{\mu_2-1} \\ \multicolumn{2}{c}{-\boldsymbol{d}^2} \end{pmatrix} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \begin{pmatrix} \boldsymbol{0}_{\mu_m-1,1} & \boldsymbol{I}_{\mu_m-1} \\ \multicolumn{2}{c}{-\boldsymbol{d}^m} \end{pmatrix} \end{pmatrix}. \tag{16}$$

It is noting that the eigenvalues of $\boldsymbol{A}_Z$ are the same as the desired closed-loop poles. From the equations (13) and/or (14) it is clear that for nonsingular matrix $\boldsymbol{A}$ the desired matrix $\boldsymbol{A}_Z$ must be also nonsingular as the matrices $(\boldsymbol{I}_n + \boldsymbol{BK})$ and $\boldsymbol{Q}$ are of full rank.

From the derivation of the state-derivative feedback pole placement the necessary conditions for arbitrary pole placement with nonzero eigenvalues can be described in the following lemma.

Lemma 1. If the pole placement problem with nonzero self-conjugate desired poles for the real pair $(\boldsymbol{A},\boldsymbol{B})$ is solvable, then $(\boldsymbol{A},\boldsymbol{B})$ is completely controllable, that is

$$\operatorname{rank}[\boldsymbol{B},\boldsymbol{AB},\dots,\boldsymbol{A}^{n-1}\boldsymbol{B}] = n, \tag{17}$$

and $\boldsymbol{A}$ is nonsingular.

Proof. Suppose that $(\boldsymbol{A},\boldsymbol{B})$ is not completely controllable. Then there exist an eigenvalue, say λ, of $\boldsymbol{A}$ and a vector $\boldsymbol{w} \neq 0$ such that

$$\boldsymbol{w}^{\mathrm{T}}\boldsymbol{A} = \lambda\boldsymbol{w}^{\mathrm{T}}, \quad \boldsymbol{w}^{\mathrm{T}}\boldsymbol{B} = 0.$$

Then
$$\boldsymbol{w}^{\mathrm{T}}\left[\left(\boldsymbol{I}_n-\boldsymbol{B}\boldsymbol{K}\right)\lambda-\boldsymbol{A}\right]=\boldsymbol{w}^{\mathrm{T}}\lambda-\lambda\boldsymbol{w}^{\mathrm{T}}=0$$
so that λ is a closed-loop eigenvalue as well, contradicting the change of poles to desired ones. For the transformation into the Frobenius canonical form and/or computing the feedback gain the controllability matrix $\boldsymbol{R}$ must be of full rank (the open-loop system must be controllable).

From the condition that the closed-loop matrix in equation (3) must be defined it follows that $(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K})$ must be of full rank. The equation (13) is easy to be rewritten as
$$\boldsymbol{A}_Z=\left(\boldsymbol{I}_n+\boldsymbol{B}_F\boldsymbol{K}_F\right)^{-1}\boldsymbol{A}_F,\quad \boldsymbol{K}_F=\boldsymbol{K}\boldsymbol{Q}. \tag{18}$$
In order that the matrix $(\boldsymbol{I}_n+\boldsymbol{B}_F\boldsymbol{K}_F)$ has a full rank, the matrices $\boldsymbol{A}_F$ and $\boldsymbol{A}_Z$ must be both either nonsingular or singular. Thus if $\boldsymbol{A}_Z$ is nonsingular, i. e. the desired poles are nonzero, then the matrix $\boldsymbol{A}_F$ must be also nonsingular, i. e. $\boldsymbol{A}$ is nonsingular. □

Equation (14) can be rewritten in terms of the row vectors $\boldsymbol{q}_j$ $(j=1,\dots,m)$ of $\boldsymbol{Q}^{-1}$ as
$$\begin{aligned}
\boldsymbol{q}_1\boldsymbol{A}^i\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right)&=\boldsymbol{q}_1\boldsymbol{A}^i, \qquad i=0,\dots,\mu_1-2,\\
\sum_{i=0}^{\mu_1-1}\left(-d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right)&=\boldsymbol{q}_1\boldsymbol{A}^{\mu_1-2},\\
\boldsymbol{q}_2\boldsymbol{A}^i\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right)&=\boldsymbol{q}_2\boldsymbol{A}^i, \qquad i=0,\dots,\mu_2-2,\\
\sum_{i=0}^{\mu_2-1}\left(-d_i^2\boldsymbol{q}_2\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right)&=\boldsymbol{q}_2\boldsymbol{A}^{\mu_2-1},\\
&\dots,\\
\boldsymbol{q}_m\boldsymbol{A}^i(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K})&=\boldsymbol{q}_m\boldsymbol{A}^i, \qquad i=0,\dots,\mu_m-2,\\
\sum_{i=0}^{\mu_m-1}\left(-d_i^m\boldsymbol{q}_m\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right)&=\boldsymbol{q}_m\boldsymbol{A}^{\mu_m-1}.
\end{aligned} \tag{19}$$

Based on the definition of the transformation matrix $\boldsymbol{Q}^{-1}$, it can be easily verified that
$$\boldsymbol{q}_j\boldsymbol{A}^i\boldsymbol{B}=\boldsymbol{0}_{1,m},\quad j=1,\dots,m,\ i=0,\dots,\mu_j-2. \tag{20}$$
It is easy to write the m equations describing the closed-loop system as
$$\begin{aligned}
\sum_{i=0}^{\mu_1-1}\left(-d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right) &= \boldsymbol{q}_1\boldsymbol{A}^{\mu_1-1},\\
\sum_{i=0}^{\mu_2-1}\left(-d_i^2\boldsymbol{q}_1\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right) &= \boldsymbol{q}_2\boldsymbol{A}^{\mu_2-1},\\
&\dots,\\
\sum_{i=0}^{\mu_m-1}\left(-d_i^m\boldsymbol{q}_1\boldsymbol{A}^{i-1}\right)\left(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}\right) &= \boldsymbol{q}_m\boldsymbol{A}^{\mu_m-1}.
\end{aligned} \tag{21}$$

These equations can be put in a matrix form and solved algebraically. Then, the feedback gain matrix $\boldsymbol{K}$ for the time-invariant system can be written as

$$\boldsymbol{K} = \begin{pmatrix} \left(\sum_{i=0}^{\mu_1-1}(-d_i^1\boldsymbol{q}_i\boldsymbol{A}^{i-1})\right)\boldsymbol{B} \\ \vdots \\ \left(\sum_{i=0}^{\mu_m-1}(-d_i^m\boldsymbol{q}_m\boldsymbol{A}^{i-1})\right)\boldsymbol{B} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{q}_1\boldsymbol{A}^{\mu_1-1}+\sum_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1}) \\ \vdots \\ \boldsymbol{q}_m\boldsymbol{A}^{\mu_m-1}+\sum_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_m\boldsymbol{A}^{i-1}) \end{pmatrix}$$
$$= \boldsymbol{M}_1^{-1}\begin{pmatrix} \boldsymbol{q}_1\boldsymbol{A}^{\mu_i-1}+\sum_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1}) \\ \vdots \\ \boldsymbol{q}_m\boldsymbol{A}^{\mu_m-1}+\sum_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_m\boldsymbol{A}^{i-1}) \end{pmatrix}. \tag{22}$$

Utilizing (20) then matrix $\boldsymbol{M}_1$ can be given by

$$\boldsymbol{M}_1 = \begin{pmatrix} \sum_{i=0}^{\mu_1-1}(-d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1}) \\ \vdots \\ \sum_{i=0}^{\mu_m-1}(-d_i^m\boldsymbol{q}_1\boldsymbol{A}^{i-1}) \end{pmatrix}\boldsymbol{B} = -\begin{pmatrix} d_0^1 & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & d_0^m \end{pmatrix}\begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_m \end{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B}. \tag{23}$$

Therefore, $\boldsymbol{M}_1$ is nonsingular if $\boldsymbol{A}$ has full rank and $\boldsymbol{B}$ has full column rank and all the desired poles are non-zero.

The gain matrix can be given by

$$\boldsymbol{K} = -\left(\begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_m \end{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\begin{pmatrix} \frac{1}{d_0^1}\left(\boldsymbol{q}_m\boldsymbol{A}^{\mu_1-1}+\sum_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_1\boldsymbol{A}^{i-1})\right) \\ \vdots \\ \frac{1}{d_0^m}\left(\boldsymbol{q}_1\boldsymbol{A}^{\mu_m-1}+\sum_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_m\boldsymbol{A}^{i-1})\right) \end{pmatrix}. \tag{24}$$

The gain matrix can be rewritten in a simple form as

$$\boldsymbol{K} = -\left(\begin{pmatrix} \boldsymbol{e}_{\mu_1}^{\mathrm{T}}(\boldsymbol{AR})^{-1} \\ \vdots \\ \boldsymbol{e}_n^{\mathrm{T}}(\boldsymbol{AR})^{-1} \end{pmatrix}\boldsymbol{B}\right)^{-1}\begin{pmatrix} \frac{1}{d_0^1}\left(\boldsymbol{e}_{\mu_1}^{\mathrm{T}}(\boldsymbol{AR})^{-1}\boldsymbol{D}_1(\boldsymbol{A})\right) \\ \vdots \\ \frac{1}{d_0^m}\left(\boldsymbol{e}_n^{\mathrm{T}}(\boldsymbol{AR})^{-1}\boldsymbol{D}_m(\boldsymbol{A})\right) \end{pmatrix} \tag{25}$$

where $\boldsymbol{D}_j(\boldsymbol{A}) \in \mathbb{R}^{n\times n}$ is the evaluation of the desired characteristic polynomial $\boldsymbol{D}_j$ with the state matrix $\boldsymbol{A}$ and computed as

$$\boldsymbol{D}_j(\boldsymbol{A}) = \boldsymbol{A}^{\mu_j} + d_{\mu_j-1}^j\boldsymbol{A}^{\mu_j-1} + \cdots + d_1^j\boldsymbol{A} + d_0^j\boldsymbol{I}_n, \quad j = 1,\ldots,m. \tag{26}$$

Now, it is considered the stabilizing feedback control defined by a set of desired eigenvalues λ_i, $i = 1,\ldots,n$, instead of the evaluated coefficients of the characteristic equation. The desired eigenvalues are divided into self-conjugate m groups

$\{\lambda^1\}, \ldots, \{\lambda^m\}$ with μ_j poles in each and distributed these poles among the blocks. The feedback gain matrix is

$$\boldsymbol{K} = -\left(\begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_m \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \begin{pmatrix} \prod_{i=1}^{\mu_1}\left(\frac{-1}{\lambda_i^1}\right)\left(\boldsymbol{q}_1\boldsymbol{A}^{-1}\prod_{i=1}^{\mu_1}(\boldsymbol{A}-\lambda_i^1\boldsymbol{I}_n)\right) \\ \vdots \\ \prod_{i=1}^{\mu_m}\left(\frac{-1}{\lambda_1^m}\right)\left(\boldsymbol{q}_m\boldsymbol{A}^{-1}\prod_{i=1}^{\mu_m}(\boldsymbol{A}-\lambda_m^m\boldsymbol{I}_n)\right) \end{pmatrix}. \tag{27}$$

An efficient numerical algorithm for computing the feedback gain matrix $\boldsymbol{K}$ is

$$\boldsymbol{K} = -\left(\begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_m \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \begin{pmatrix} \prod_{i=1}^{\mu_1}\left(\frac{1}{\lambda_i^1}\right)\boldsymbol{q}_{\mu_1}^{1'} \\ \vdots \\ \prod_{i=1}^{\mu_m}\left(\frac{1}{\lambda_i^m}\right)\boldsymbol{q}_{\mu_m}^{m'} \end{pmatrix}, \tag{28}$$

where

$$\boldsymbol{q}_U^{j'} = \boldsymbol{e}_{r_j}^{\mathrm{T}}(\boldsymbol{AR})^{-1}, \quad \boldsymbol{q}_i^{j'} = \boldsymbol{q}_{i-1}^{j'}(\boldsymbol{A}-\lambda_i^j\boldsymbol{I}_n), \quad j = 1, \ldots, m,\ i = 1, \ldots, \mu_j.$$

One can notice that the proposed algorithm is straightforward, easy to be implemented and the feedback gain calculations are not done in the intermediate Frobenius form and direct implementation is performed in the original state space. The above algorithm is valid for desired eigenvalues that are real, complex-conjugate and repeated poles. Note that, the complex-conjugate eigenvalues should be placed within the same block. It should be pointed out that different sequence of the desired poles will lead to different feedback gain matrices. For smoothing and minimizing undesirable transient variations, the largest poles can lie within the smallest block [19]. The transformation matrix $\boldsymbol{Q}^{-1}$ plays an important role to solve this problem.

Remark 1. For the case of $(m = n)$ and utilizing (14) the feedback gain can be given by:

$$\boldsymbol{K} = \boldsymbol{B}^{-1}\left(\boldsymbol{AQA}_Z^{-1} - \boldsymbol{I}_n\right) \tag{29}$$

where $\boldsymbol{A}_Z$ is in Jordan canonical form with the desired eigenvalues on the diagonal.

Remark 2. For single-input case $(m = 1)$, the state-derivative feedback gain can be written as:

If the coefficients d_i, $i = 1, \ldots, n$, of the characteristic equation are given [1, 2]

$$\boldsymbol{K} = \left(\frac{\det(-\boldsymbol{A})}{d_0}\right)\left(\boldsymbol{q}_n' + \sum_{i=0}^{n-1}(d_i\boldsymbol{q}_i')\right), \tag{30}$$

where

$$\boldsymbol{q}_0' = \boldsymbol{e}_n^{\mathrm{T}}(\boldsymbol{AR})^{-1}, \quad \boldsymbol{q}_i' = \boldsymbol{q}_{i-1}'\boldsymbol{A}.$$

Furthermore, if a set of desired eigenvalues λ_i, $i = 1, \ldots, n$, are given [1, 2]

$$\boldsymbol{K} = \frac{\det(\boldsymbol{A})}{\prod_{i=1}^{n}\lambda_i}\boldsymbol{q}_n', \tag{31}$$

where
$$\boldsymbol{q}_0' = \boldsymbol{e}_n^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{R})^{-1}, \quad \boldsymbol{q}_i' = \boldsymbol{q}_{i-1}'(\boldsymbol{A} - \lambda_i \boldsymbol{I}_n).$$

With the above results, we are now in the position to present the first main result of this work.

Theorem 1. Consider the controllable multi-input time-invariant linear system (1). If system matrix $\boldsymbol{A}$ is nonsingular and $\boldsymbol{B}$ has full column rank, then the system (1) with the state-derivative feedback (2) can be stabilized with the unique feedback gain $\boldsymbol{K}$ (28) or (25) with the prescribed non-zero eigenvalues $\{\lambda^1\}, \ldots, \{\lambda^m\}$, with self-conjugate μ_j poles in each block, or with the real non-zero coefficients $\{\boldsymbol{d}^1\}, \ldots, \{\boldsymbol{d}^m\}$. For single-input case $(m = 1)$, the feedback gain can be given by (30) or (31).

However, on the other hand the control effort $\boldsymbol{u}(t)$ is the same for both state feedback and state-derivative feedback. This can be derived from (14), (12), (11) and the fact that the system has after the application of the feedback $\boldsymbol{K}$ the desired dynamic properties
$$\begin{aligned} \boldsymbol{u}(t) &= -\boldsymbol{K}\dot{\boldsymbol{x}}(t) = -\boldsymbol{B}^{+}\left(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}_Z^{-1}\boldsymbol{Q}^{-1} - \boldsymbol{I}_n\right)\boldsymbol{Q}\boldsymbol{A}_Z\boldsymbol{z}(t) \\ &= -\boldsymbol{B}^{+}(\boldsymbol{A}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{A}_Z)\,\boldsymbol{u}(t) = -\boldsymbol{B}^{+}(\boldsymbol{A} - \boldsymbol{Q}\boldsymbol{A}_Z\boldsymbol{Q}^{-1})\,\boldsymbol{x}(t) \\ &= -\boldsymbol{K}_S\boldsymbol{x}(t) \end{aligned} \tag{32}$$

where $(\cdot)^{+}$ denote the Moore–Penrose generalized inverse. The last expression is exactly the traditional state feedback for the change from original system poles to the desired ones and the same state transformation matrix $\boldsymbol{Q}^{-1}$.

Further, the transient response for state-derivative feedback is obtained by utilizing (13)
$$(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{A}_Z\boldsymbol{Q}^{-1}. \tag{33}$$

Therefore, the closed-loop system is
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{Q}\boldsymbol{A}_Z\boldsymbol{Q}^{-1}\boldsymbol{x}(t) \tag{34}$$

which is the identical response for state feedback with the same desired poles and transformation matrix.

The above formulation is devoted for completely controllable systems. In the following remark uncontrollable systems can be stabilized via state-derivative feedback.

Remark 3. If system (1) is not completely controllable, then by using a nonsingular state transformation matrix $\boldsymbol{T} \in \mathbb{R}^{n\times n}$
$$\boldsymbol{z}(t) = \boldsymbol{T}\boldsymbol{x}(t) \tag{35}$$
we can obtain that
$$\dot{\boldsymbol{z}}(t) = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix}\boldsymbol{z}(t) + \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{0} \end{pmatrix}\boldsymbol{u}(t), \quad \boldsymbol{z}(t) = \begin{pmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{pmatrix} \tag{36}$$

where the pair $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is controllable and the vector $\boldsymbol{x}_1(t) \in \mathbb{R}^c$ has dimension $c = \operatorname{rank}[\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{B}] < n$, whilst the vector $\boldsymbol{x}_2(t) \in \mathbb{R}^{n-c}$ contains the state components which are completely uncontrollable. The poles of matrix $\boldsymbol{A}_{22}$ are referred to as uncontrollable poles of the system.

Let the control law be taken as

$$\boldsymbol{u}(t) = -[\boldsymbol{K}_1, \boldsymbol{K}_2]\, \dot{\boldsymbol{z}}(t) \tag{37}$$

where $\boldsymbol{K}_1 \in \mathbb{R}^{m\times c}$ and $\boldsymbol{K}_2 \in \mathbb{R}^{m\times n-c}$.

Then, the transformed closed-loop system can be described by

$$\begin{pmatrix} \boldsymbol{I}_c + \boldsymbol{B}_1\boldsymbol{K}_1 & \boldsymbol{B}_1\boldsymbol{K}_2 \\ \boldsymbol{0} & \boldsymbol{I}_{n-c} \end{pmatrix} \dot{\boldsymbol{z}}(t) = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix} \boldsymbol{z}(t). \tag{38}$$

Therefore

$$\dot{\boldsymbol{z}}(t) = \begin{pmatrix} (\boldsymbol{I}_C + \boldsymbol{B}_1\boldsymbol{K}_1)^{-1} & \boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I}_{n-c} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix} \boldsymbol{z}(t) \tag{39}$$

where $\boldsymbol{N} \in \mathbb{R}^{c\times n-c}$.

Continuing the derivation, it is easy to obtain

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_Z \boldsymbol{z}(t), \quad \boldsymbol{A}_Z = \begin{pmatrix} (\boldsymbol{I}_C + \boldsymbol{B}_1\boldsymbol{K}_1)^{-1}\boldsymbol{A}_{11} & (\boldsymbol{I}_n + \boldsymbol{B}_1\boldsymbol{K}_1)^{-1}\boldsymbol{A}_{12} + \boldsymbol{N}\boldsymbol{A}_{22} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix}. \tag{40}$$

Then, the eigenvalues of matrix $\boldsymbol{A}_Z$ are those of $(\boldsymbol{I}_C + \boldsymbol{B}_1\boldsymbol{K}_1)^{-1}\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$. Therefore the state-derivative feedback affects only the controllable part of the system. The controllable poles can be assigned at desired values using the above algorithm, while the uncontrollable poles are not altered by feedback. If the matrix $\boldsymbol{A}_{22}$ is stable, the system is said to be stabilized and it is possible to find the feedback gains for which the closed-loop system is asymptotically stable. The matrix $\boldsymbol{K}_2$ does not affect the closed-loop poles and may be arbitrarily chosen as $\boldsymbol{K}_2 = \boldsymbol{0}$.

Finally, the state-derivative feedback gain can be given by

$$\boldsymbol{K} = [\boldsymbol{K}_1,\, \boldsymbol{0}_{n-c}]\, \boldsymbol{T}. \tag{41}$$

Therefore, the controllable eigenvalues can be reassigned with desired values.

3. POLE PLACEMENT BY STATE–DERIVATIVE FEEDBACK FOR MULTI–INPUT TIME–VARYING SYSTEMS

In this section, we extended the above methodology for the general multi-input linear time-varying dynamic systems. Consider the multi-input time-varying linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\, \boldsymbol{x}(t) + \boldsymbol{B}(t)\, \boldsymbol{u}(t) \tag{42}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{u}(t) \in \mathbb{R}^m$ are the state and the control vectors, respectively, while $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}(t) \in \mathbb{R}^{n\times m}$ are the system and control gain matrices, respectively. The sufficient conditions for the existence and unique solution is to require that all elements of matrices $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$ are bounded and

n-times continuously differentiable with bounded derivatives, $\boldsymbol{A}(t)$ is of full rank and $\boldsymbol{B}(t) \equiv [\boldsymbol{b}_1(t), \ldots, \boldsymbol{b}_m(t)]$ is of full column rank in the time interval of interest, $t \in [t_0, \infty]$.

The objective here is to find a time-dependent linear feedback gain matrix that stabilize the system by the time-varying state-derivative feedback control law

$$\boldsymbol{u}(t) = -\boldsymbol{K}(t)\, \dot{\boldsymbol{x}}(t). \tag{43}$$

Then the closed-loop system can be written as

$$\dot{\boldsymbol{x}}(t) = \left(\boldsymbol{I}_n + \boldsymbol{B}(t)\, \boldsymbol{K}(t)\right)^{-1} \boldsymbol{A}(t)\, \boldsymbol{x}(t). \tag{44}$$

Similar to the time-invariant case, matrix $(\boldsymbol{I}_n + \boldsymbol{B}(t)\, \boldsymbol{K}(t))$ is assumed to have a full rank in order that the closed-loop system is well defined.

One important difference between linear time-varying and time-invariant systems is stability criteria. Linear time-invariant systems are stable if and only if all of the system's eigenvalues are negative. On the other hand, linear time-varying systems may be unstable even if all of the system's "frozen-time" eigenvalues (the eigenvalues of the system at any fixed time) are negative for all time. In this work a stabilization of linear time-varying system is introduced. The scheme could be used to determine stability of time-varying systems easily as well as to provide a new horizon of designing controllers via state-derivative feedback. It is shown that the performance for linear time-varying systems can be appropriately assigning the closed-loop eigenvalues of linear time-varying systems such as linear time-invariant cases.

The objective now is to construct the varying feedback gain matrix $\boldsymbol{K}(t)$ in order to stabilize the system. In this treatment, it is utilized the Frobenius transformation as an intermediate step to enable us to apply the pole placement approach according to [19, 20] for stabilization of time-varying systems.

Let us take the following time-dependent state transformation that transforms the system into a new state variable $\boldsymbol{z}(t)$ as

$$\boldsymbol{z}(t) = \boldsymbol{Q}^{-1}(t)\, \boldsymbol{x}(t), \quad \boldsymbol{x}(t) = \boldsymbol{Q}(t)\, \boldsymbol{z}(t) \tag{45}$$

then the system is transformed to the Frobenius canonical form and the system matrices can be computed as

$$\boldsymbol{A}_F(t) = \boldsymbol{Q}^{-1}(\boldsymbol{A}\boldsymbol{Q} - \dot{\boldsymbol{Q}}), \quad \boldsymbol{B}_F(t) = \boldsymbol{Q}^{-1}\boldsymbol{B} \tag{46}$$

where $\boldsymbol{A}_F(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}_F(t) \in \mathbb{R}^{n\times m}$ are the transformed system and control gain matrices, respectively. The transformed system is the same as (7). Note that, the eigenvalues of the time-varying dynamic system do not have the classical meaning regarding its behavior nor its stability features.

The state transformation matrix $\boldsymbol{Q}^{-1}(t) \in \mathbb{R}^{n\times n}$ can be calculated as follows

$$\boldsymbol{Q}^{-1}(t) = \mathrm{rows}\left(\boldsymbol{q}_1^1\ \boldsymbol{q}_2^1 \cdots \boldsymbol{q}_{\mu_1}^1\ \boldsymbol{q}_1^2\ \boldsymbol{q}_2^2 \cdots \boldsymbol{q}_{\mu_2}^2 \cdots \boldsymbol{q}_1^m\ \boldsymbol{q}_2^m \cdots \boldsymbol{q}_{\mu_m}^m\right) \tag{47}$$

where $\boldsymbol{q}_i^j(t) \in \mathbb{R}^{1\times n}$ is computed by using the recursive computations of the rows as follows

$$\boldsymbol{q}_1^j = \boldsymbol{e}_{r_j}^{\mathrm{T}} \boldsymbol{R}^{-1}, \quad \boldsymbol{q}_{i+1}^j = \boldsymbol{q}_i^j \boldsymbol{A} + \dot{\boldsymbol{q}}_i^j, \quad \boldsymbol{r}_j = \sum_{k=1}^{j} \mu_k, \quad j = 1, \ldots, m, \; i = 1, \ldots, \mu_1 - 1, \tag{48}$$

where μ_j is the controllability index and satisfy $\mu_j + \cdots + \mu_m = n$.

The controllability matrix for the time-varying system $\boldsymbol{R}(t) \in \mathbb{R}^{n \times n}$ is formed as

$$\boldsymbol{R}(t) = (\boldsymbol{r}_{11} \; \boldsymbol{r}_{12} \cdots \boldsymbol{r}_{1\mu_1} \; \boldsymbol{r}_{21} \; \boldsymbol{r}_{22} \cdots \boldsymbol{r}_{2\mu_2} \cdots \boldsymbol{r}_{m1} \; \boldsymbol{r}_{m2} \cdots \boldsymbol{r}_{m\mu_m}) \tag{49}$$

where $\boldsymbol{r}_{ji}(t) \in \mathbb{R}^n$ can be computed algebraically using the recursion

$$\boldsymbol{r}_{j1} = \boldsymbol{b}_j, \quad \boldsymbol{r}_{j,i+1} = \boldsymbol{A}\boldsymbol{r}_{ji} - \dot{\boldsymbol{r}}_{ji}, \quad j = 1, \ldots, m, \; i = 1, \ldots \mu_j - 1. \tag{50}$$

The fundamental assumption imposed on the system is that, the controllability matrix is of full rank with some choice of indices μ_j fixed in the studied time interval $t \in [t_0, \infty]$. This means this controllable system is lexicographically-fixed [19, 20].

If $\boldsymbol{Q}(t)$, $\boldsymbol{Q}^{-1}(t)$, and $\mathrm{d}\boldsymbol{Q}(t)/\mathrm{d}t$ are continuous and bounded matrices and $\boldsymbol{Q}^{-1}(t)$ has a full rank at the time interval of interest, $t \in [t_0, \infty]$, then this transformation is called a *Lyapunov transformation*. One way of observing this boundedness is to check on the magnitude of the maximum singular value of $\boldsymbol{Q}(t)$ in this interval. It is worth to note that, the Lyapunov transformation means that the transformation from one system to the other preserves the property of stability.

Therefore, the stabilization of time-varying systems by pole placement approach is based on computation of such time-varying feedback gain that modifies the original system into the new system, which is Lyapunov equivalent to linear time-invariant system. This linear time-invariant system is the Frobenius canonical form of the modified system, the Laypunov transformation is the transformation into Frobenius canonical form and the linear time-invariant system has the prescribed desired poles that guarantee the stability and desired behaviour. This stable behaviour is a reflection of that with constant and prescribed eigenvalues.

Assuming that the above transformation is a Lyapunov type and the controllability matrix of the system is lexicographically-fixed, then the pole placement technique that introduced in the previous section can be applied. In this treatment, the similar steps as described in Section 2 for the time-invariant system to derive explicit expression for the feedback gain for the time-varying system are used.

By differentiating the transformation equation (45) and substitute (44), the resulting closed-loop system is

$$\dot{\boldsymbol{z}} = \boldsymbol{Q}^{-1}\dot{\boldsymbol{x}} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\,\boldsymbol{x} = \left(\boldsymbol{Q}^{-1}(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{A} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\right)\boldsymbol{Q}\boldsymbol{z} = \boldsymbol{A}_Z\boldsymbol{z}, \tag{51}$$

where $\boldsymbol{A}_Z \in \mathbb{R}^{n \times n}$ is the closed-loop system matrix which given as (16) and can be computed as

$$\boldsymbol{A}_Z = \left(\boldsymbol{Q}^{-1}(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{A} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\right)\boldsymbol{Q}. \tag{52}$$

Hence, the above equation can be reformulated as

$$\left(\boldsymbol{A}_Z\boldsymbol{Q}^{-1} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\right)\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{K}) = \boldsymbol{Q}^{-1}. \tag{53}$$

Applying the same procedure for the time-invariant system, it is easy to write the m equations describing the system in terms of the row vectors $\boldsymbol{q}_i^j$ ($i = 1, \ldots, \mu_j$, $j = 1, \ldots, m$) of $\boldsymbol{Q}^{-1}(t)$ as

$$\begin{aligned}
\left(\sum_{i=0}^{\mu_1-1}(-d_i^1\boldsymbol{q}_{i+1}^1) - \dot{\boldsymbol{q}}_{\mu_1}^1\right)\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK}) &= \boldsymbol{q}_{\mu_1}^1,\\
\left(\sum_{i=0}^{\mu_2-1}(-d_i^2\boldsymbol{q}_{i+1}^2) - \dot{\boldsymbol{q}}_{\mu_2}^2\right)\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK}) &= \boldsymbol{q}_{\mu_2}^2,\\
&\vdots\\
\left(\sum_{i=0}^{\mu_m-1}(-d_i^m\boldsymbol{q}_{i+1}^m) - \dot{\boldsymbol{q}}_{\mu_m}^m\right)\boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{BK}) &= \boldsymbol{q}_{\mu_m}^m
\end{aligned} \tag{54}$$

with the desired (*Hurwitz*) constant characteristic coefficients d_i^j ($i = 0, \ldots, \mu_j - 1$, $j = 1, \ldots, m$) for the m groups. The simple reason for distributing these poles into different groups is to obtain the smoother transient behavior of the system.

Continuing this procedure, these equations can be put in a matrix form. Therefore, the feedback gain matrix $\boldsymbol{K}(t)$ for the time-varying system can be written as

$$\boldsymbol{K}(t) = \begin{pmatrix} \left(\sum\limits_{i=0}^{\mu_1-1}(-d_i^1\boldsymbol{q}_{i+1}^1) - \dot{\boldsymbol{q}}_{\mu_1}^1\right)\boldsymbol{A}^{-1}\boldsymbol{B}\\ \vdots\\ \left(\sum\limits_{i=0}^{\mu_m-1}(-d_i^m\boldsymbol{q}_{i+1}^m) - \dot{\boldsymbol{q}}_{\mu_m}^m\right)\boldsymbol{A}^{-1}\boldsymbol{B} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{q}_{\mu_1}^1 + \left(\sum\limits_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_{i+1}^1) + \dot{\boldsymbol{q}}_{\mu_1}^1\right)\boldsymbol{A}^{-1}\\ \vdots\\ \boldsymbol{q}_{\mu_m}^m + \left(\sum\limits_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_{i+1}^m) + \dot{\boldsymbol{q}}_{\mu_m}^m\right)\boldsymbol{A}^{-1} \end{pmatrix}. \tag{55}$$

The feedback gain matrix $\boldsymbol{K}(t)$ can be rewritten as

$$\boldsymbol{K}(t) = \begin{pmatrix} \left(\boldsymbol{q}_{\mu_1}^1 - \left(\boldsymbol{q}_{\mu_1+1}^1 + \sum\limits_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_{i+1}^1)\right)\boldsymbol{A}^{-1}\right)\boldsymbol{B}\\ \vdots\\ \left(\boldsymbol{q}_{\mu_m}^m - \left(\boldsymbol{q}_{\mu_m+1}^m + \sum\limits_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_{i+1}^m)\right)\boldsymbol{A}^{-1}\right)\boldsymbol{B} \end{pmatrix}^{-1} \begin{pmatrix} \left(\boldsymbol{q}_{\mu_1+1}^1 + \sum\limits_{i=0}^{\mu_1-1}(d_i^1\boldsymbol{q}_{i+1}^1)\right)\boldsymbol{A}^{-1}\\ \vdots\\ \left(\boldsymbol{q}_{\mu_m+1}^m + \sum\limits_{i=0}^{\mu_m-1}(d_i^m\boldsymbol{q}_{i+1}^m)\right)\boldsymbol{A}^{-1} \end{pmatrix}. \tag{56}$$

Next, we consider a stabilizing feedback control defined by the desired m group eigenvalues $\{\lambda^1\}, \ldots, \{\lambda^m\}$, with μ_j in each. An efficient numerical algorithm as computes the gain is

$$\boldsymbol{K}(t) = \left(\begin{pmatrix} \boldsymbol{q}^1_{\mu_1} \\ \vdots \\ \boldsymbol{q}^m_{\mu_m} \end{pmatrix} \boldsymbol{B} - \begin{pmatrix} \boldsymbol{q}^{1'}_{\mu_1+1} \\ \vdots \\ \boldsymbol{q}^{m'}_{\mu_m+1} \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \begin{pmatrix} \boldsymbol{q}^{1'}_{\mu_1+1} \\ \vdots \\ \boldsymbol{q}^{m'}_{\mu_m+1} \end{pmatrix} \boldsymbol{A}^{-1} = \boldsymbol{M}_2^{-1} \begin{pmatrix} \boldsymbol{q}^{1'}_{\mu_1+1} \\ \vdots \\ \boldsymbol{q}^{m'}_{\mu_m+1} \end{pmatrix} \boldsymbol{A}^{-1} \tag{57}$$

where

$$\boldsymbol{q}^{j'}_1 = \boldsymbol{e}^{\mathrm{T}}_{r_j} \boldsymbol{R}^{-1}, \quad \boldsymbol{q}^{j'}_{i+1} = \boldsymbol{q}^{j'}_i (\boldsymbol{A} - \lambda^j_i \boldsymbol{I}_n) + \dot{\boldsymbol{q}}^{j'}_i, \quad j = 1, \ldots, m,\ i = 1, \ldots, \mu_j. \tag{58}$$

Remark 4. The matrix $(\,(\boldsymbol{q}^1_{\mu_1})^{\mathrm{T}} \cdots (\boldsymbol{q}^m_{\mu_m})^{\mathrm{T}}\,)^{\mathrm{T}} \boldsymbol{B}$ is upper triangular matrix and all diagonal elements are one. The rows of this matrix are the μ_1, $\mu_1 + \mu_2, \ldots, n$ rows of matrix $\boldsymbol{B}_F(t)$ and since we assume that $\boldsymbol{B}(t)$ is of full rank therefore this matrix is nonsingular for the time interval of interest, $t \in [t_0, \infty)$. Also matrix $\boldsymbol{M}_2(t)$ can be reformulated as

$$\begin{aligned} \boldsymbol{M}_2 &= \begin{pmatrix} \sum\limits_{i=0}^{\mu_1-1} (-d^1_i \boldsymbol{q}^1_{i+1}) - \dot{\boldsymbol{q}}^1_{\mu_1} \\ \vdots \\ \sum\limits_{i=0}^{\mu_m-1} (-d^m_i \boldsymbol{q}^m_{i+1}) - \dot{\boldsymbol{q}}^m_{\mu_m} \end{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{B} \\ &= -\left(\begin{pmatrix} \boldsymbol{d}^1 & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{d}^m \end{pmatrix} \boldsymbol{Q}^{-1} + \begin{pmatrix} \dot{\boldsymbol{q}}^1_{\mu_1} \\ \vdots \\ \dot{\boldsymbol{q}}^m_{\mu_m} \end{pmatrix}\right) \boldsymbol{A}^{-1}\boldsymbol{B} = -\boldsymbol{M}_3 \boldsymbol{A}^{-1}\boldsymbol{B}. \end{aligned} \tag{59}$$

Therefore this matrix in nonsingular if $\boldsymbol{A}(t)$ is nonsingular, matrices $\boldsymbol{B}(t)$ and $\boldsymbol{M}_3(t)$ have a full rank m and all desired poles are non-zero at the time interval of interest $t \in [t_0, \infty)$.

These derivations solve the problem for time-varying linear system if matrix $\boldsymbol{A}(t)$ is nonsingular. Obviously, the technique presented here is directly implemented in the state space with simple and efficient calculations.

Remark 5. For the case of $(m = n)$ and utilizing (52), the feedback gain can be computed by

$$\boldsymbol{K}(t) = \boldsymbol{B}^{-1} \left(\boldsymbol{A} \left(\boldsymbol{A}_Z \boldsymbol{Q}^{-1} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1}) \right)^{-1} \boldsymbol{Q}^{-1} - \boldsymbol{I}_n \right) \tag{60}$$

where $\boldsymbol{A}_Z$ is in Jordan canonical form with the desired eigenvalues on the diagonal.

Remark 6. For single-input case ($m = 1$), the state-derivative feedback gain $\boldsymbol{K}(t)$ can be given by:

If the coefficients $d_i,\ i = 1, \dots, n$, of the characteristic equation are given [1]

$$\boldsymbol{K}(t) \left(1 - \left(\boldsymbol{q}_{n+1} + \sum_{i=0}^{n-1}(d_i \boldsymbol{q}_{i+1})\right) \boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \left(\boldsymbol{q}_{n+1} + \sum_{i=0}^{n-1}(d_i \boldsymbol{q}_{i+1})\right) \boldsymbol{A}^{-1} \tag{61}$$

where

$$\boldsymbol{q}_1 = \boldsymbol{e}_n^{\mathrm{T}} \boldsymbol{R}^{-1}, \quad \boldsymbol{q}_{i+1} = \boldsymbol{q}_i \boldsymbol{A} + \dot{\boldsymbol{q}}_i, \quad i = 1, \dots, n.$$

Further, if the desired poles λ_i, $i = 1, \dots, n$ are given [1]

$$\boldsymbol{K}(t) = \left(1 - \boldsymbol{q}'_{n+1}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} \boldsymbol{q}'_{n+1}\boldsymbol{A}^{-1} \tag{62}$$

where

$$\boldsymbol{q}'_1 = \boldsymbol{e}_n^{\mathrm{T}} \boldsymbol{R}^{-1}, \quad \boldsymbol{q}'_{i+1} = \boldsymbol{q}'_i(\boldsymbol{A} - \lambda_i \boldsymbol{I}_n) + \dot{\boldsymbol{q}}'_i, \quad i = 1, \dots, n.$$

Based on that the following theorem for a multi-input time-varying system can be formulated.

Theorem 2. Consider the lexicographically-fixed completely controllable, multi-input time-varying linear control system (42). If the transformation $\boldsymbol{Q}(t)$ is of Lyapunov kind, i. e. $\boldsymbol{Q}(t), \boldsymbol{Q}^{-1}(t)$ and $\mathrm{d}\boldsymbol{Q}(t)/\mathrm{d}t$ are continuous and bounded and $\boldsymbol{Q}^{-1}(t)$ is full rank, and $\boldsymbol{A}(t)$ is nonsingular and its inverse is bounded, $\boldsymbol{B}(t)$ is full column rank, and $\boldsymbol{M}_3(t)$ has full rank, then the system (42) with the state-derivative feedback (43) can be stabilized with the unique time-dependent feedback gain $\boldsymbol{K}(t)$ (56) or (57). For single-input case ($m = 1$), the feedback gain can be computed by (62)–(63). Everything must be valid at the time interval of interest $t \in [t_0, \infty)$.

Further, the control effort $\boldsymbol{u}(t)$ and transient response $\boldsymbol{x}(t)$ can be derived from (51) and (53) as

$$\begin{aligned}
\boldsymbol{u}(t) &= -\boldsymbol{K}(t)\,\dot{\boldsymbol{x}}(t) = -\boldsymbol{B}^{+}\left(\boldsymbol{A}\left(\boldsymbol{A}_Z\boldsymbol{Q}^{-1} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q})^{-1}\right)^{-1} - \boldsymbol{I}_n\right)\left(\boldsymbol{Q}\dot{\boldsymbol{z}}(t) + \dot{\boldsymbol{Q}}\boldsymbol{z}(t)\right) \\
&= -\boldsymbol{B}^{+}\left(\boldsymbol{A}(\boldsymbol{A}_z\boldsymbol{Q}^{-1} + \boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1})^{-1}\boldsymbol{Q}^{-1} - \boldsymbol{I}_n\right)(\boldsymbol{Q}\boldsymbol{A}_Z + \dot{\boldsymbol{Q}})\,\boldsymbol{z}(t) \\
&= -\boldsymbol{B}^{+}\left(\boldsymbol{A}\boldsymbol{Q}(\boldsymbol{Q}\boldsymbol{A}_Z + \dot{\boldsymbol{Q}})^{-1} - \boldsymbol{I}_n\right)(\boldsymbol{Q}\boldsymbol{A}_Z + \dot{\boldsymbol{Q}})\,\boldsymbol{z}(t) \\
&= -\boldsymbol{B}^{+}(\boldsymbol{A}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{A}_Z - \dot{\boldsymbol{Q}})\,\boldsymbol{z}(t) = -\boldsymbol{B}^{+}\left(\boldsymbol{A} - \boldsymbol{Q}\left(\boldsymbol{A}_Z\boldsymbol{Q}^{-1} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\right)\right)\boldsymbol{x}(t) \\
&= -\boldsymbol{K}_S(t)\,\boldsymbol{x}(t)
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (\boldsymbol{I}_n + \boldsymbol{B}(t)\,\boldsymbol{K}(t))^{-1}\boldsymbol{A}(t)\,\boldsymbol{x}(t) \\
&= \boldsymbol{Q}\left(\boldsymbol{A}_Z\boldsymbol{Q}^{-1} - \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1})\right)\boldsymbol{x}(t).
\end{aligned} \tag{64}$$

Similar to the case of time-invariant system, the last expressions (63) and (64) are exactly for the time varying system via state feedback with the same desired eigenvalues and transformation matrix $\boldsymbol{Q}^{-1}(t)$.

4. ILLUSTRATIVE EXAMPLES

In this section, the proposed pole placement techniques are applied and simulated to several systems to demonstrate the feasibility and effectiveness of the previous results.

Example 1. The configuration of the mechanical system and its parameters are shown in Figure 1. The dynamic equation of this system, assuming small angle φ, can be described in the state-space from using the state vector $\boldsymbol{x}(t) = [\, x_1 \; x_2 \; \dot{x}_1 \; \dot{x}_2 \,]$ as:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1c_1 & -k_2c_2 & -b_1c_1 & -b_2c_2 \\ -k_1c_1 & -k_2c_1 & -b_1c_2 & -b_2c_1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \\ c_2 & c_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where

$$c_1 = \frac{1}{m} + \frac{L^2}{I}, \quad c_1 = \frac{1}{m} - \frac{L^2}{I}, \quad x_3 = 0.5(x_1 + x_2) \quad \text{and} \quad \varphi = 0.5(x_1 - x_2)/L$$

where m and I represent the mass and inertia of the mass, k_1 and k_2 are the spring constants, b_1 and b_2 are the damper constants, x_1 and x_2 are the mass displacement from both sides, x_3 is the vertical displacement of the center of mass, φ is the inclination angle of the mass with the horizontal, $2L$ is the distance between two supporting points, and u_1 andu_2 are the control inputs.

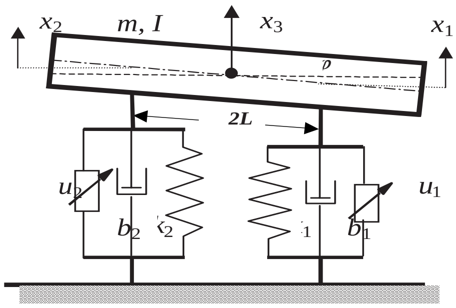

Fig. 1.

The model parameters are taken as $m = 10\,\mathrm{kg}$, $I = 1\,\mathrm{Kg\cdot m^2}$, $L = 1\,\mathrm{m}$, $k_1 = 500\,\mathrm{N/m}$, $k_2 = 700\,\mathrm{N/m}$, $b_1 = 10\,\mathrm{N\cdot s/m}$ and $b_2 = 20\,\mathrm{N\cdot s/m}$.

The original system poles are $-15.1384 \pm 31.1738i$ and $-1.3616 \pm 10.7106i$. The desired closed-loop eigenvalues are selected as, $\{\lambda_1^1, \lambda_2^1\} = -5 \pm 2i$, for the first block, while the second block are $\{\lambda_1^2, \lambda_2^2\} = -10 \pm 5i$. The transformation matrix and the equivalent Frobenius canonical form are

$$\boldsymbol{Q}^{-1} = \begin{pmatrix} 2.75 & 2.25 & 0 & 0 \\ 0 & 0 & 2.75 & 2.25 \\ 2.25 & 2.25 & 0 & 0 \\ 0 & 0 & 2.25 & 2.75 \end{pmatrix}$$

and
$$\dot{\boldsymbol{z}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -550 & -11 & 450 & 9 \\ 0 & 0 & 0 & 1 \\ 630 & 18 & -770 & -22 \end{pmatrix} \boldsymbol{z} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then the computed state-derivative feedback gain is

$$\boldsymbol{K} = \begin{pmatrix} 349.5517 & 228.7241 & 41.3052 & 30.5224 \\ -320.2138 & -169.9931 & -48.1314 & -34.6893 \end{pmatrix}.$$

Applying the control synthesis procedure of pole placement from Section 2 to this system. In this simulation, the initial conditions of the states are taken as, $\boldsymbol{x}(t_0) = [-0.01,\ 0.02\ \ -0.02,\ 0.01]^{\mathrm{T}}$ The transient response and control input are shown in Figure 2. In addition, the vertical displacement and inclination angle are displayed in Figures 3.

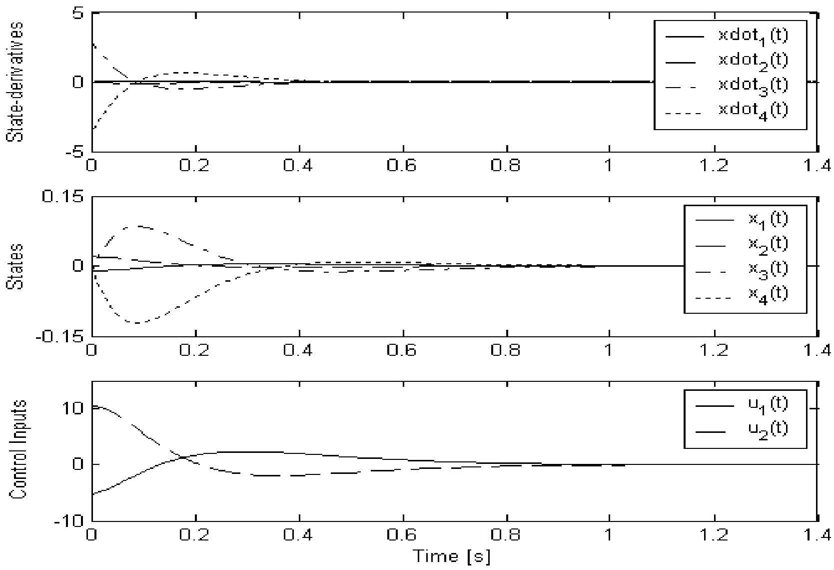

Fig. 2. Transient response and control input of the system via state-derivative feedback.

For a comparison, the computed state feedback gain matrix for the same desired system poles using [19] is

$$\boldsymbol{K}_S = \begin{pmatrix} -420.2500 & 65.2500 & 17.5000 & 22.5000 \\ 281.2500 & -356.2500 & 45.0000 & 35.0000 \end{pmatrix}.$$

The simulation results illustrated that the performance of both cases are identical as the system obtains the same poles. Therefore the control inputs are in both cases identical and this means that the robustness properties of both feedback controllers

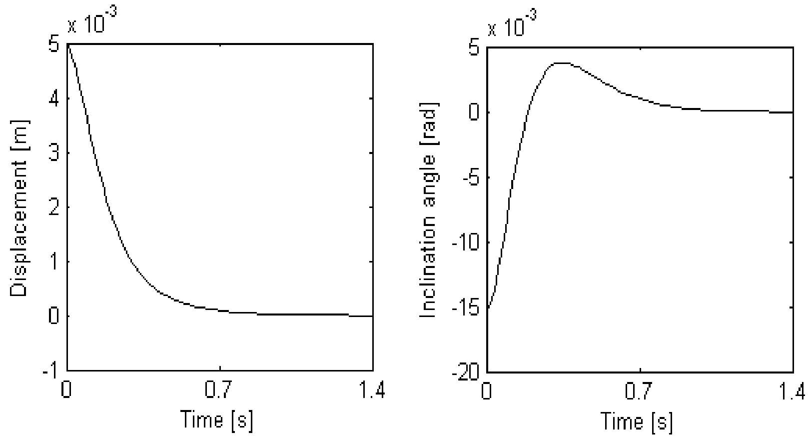

Fig. 3. Vertical displacement of center of mass and the inclination angle with the horizontal.

are the same if properties of sensors are the same. It is also clear that the same performance is achieved in case of state-derivative feedback controller with lower gain matrix elements than by the state feedback, i. e. $\|\boldsymbol{K}_2\|_2 = 558.6532$ and $\|\boldsymbol{K}_S\|_2 = 625.2565$.

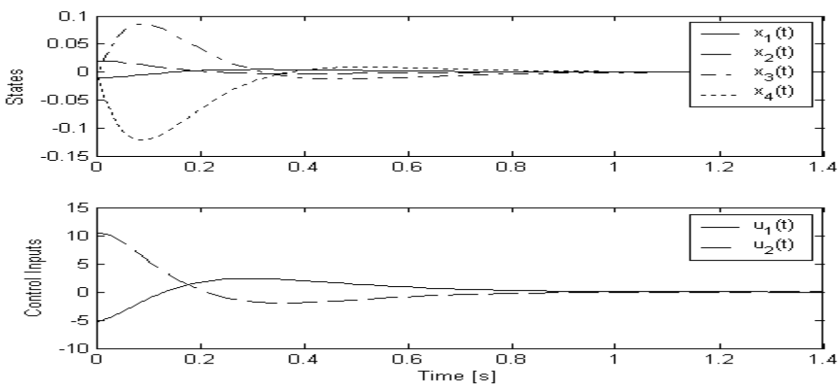

Fig. 4. Transient response and control input of the system via state feedback [19].

Example 2. Consider the dynamic equation of the multi-input linear time-varying system

$$\dot{\boldsymbol{x}}(t) = \begin{pmatrix} 0.1e^{-2t} & -0.1 & 0 \\ 0.1 & 0.1 & 0.1e^{-t} \\ 0.1e^{-t} & 0 & 0.1 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 0 & 0 \\ 0.1 & 0.1e^{-t} \\ 0 & 0.1 \end{pmatrix} \boldsymbol{u}(t).$$

This system is unstable and the zero-input transient response of open-loop system is shown in Figure 5.

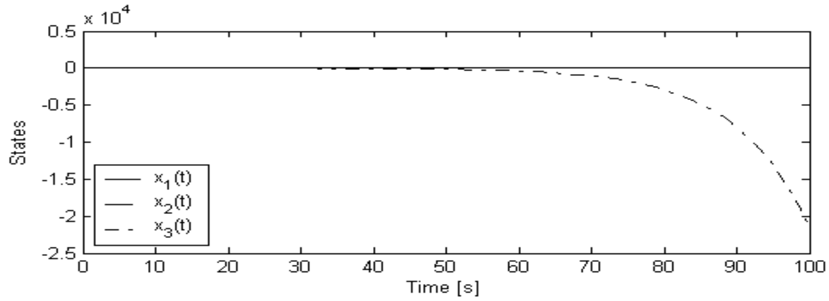

Fig. 5. Zero-input transient response of open-loop system.

The controllability indices are $\mu_1 = 2$ and $\mu_2 = 1$. The controllability matrix and its inverse are

$$\boldsymbol{R}(t) = \begin{pmatrix} 0 & -0.01 & 0 \\ 0.1 & 0.01 & 0.1e^{-t} \\ 0 & 0 & 0.1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{R}^{-1}(t) = \begin{pmatrix} 10 & 10 & -10e^{-t} \\ -100 & 0 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

It is clear that the controllability matrix is a full rank in the time interval of interest $t \in [t_0, \infty)$ and the system is *lexicographically-fixed*. The rows of the transformation matrix can be computed as

$$\boldsymbol{q}_1^1 = e_{r_1}^{\mathrm{T}} \boldsymbol{R}^{-1} = (\,-100\ 0\ 0\,), \quad \boldsymbol{q}_2^1 = \boldsymbol{q}_1^1 \boldsymbol{A} + \dot{\boldsymbol{q}}_1^1 = (\,-10e^{-2t}\ 10\ 0\,),$$

$$\boldsymbol{q}_1^2 = \boldsymbol{e}_{r_2}^{\mathrm{T}} \boldsymbol{R}^{-1} = (\,0\ 0\ 10\,).$$

Then, the transformation matrix, inverse, and derivative are

$$\boldsymbol{Q}^{-1}(t) = \begin{pmatrix} \boldsymbol{q}_1^1 \\ \boldsymbol{q}_2^1 \\ \boldsymbol{q}_1^2 \end{pmatrix} = \begin{pmatrix} -100 & 0 & 0 \\ -10e^{-2t} & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \quad \boldsymbol{Q}(t) = \begin{pmatrix} -0.01 & 0 & 0 \\ -0.01e^{-2t} & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^{-1}(t)) = \begin{pmatrix} 0 & 0 & 0 \\ 20e^{-2t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \dot{\boldsymbol{Q}}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0.02e^{-2t} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices are continuous and bounded and the transformation matrix has a full rank at the time interval of interest $t \in [t_0, \infty)$, then this is a *Lyapunov equivalent transformation* and the proposed technique can be applied. The computation of the feedback gain matrix can be done as follows:

First, the vectors $\boldsymbol{q}_i^{j'}$, $j = 1, \dots, m$, $i = 1, \dots, \mu_j + 1$.

$$
\begin{aligned}
\boldsymbol{q}_1^{1'} &= \boldsymbol{q}_1^1 = (\,-100\ 0\ 0\,),\\
\boldsymbol{q}_2^{1'} &= \boldsymbol{q}_1^{1'}(\boldsymbol{A}-\lambda_1^1\boldsymbol{I}_n)+\dot{\boldsymbol{q}}_1^{1'} = (\,-10e^{-2t}+100\lambda_1^1\ \ 10\ \ 0\,)\\
\boldsymbol{q}_3^{1'} &= \boldsymbol{q}_2^{1'}(\boldsymbol{A}-\lambda_2^1\boldsymbol{I}_n)+\dot{\boldsymbol{q}}_2^{1'}\\
&= \left(-e^{-4t}+10e^{-2t}(\lambda_1^1+\lambda_2^1+2)-100\lambda_1^1\lambda_2^1+1 \quad e^{-2t}-10(\lambda_1^1-\lambda_2^1)+1 \quad e^{-t}\right),\\
\boldsymbol{q}_1^{2'} &= \boldsymbol{q}_1^2 = (\,0\ 0\ 10\,),\\
\boldsymbol{q}_2^{2'} &= \boldsymbol{q}_1^{2'}(\boldsymbol{A}-\lambda_1^2\boldsymbol{I}_n)+\dot{\boldsymbol{q}}_1^{2'} = (\,e^{-t}\ \ 0\ \ 1-10\lambda_1^2\,),
\end{aligned}
$$

$$
\begin{pmatrix}\boldsymbol{q}_2^1\\ \boldsymbol{q}_1^2\end{pmatrix}\boldsymbol{B} = \begin{pmatrix}1 & e^{-t}\\ 0 & 1\end{pmatrix},
$$

$$
\begin{pmatrix}\boldsymbol{q}_3^{1'}\\ \boldsymbol{q}_2^{2'}\end{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B} = \begin{pmatrix}20e^{-2t}-100\lambda_1^1\lambda_2^1+1 & e^{-t}\\ 10\lambda_1^2e^{-t} & 1-10\lambda_1^2\end{pmatrix},
$$

$$
\left(\begin{pmatrix}\boldsymbol{q}_2^1\\ \boldsymbol{q}_1^2\end{pmatrix}\boldsymbol{B}-\begin{pmatrix}\boldsymbol{q}_3^{1'}\\ \boldsymbol{q}_2^{2'}\end{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} = \begin{pmatrix}\dfrac{1}{-20e^{-2t}+100\lambda_1^1\lambda_2^1} & 0\\ \dfrac{e^{-t}}{-20e^{-2t}+100\lambda_1^1\lambda_2^1} & \dfrac{1}{10\lambda_1^2},\end{pmatrix}
$$

$$
\begin{pmatrix}\boldsymbol{q}_3^{1'}\\ \boldsymbol{q}_2^{2'}\end{pmatrix}\boldsymbol{A}^{-1} = 100\begin{pmatrix}1.9e^{-2t}+\lambda_1^1+\lambda_2^1-10\lambda_1^1\lambda_2^1 & 2e^{-2t}-10\lambda_1^1\lambda_2^1+0.1 & -2e^{-3t}+10\lambda_1^1\lambda_2^1e^{-t}\\ \lambda_1^2e^{-t} & \lambda_1^2e^{-t} & 0.1-\lambda_1^2(e^{-2t}+1)\end{pmatrix}.
$$

Finally, the state-derivative feedback gain matrix is

$$
\begin{aligned}
\boldsymbol{K}(t) &= \left(\begin{pmatrix}\boldsymbol{q}_2^1\\ \boldsymbol{q}_1^2\end{pmatrix}\boldsymbol{B}-\begin{pmatrix}\boldsymbol{q}_3^{1'}\\ \boldsymbol{q}_2^{2'}\end{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\begin{pmatrix}\boldsymbol{q}_3^{1'}\\ \boldsymbol{q}_2^{2'}\end{pmatrix}\boldsymbol{A}^{-1}\\
&= \begin{pmatrix}\dfrac{e^{-2t}-10(\lambda_1^1+\lambda_2^1)}{2e^{-2t}-10\lambda_1^1+\lambda_2^1}-10 & \dfrac{-1}{2e^{-2t}-10\lambda_1^1\lambda_2^1-10} & 10e^{-t}\\ \dfrac{e^{-3t}-10(\lambda_1^1+\lambda_2^2)\,e^{-t}}{2e^{-2t}-10\lambda_1^1\lambda_2^1} & \dfrac{-e^{-t}}{2e^{-2t}-10\lambda_1^1\lambda_2^1} & \dfrac{1}{\lambda_1}-10\end{pmatrix}.
\end{aligned}
$$

Given the desired closed-loop eigenvalues of the first block $\{\lambda_1^1,\lambda_2^1\}=-2\pm i$, while, the second block $\lambda_1^2=-3$ and the initial state conditions $\boldsymbol{x}(t_0)=[0.2,\,-0.1,\,-1]^{\mathrm{T}}$. The transient response and control input are shown in Figure 6. The elements of gain matrix are shown in Figure 7a.

As a comparison with the state feedback, the elements of the state gain matrix is calculated from [19], and displayed in Figure 7b

$$
\boldsymbol{K}_S(t)=\begin{pmatrix}-e^{-4t}+10e^{-2t}(\lambda_1^1+\lambda_2^1+1.9)-100\lambda_1^1\lambda_2^1+1 & e^{-2t}-10(\lambda_1^1+\lambda_2^1)+1 & 10\lambda_1^2e^{-t}\\ e^{-t} & 0.0 & 1-10\lambda_1^2\end{pmatrix}.
$$

From these results, we notice the high reduction in the state-derivative feedback gain matrix compared to the well-known state feedback approach with the same performance for the time-invariant and time-varying systems.

In this work, it is shown that how the pole placement approaches can be used to design a controller-based state-derivative feedback control, which yields a closed-loop system with specified characteristics. The approach is relevant for design with

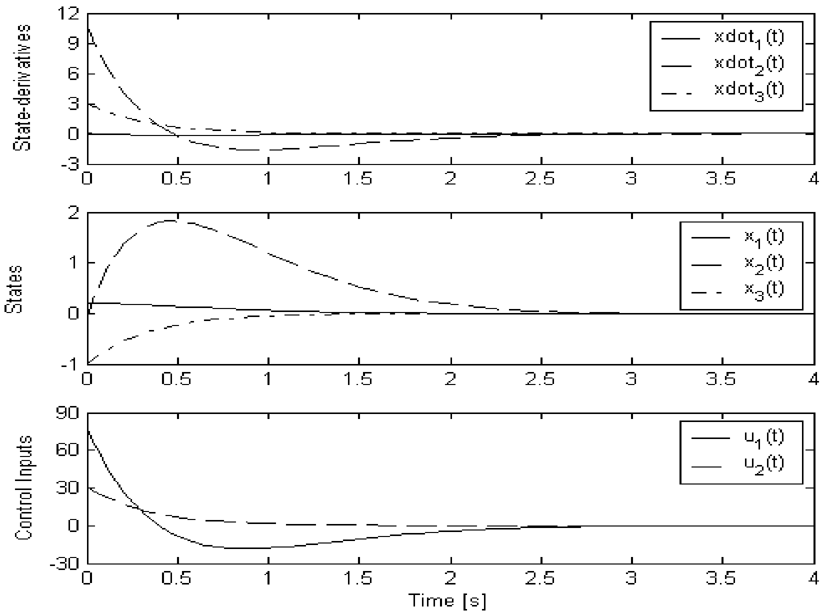

Fig. 6. Transient response of the closed-loop system via state-derivative feedback.

preservation of stability when some necessary and sufficient conditions are provided. Compared to state feedback, the state-derivative feedback controller in some cases achieves the same performance with lower gain elements. From practical point of view, it is desirable to determine feedback matrices with smaller gains. Intuitively, this must be advantageous since small gains are beneficial to reduce noise amplification.

5. CONCLUSIONS

This paper has presented a new technique and tool for solving the pole placement problem. The main result of this work is a computationally efficient algorithm for solving the pole placement problem of linear multi-input systems with nonsingular system matrix by state-derivative feedback. This problem is treated both for a linear time-invariant and time-varying multi-input systems. It is the first general treatment for multi-input pole placement by state-derivative feedback in the literature.

The technique is based on the transformation of a linear system into canonical form to derive the feedback gain matrix. This transformation provides a great simplification to this problem. The desired poles are placed within both a linear time-invariant and time-varying multi-input systems in such a way that the smoother transient response characteristics are preserved.

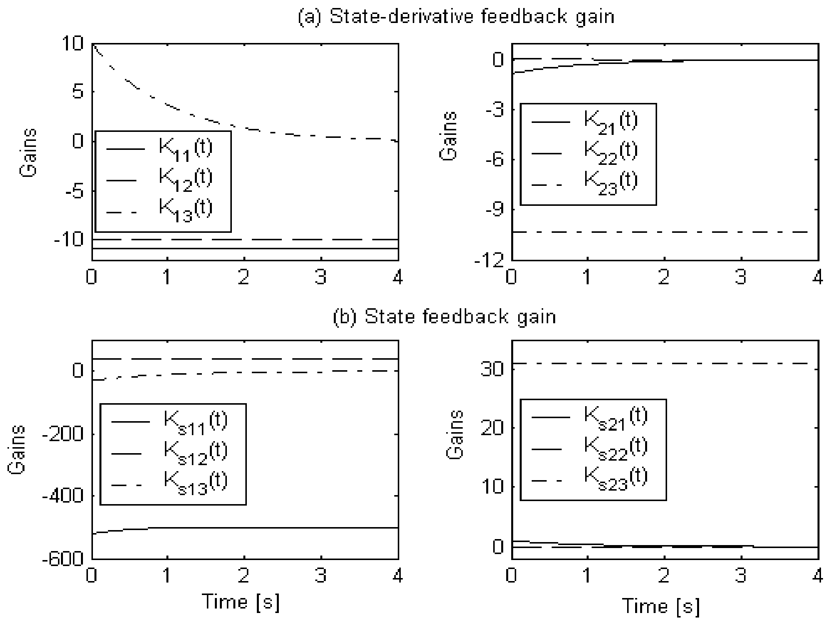

Fig. 7. (a) State-derivative feedback gain elements and
(b) State feedback gain elements.

The simulation results prove the feasibility and effectiveness of the proposed technique. The achieved state-derivative controllers provide the same performance that can be obtained by state feedback. An interesting feature of the state-derivative feedback is that it gives in many cases the feedback gains with smaller absolute values than traditional state feedback gains.

(Received April 26, 2004.)

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