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## A NECESSARY AND SUFFICIENT CONDITION FOR STATIC OUTPUT FEEDBACK STABILIZABILITY OF LINEAR DISCRETE-TIME SYSTEMS<sup>1</sup>

DANICA ROSINOVÁ, VOJTECH VESELÝ AND VLADIMÍR KUČERA

Necessary and sufficient conditions for a discrete-time system to be stabilizable via static output feedback are established. The conditions include a Riccati equation. An iterative as well as non-iterative LMI based algorithm with guaranteed cost for the computation of output stabilizing feedback gains is proposed and introduces the novel LMI approach to compute the stabilizing output feedback gain matrix. The results provide the discrete-time counterpart to the results by Kučera and De Souza [8].

*Keywords:* discrete-time systems, output feedback, stabilizability, stabilizing feedback, Riccati equations, LMI approach

*AMS Subject Classification:* 93D15

### 1. INTRODUCTION

Stabilization of linear systems using static output feedback has attracted considerable interest during the past decades. Various approaches have been used to study two aspects of the stabilization problem, namely conditions under which the linear system described in state-space can be stabilized via static output feedback and the respective procedure to obtain a stabilizing control law. A body of literature deals with the output stabilization problem for the continuous-time systems. An approach based on linear-quadratic regulator theory applying Lyapunov results to output stabilization was presented in Levine and Athans [9], leading to an iterative solution of three coupled matrix equations. Trinh and Aldeen [11] indicate an iterative algorithm to find output control gains derived from state-feedback solution to the corresponding Riccati equation. The existence of a solution or convergence of the algorithm is not discussed. Various other approaches and results for continuous-time systems are surveyed in Kučera and De Souza [8]. In the above paper Kučera and De Souza [8] found necessary and sufficient conditions for the existence of output feedback stabilizing control for continuous-time systems and proposed an iterative algorithm to find a stabilizing feedback gain.

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<sup>1</sup>Preliminary version of this paper was presented at the IFAC Conference on Control Systems Design held in Bratislava on June 18–20, 2000.

Output feedback stabilization of discrete-time systems employing LQ regulator theory can be found in Kolla and Farison [7], Sharav-Schapiro et al. [10], De Souza and Trofino [3]. The latter papers are devoted to a more complex problem including robustness aspects. The output stabilization is discussed as a special case. Kolla and Farison [7] derived the system of five matrix equations that should be solved by gradient methods to obtain control gains. Sharav-Schapiro et al. [10] study the output stabilizing robust control problem. They treat both Lyapunov and Riccati equation based controllers and develop the criterion for the existence of the so called output min-max controller. Crusius and Trofino [2], De Souza and Trofino [3] provide the sufficient conditions for output feedback stabilization that are convex and given in terms of linear matrix inequalities (LMIs).

The crucial point in the stabilizing output feedback statement is non-convex problem formulation. The existing approaches either use iterative algorithms to cope with non-convexity or add the appropriate constraint to restrict the problem to a convex one appropriate for LMI solution and thus the necessary and sufficient conditions are reduced to sufficient ones. Another approach employs bilinear matrix inequalities (BMIs), see Goh, Safonov and Papavassilopoulos, [5].

In this paper the linear discrete-time systems counterpart to the results of Kučera and De Souza [8] is presented in Section 2, and modified in Section 3 to provide the necessary and sufficient conditions for static output feedback stabilization. The corresponding iterative and also non iterative LMI based algorithm to compute a stabilizing static output feedback gain matrix with guaranteed cost is proposed. In Section 3 the novel approach to LMI algorithm proposal is developed to cope with non-linear terms and avoid iterative procedures. The use of LMI approach is motivated by the existence of standard packages and efficient LMI solvers as well as possibility to extend the results to robust static output feedback stabilization of linear time invariant (LTI) systems with polytopic models. The notation is standard, and will be defined as the need arises. Much of the notation and terminology follows references Kučera and De Souza [8], Zhou, Doyle and Glover [13].

## 2. MAIN RESULTS

Consider a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \quad (1)$$

$$y(k) = Cx(k)$$

with static output feedback

$$u(k) = Ky(k) \quad (2)$$

where  $x(k) \in \mathbb{R}^n$ ,  $u(k) \in \mathbb{R}^m$ ,  $y(k) \in \mathbb{R}^p$  are state, control and output vector respectively, and  $A, B, C, K$  are real matrices of corresponding dimensions.

Let us recall several commonly used notions. Matrix  $X$  is called *stable* when all its eigenvalues have modulus less than 1. System (1) with a stable matrix  $A$  is called *stable*. System (1) or pair  $(A, B)$  is called *stabilizable* if there exists a real state feedback gain matrix  $F$  such that  $A + BF$  is a stable matrix. System (1) is called

*output feedback stabilizable* if there exists a real output feedback gain matrix  $L$  such that  $A + BLC$  is a stable matrix. The pair  $(A, C)$  is called *detectable* if there exists a real matrix  $Y$  such that  $A + YC$  is stable.

The basic aim of applying control law (2) to the system (1) is to achieve stability of the closed-loop system. In the following theorem, necessary and sufficient conditions for static output feedback stabilizability of the studied system (1) are given.

**Theorem 1.** The discrete-time system (1) is static output feedback stabilizable if and only if

(i) the pair  $(A, B)$  is stabilizable, the pair  $(A, C)$  is detectable, and either of the following statements holds

(ii-a) there exist real matrices  $K$  and  $G$  such that

$$G = KC + (B^T PB + R)^{-1} B^T PA \quad (3)$$

where  $P$  is the real symmetric nonnegative definite solution of

$$A^T PA - P - A^T PB(B^T PB + R)^{-1} B^T PA + C^T C + G^T (B^T PB + R)G = 0 \quad (4)$$

and  $R$  is a real symmetric positive definite matrix of appropriate dimensions

(ii-b) there exist real matrices  $K_1$  and  $G_1$  such that

$$G_1 = (B^T P_1 B + R)^{-\frac{1}{2}} B^T P_1 A + (B^T P_1 B + R)^{\frac{1}{2}} K_1 C \quad (5)$$

where  $P_1$  is the real symmetric nonnegative definite solution of

$$A^T P_1 A - P_1 - A^T P_1 B(B^T P_1 B + R)^{-1} B^T P_1 A + Q + G_1^T G_1 = 0 \quad (6)$$

and  $Q$  and  $R$  are real symmetric positive definite matrices of appropriate dimensions.

*Proof.* Let us start with the first alternative of Theorem 1 that (i) and (ii-a) are necessary and sufficient conditions for static output feedback stabilizability of the system (1).

*Necessity.* Suppose that  $A + BKC$  is stable for some  $K$ . Then  $(A, B)$  is stabilizable since  $A + BF$  is stable for  $F = KC$  and  $(A, C)$  is detectable since  $A + LC$  is stable for  $L = BK$ . Thus (i) is proved. Since  $A + BKC$  is stable, it is known (Zhou, Doyle and Glover [13], Lemma 21.6) that there exists a unique symmetric nonnegative definite matrix  $P$  such that

$$(A + BKC)^T P(A + BKC) - P + C^T C + C^T K^T RKC = 0 \quad (7)$$

for some real symmetric nonnegative definite matrix  $R$ . After rearranging, (7) yields

$$\begin{aligned} A^T PA - P + C^T K^T B^T PA + A^T PBKC + \\ C^T K^T B^T PBKC + C^T C + C^T K^T RKC = 0 \end{aligned} \quad (8)$$

For  $G$  defined by (3) we obtain

$$G^T(B^T PB + R)G = C^T K^T B^T P B K C + \quad (9)$$

$$C^T K^T R K C + C^T K^T B^T P A + A^T P B K C + A^T P B (B^T P B + R)^{-1} B^T P A$$

The combination of (8) and (9) proves the equivalence of (7) and (4), with  $G$  given by (3).

*Sufficiency.* Suppose that (i) and (ii-a) hold. After substitution from (3) to (4) the Lyapunov equation (7) is obtained, where  $P$  is symmetric nonnegative definite matrix. Obviously  $C^T C + C^T K^T R K C$  is nonnegative definite. From detectability of  $(A, C)$  the existence of  $L$  such that  $A + LC$  is stable is guaranteed. Thus also

$$\begin{bmatrix} A + B K C & [C^T & (K C)^T]^T \end{bmatrix}$$

is detectable since

$$A + LC = (A + B K C) + [L \quad -B][C^T \quad (K C)^T]^T$$

Therefore from (7), considering the previous arguments, stability of  $A + B K C$  is obtained, see Zhou, Doyle and Glover [13].

It remains to prove that (i) and (ii-b) are necessary and sufficient conditions for static output feedback stabilizability as well. This part of the proof can be completed using the same arguments as above.  $\square$

**Remark 1.** Equation (3) is equivalent to (5) for

$$G_1 = (B^T P B + R)^{\frac{1}{2}} G.$$

**Remark 2.** The difference between (4) and (6) is then only in constant terms:  $C^T C$  in (4) and  $Q$  in (6). Due to this difference generally different solutions  $P, P_1$  are obtained from (4) and (6) and therefore the corresponding stabilizing gains  $K, K_1$  are also different in general.

In Theorem 1 the output feedback stabilization problem is analyzed using the linear-quadratic theory tools. The output feedback matrix  $K$  is tightly connected, through equations (3) and (4), with the LQ optimal state feedback control gain matrix. The weighting matrix  $G$  shows in certain sense “the difference” between the LQ optimal state feedback gain and the proposed output gain (see equation (3)). However, similarly to the existing literature, Theorem 1 is existential and does not solve the computational aspects of the problem.

The discrete-time counterpart algorithm to that given in Kučera and De Souza [8] is proposed here in two alternatives corresponding to (ii-a) and (ii-b) of Theorem 1. The idea behind the algorithm is to start from ideal case when  $G = 0$ , which corresponds to the optimal LQ control law,

$$K C = -(B^T P B + R)^{-1} B^T P A$$

where  $P$  is a solution to Riccati equation (4). However such a matrix  $P$  does not necessarily exist for  $C \neq I$ , therefore an iterative procedure is proposed to find  $G$  and  $P$  such that the constraints (3), (4) are both satisfied.

**Algorithm A.**

Step 1. Set  $i = 0$ ,  $G_i = 0$ .

Step 2. Solve the Riccati equation

$$A^T P_{i+1} A - P_{i+1} - A^T P_{i+1} B (B^T P_{i+1} B + R)^{-1} B^T P_{i+1} A + \\ C^T C + G_i^T (B^T P_{i+1} B + R) G_i = 0$$

for  $P_{i+1}$  symmetric and nonnegative definite.

Step 3. Put

$$G_{i+1} = (B^T P_{i+1} B + R)^{-1} B^T P_{i+1} A [I - C^T (C C^T)^{-1} C].$$

Step 4. Increase  $i$  by one and go to Step 2.

If the sequence  $\{P_i\}$  converges to some  $P$ , an output feedback matrix that satisfies (3) and (4) is given by

$$K = -(B^T P B + R)^{-1} B^T P A C^T (C C^T)^{-1}.$$

**Algorithm B.**

Step 1. Set  $i = 0$ ,  $G_{1,i} = 0$ .

Step 2. Solve the Riccati equation

$$A^T P_{1,i+1} A - P_{1,i+1} - A^T P_{1,i+1} B (B^T P_{1,i+1} B + R)^{-1} \\ B^T P_{1,i+1} A + Q + G_{1,i}^T G_{1,i} = 0$$

for  $P_{1,i+1}$  symmetric and nonnegative definite.

Step 3. Put

$$G_{1,i+1} = (B^T P_{1,i+1} B + R)^{-\frac{1}{2}} B^T P_{1,i+1} A [I - C^T (C C^T)^{-1} C]$$

Step 4. Increase  $i$  by one and go to Step 2.

If the sequence  $\{P_{1,i}\}$  converges to some  $P_1$ , an output feedback gain matrix that satisfies (5) and (6) is given by

$$K_1 = -(B^T P_1 B + R)^{-1} B^T P_1 A C^T (C C^T)^{-1}.$$

Notice that while alternatives (ii-a) and (ii-b) in Theorem 1 are equivalent as far as the *existence* of corresponding solutions is concerned, it is not necessarily the case for Algorithm A and Algorithm B since there is a difference in Step 3 (solution  $P_i$  is generally different from  $P_{1,i}$ ). The proposed algorithms are computationally simple; however the question of convergence of the above algorithms still remains open and limits their efficiency. Therefore an LMI approach will be developed in the next section that provides a way to non-iterative computation of stabilizing output feedback.

### 3. GUARANTEED COST OUTPUT FEEDBACK CONTROL

Consider the linear discrete time system (1) with output feedback (2) and the cost function

$$J = \sum_{k=0}^{\infty} [x(k)^T Q x(k) + u(k)^T R u(k)] \quad (10)$$

where  $Q \in \mathbb{R}^{n \times n}$ ,  $R \in \mathbb{R}^{m \times m}$  are real symmetric positive definite matrices. The results on output feedback stabilization from Section 2 can be slightly modified to get an upper bound on the closed loop value of the quadratic cost function. In this way the so called guaranteed cost control for output feedback is given in this section. The results are provided in terms of algebraic linear matrix inequalities to indicate a possibility of LMI solution (Boyd, El Ghaoui, Feron and Balakrishnan, [1]).

**Theorem 2.** Consider system (1) and cost function (10). Then the following statements are equivalent.

1. System (1) is static output feedback stabilizable with guaranteed cost

$$J \leq x_0^T P x_0 \quad (11)$$

where  $P$  is a real symmetric positive definite matrix,  $x_0 = x(0)$  is initial value of the state vector  $x(k)$ .

2. The pair  $(A, B)$  is stabilizable, the pair  $(A, C)$  is detectable and there exist real matrices  $K$  and  $G$  such that

$$G = (B^T P B + R)^{-\frac{1}{2}} B^T P A + (B^T P B + R)^{\frac{1}{2}} K C \quad (12)$$

where  $P$  is the real symmetric positive definite solution of

$$A^T P A - P - A^T P B (B^T P B + R)^{-1} B^T P A + Q + G^T G \leq 0. \quad (13)$$

*Proof.*

2.  $\rightarrow$  1.

The first part of proof follows from Theorem 1. When conditions in statement 2. hold that implies output feedback stabilizability of system (1). Note that (13) is for  $G$  given by (12) equivalent to

$$(A + B K C)^T P (A + B K C) - P + C^T K^T R K C + Q \leq 0 \quad (14)$$

Inequality (14) is furthermore equivalent to stability of closed loop system  $(A + B K C)$ . The upper bound on the cost function is then derived in standard way. Let us define the function

$$V(k) = x(k)^T P x(k)$$

From (14)

$$x(k)^T [(A + B K C)^T P (A + B K C) - P + C^T K^T R K C + Q] x(k) \leq 0$$

Considering system description (1) and (2)

$$x(k+1)^T Px(k+1) - x(k)^T Px(k) \leq -[x(k)^T Qx(k) + u(k)^T Ru(k)]$$

and

$$\sum_{k=0}^{\infty} [x(k)^T Px(k) - x(k+1)^T Px(k+1)] \geq \sum_{k=0}^{\infty} [x(k)^T Qx(k) + u(k)^T Ru(k)]$$

Since  $(A + BKC)$  is stable,  $x(k) \rightarrow 0$  for  $k \rightarrow \infty$  and we finally obtain

$$J \leq x(0)^T Px(0).$$

1.  $\rightarrow$  2.

We will show that assuming 1. holds and 2. does not hold in Theorem 2 leads to a contradiction. Assume that (1) is stabilizable by output feedback, i.e. there exists  $K$  such that  $(A + BKC)$  is stable and  $J \leq x(0)^T P^* x(0)$  for a symmetric positive definite matrix  $P^*$ . Suppose that 2. does not hold for  $P^*$  or, equivalently

$$(A + BKC)^T P^* (A + BKC) - P^* + C^T K^T RKC + Q > 0 \quad (15)$$

since (13) is equivalent to (14). Let us define the function  $V(k)^* = x(k)^T P^* x(k)$ . Then following the same steps as in the previous part of the proof for  $V(k)^*$  and inequality (15) we obtain

$$\sum_{k=0}^{\infty} [x(k+1)^T P^* x(k+1) - x(k)^T P^* x(k)] > - \sum_{k=0}^{\infty} x(k)^T [Q + C^T K^T RKC] x(k)$$

Since according to the assumption  $(A + BKC)$  is stable,  $x(k) \rightarrow 0$  for  $k \rightarrow \infty$  and the last inequality reduces to

$$x(0)^T P^* x(0) < \sum_{k=0}^{\infty} [x(k)^T Qx(k) + u(k)^T Ru(k)] = J$$

that contradicts to the assumption that  $P^*$  provides an upper bound on  $J$ .  $\square$

Inequality (13) with (12) in Theorem 2 corresponds to (6) with (5) from Theorem 1. Similarly (4) with (3) from Theorem 1 can be modified to include  $Q$  instead of the  $C^T C$  term. However then according to Remark 1 and 2 the same results are obtained as those in (13) with (12).

The following corollary outlines a way to develop an LMI solution for output feedback design.

**Corollary 1.** System (1) with cost function (10) is output feedback stabilizable with guaranteed cost

$$J \leq x_0^T P x_0, \quad P > 0$$



if and only if

$$\Phi_d = A^T P A - P + Q - A^T P B (B^T P B + R)^{-1} B^T P A \leq 0 \quad (16)$$

and

$$\begin{bmatrix} -I & G \\ G^T & \Phi_d \end{bmatrix} \leq 0 \quad (17)$$

where  $G$  is given in (12).

In Corollary 1 the inequality (13) is split into (16) and (17) so that  $G$  and  $P$  are formally “separated”. Thus once a  $P$  is obtained satisfying (16), output feedback matrix can be computed from (17) as in Algorithm C below.

### Algorithm C.

- Step 1.* Find  $P$  as a solution to (16). If (16) is not feasible the considered system is not output feedback stabilizable.
- Step 2.* Compute  $K$  from (17) for  $P$  from Step 1. If (17) is feasible, a stabilizing output feedback  $K$  is found for guaranteed cost given by  $P$ . If (17) is infeasible, another  $P$  verifying (16) can be checked or cost function  $Q, R$  modified.

The above non-iterative Algorithm C with guaranteed cost is a discrete-time counterpart to the algorithm introduced in Veselý [12] for continuous-time systems. There are two non-trivial tasks to be solved in Algorithm C. Inequality of (16) is nonlinear and does not possess any obvious convexity property. Therefore to find the LMI solution, inequality (16) is reformulated in the following way

$$P = A^T [P - P B (B^T P B + R)^{-1} B^T P] A + Q.$$

Let us denote

$$Re = B^T P B + R, \quad L = P B (Re)^{-1}$$

then the following algorithm with respect to  $P$  is obtained.

### Algorithm P.

- Step 1.*  $i = 1 \quad P_0 = I$
- Step 2.*  $P_i = A^T P_{i-1} A + Q$
- Step 3.*  $Re = B^T P_i B + R$
- Step 4.*  $L = P_i B (Re)^{-1}$
- Step 5.*  $P_{i+1} = (I - L B^T) P_i (I - L B^T)^T + L R L^T$

Step 6.  $i = i + 1$  go to Step 2.

Step 7. If matrices  $P_i$  calculated on the second step converge say to  $P$ , this is the solution of (16).

The other task is to find a matrix  $P$  subject to (16) for (17) to make it feasible. In the following we propose alternatives to solve this task. Since in the outlined Algorithm C the inequality (16) is solved separately from (17), the results may be conservative. To decrease this conservativeness and “tailor” the solution of (16) to (17) we append  $\Delta Q$  to the left hand side of (16)

$$Q_n = Q + \Delta Q \tag{18}$$

To find a “suitable”  $\Delta Q$ , the minimization of  $\|G^T G\|$  in (13) is included that brings the output control gain “as close as possible” to LQ optimal state control. Solution to

$$\min_K \|G^T G\| \tag{19}$$

where  $G$  is given in (12) yields

$$K = -(B^T P B + R)^{-1} B^T P A C^+ \tag{20}$$

where  $C^+$  is the pseudoinverse of matrix  $C$ ,  $C^+ = C^T (C C^T)^{-1}$  The term  $\Delta Q$  to be used in (18) is equal to

$$\Delta Q = (I - C^+ C)^T A^T P B (B^T P B + R)^{-1} B^T P A (I - C^+ C) \tag{21}$$

Notice that for  $C = I$ ,  $G^T G = 0$  and (13) changes to the Riccati equation form. The resulting sufficient conditions to stabilize the system (1), (2) are given by inequalities (17) and

$$\Phi_d + \Delta Q < 0 \tag{22}$$

The modified Algorithm C runs as follows:

Step 1. Find  $P$  as a solution of (22).

Step 2. Compute  $K$  from (17) for  $P$  obtained in Step 1.

The latter, modified form of Algorithm C for  $K$  given by (20) provides the non-iterative alternative to Algorithms A, B. The Algorithm C and (22) provide two alternatives of non-iterative LMI solution to find stabilizing output feedback gain matrix  $K$  for the system (1), (2) with sufficient stability conditions.

#### 4. EXAMPLES

Three examples that illustrate the use of the algorithms proposed in Sections 2 and 3 are presented. In these examples stability is indicated through a spectral radius  $\rho$  of the studied system ( $\rho(M)$  is the radius of the smallest circle centered in the origin, in which all eigenvalues of matrix  $M$  lie, or  $\rho(M) = |\lambda_M(M)|$ , where  $\lambda_M(M)$  is the eigenvalue of  $M$  with maximal modulus ).

**Example 1.** Consider a DC motor described in discrete-time state-space model by equation (1) with

$$A = \begin{bmatrix} 0.5965 & 0.0708 & 0 \\ -0.6488 & 0.9647 & 0 \\ -0.0788 & -0.0043 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0586 \\ 1.4857 \\ -0.0019 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The open loop system is not stable,  $\rho(A) = 1$ . The aim is to find an output feedback gain matrix  $K$  so that the closed loop system  $A + BKC$  is stable. The results obtained using Algorithms A, B, C are summarized as follows.

Algorithm A:

For  $R = 0.0001$  the gain matrix  $K = [0.1195 \quad 1.0679]$  is obtained and  $\rho(A+BKC) = 0.9781$ .

Algorithm B:

For  $R = 1$ ,  $Q = I$ ,  $K = [0.4034 \quad 0.1816]$  and  $\rho(A + BKC) = 0.9843$ .

For  $R = 0.1$ ,  $Q = I$ ,  $K = [0.3975 \quad 0.1769]$  and  $\rho(A + BKC) = 0.9832$ .

For  $R = 0.01$ ,  $Q = I$ ,  $K = [0.3970 \quad 0.1764]$  and  $\rho(A + BKC) = 0.9831$ .

For  $Q = \text{diag}[1 \quad 0.01 \quad 1] * 0.00001$ ;  $R = 1$  one obtains for Modified Algorithm C

$$\text{eig} CL = \{0.7774 \pm .1498i \quad 0.9998\}, \quad K = [-0.1105 \quad 0.0011]$$

and guaranteed cost  $J \leq \|x_0\|^2 1.6903$ .

Algorithm C gives an unstable closed loop system

$$\text{eig} CL = \{0.7792 \pm 0.1291i \quad 1\}, \quad K = [-0.0491 \quad 0.0001].$$

V-K iterative method (El Ghaoui and Balakrishnan, [4])

$$\text{eig} CL = \{0.6705 \quad 0.9187 \quad 0.9876\}, \quad K = [0.2673 \quad 0.0301]$$

and guaranteed cost  $J \leq \|x_0\|^2 3.1144$ , where  $\text{eig} CL$  are the closed loop eigenvalues.

**Example 2.** Consider a system described by (1), where

$$A = \begin{bmatrix} 0.8897 & 0.0920 & 0.1577 \\ 2.1211 & 0.8077 & 2.9290 \\ 0 & 0 & 0.7985 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0122 & 0.0412 \\ 0.3548 & 0.1230 \\ 0.2015 & 0.2301 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and  $\rho(A) = 1.2923$ ; the system is unstable. The results are summarized as follows.

Algorithm A:

For  $R = 0.0001 * I$  one obtains  $\rho(A + BKC) = 0.9585$  and a gain matrix

$$K = - \begin{bmatrix} 1.2799 & 7.1261 \\ 0.7825 & 0.1011 \end{bmatrix}$$

Algorithm B:

For  $R = 0.01 * I$ ,  $Q = I$  one obtains  $\rho(A + BKC) = 0.9552$  and a gain matrix

$$K = - \begin{bmatrix} 1.0243 & 6.7405 \\ 0.9717 & 0.3865 \end{bmatrix}$$

Algorithm C:

For  $Q = I$ ,  $R = 0.01 * I$  one obtains a gain matrix

$$K = - \begin{bmatrix} 0.9932 & 6.2471 \\ 0.9977 & 0.8242 \end{bmatrix}$$

and  $\rho(A + BKC) = 0.9597$ , and guaranteed cost  $J \leq \|x_0\|^2 106.17$ .

For  $Q = I$ ,  $R = 0.001 * I$  one obtains  $\rho(A + BKC) = 0.9428$ , guaranteed cost  $J \leq \|x_0\|^2 129.50$  and a gain matrix

$$K = - \begin{bmatrix} 0.9716 & 6.6139 \\ 1.1540 & 0.7018 \end{bmatrix}$$

In this example the results obtained by Algorithm A, Algorithm B and Algorithm C do not differ significantly.

**Example 3.** Consider the following discrete-time state space model of the longitudinal motion of a VTOL helicopter (Keel, Bhattacharyya and Howze, [6]) for  $T=0.01s$ .

$$A = \begin{bmatrix} 0.9999963 & 0.0002699 & 0.00016457 & -0.0045584 \\ 0.00047943 & 0.98995 & -0.00017606 & -0.040008 \\ 0.00099919 & 0.0036498 & 0.99303 & 0.014074 \\ 0.0000050006 & 0.000018301 & 0.009965 & 1.0001 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0044212 & 0.0017543 \\ 0.035272 & -0.075542 \\ -0.05494 & 0.044605 \\ -0.00027513 & 0.00022351 \end{bmatrix}, \quad C = [ 0 \quad 1 \quad 0 \quad 0 ]$$

Note that the matrix  $A$  is unstable with eigenvalues

$$eig A = \{0.9795 \quad 0.9977 \quad 1.0028 \pm 0.0026i\}$$

The results of gain matrix calculation can be summarized as follows. For  $R = r * I$ ,  $r = 1$ ;  $Q = \text{diag} [0.01, 10, 0.1, 0.1] * 0.1$  the eigenvalues of closed-loop system and

corresponding gain matrix  $K$  are as follows:

Modified Algorithm C:

$$\text{eig } CL = \{0.6748 \quad 0.0087 \quad 0.9974 \pm .007i\}, \quad K^T = [0.7026 \quad 4.4944]$$

Algorithm C:

$$\text{eig } CL = \{0.7011 \quad 0.9982 \quad 0.9977 \pm 0.0059i\}, \quad K^T = [0.0985 \quad 0.1578]$$

V-K iterative method (El Ghaoui and Balakrishnan, [4]):

$$\text{eig } CL = \{0.7915 \quad 0.9967 \quad 0.9985 \pm .0046i\}, \quad K^T = [0.8750 \quad 3.0263]$$

However, the cost is not guaranteed because LMI solution is not feasible though the closed loop system is stable.

## 5. CONCLUSION

Necessary and sufficient conditions for a discrete-time linear system to be stabilizable via static output feedback have been established in two alternatives. This result provides the discrete-time counterpart to the result of Kučera and De Souza [8]. The corresponding iterative as well as a novel non-iterative LMI based algorithm to compute a stabilizing output feedback gain matrix with guaranteed cost is proposed. Examples are presented to illustrate the use of the algorithms. In general, the algorithms yield different stabilizing control gain matrices, thus providing the designer with a possibility to choose the more appropriate one.

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