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THE ALGEBRAIC OUTPUT FEEDBACK IN THE LIGHT OF DUAL–LATTICE STRUCTURES

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The purpose of this paper is to derive constructive necessary and sufficient conditions for the problem of disturbance decoupling with algebraic output feedback. Necessary and sufficient conditions have also been derived for the same problem with internal stability. The same conditions have also been expressed by the use of invariant zeros. The main tool used is the dual-lattice structures introduced by Basile and Marro [4].

1. INTRODUCTION

It is well known that disturbance decoupling was the first problem approached with geometric techniques, by Basile and Marro [3] and, independently, by Wonham and Morse [11]. In the former of these papers the investigation was extended to output feedback using a dynamic regulator. The same problem was refined in the literature by also taking into account the stability requirement (Willems and Commault [10], Imai and Akashi [7] and, without using eigenspaces, Basile, Marro and Piazzi [5]). The problems of disturbance decoupling and disturbance decoupling with internal stability using algebraic output feedback have been solved for left invertible systems by Chen [6].

In this work we will determine constructive sufficient conditions and non constructive necessary and sufficient conditions for non invertible systems and constructive necessary and sufficient conditions for systems that are only left or only right invertible. This will be accomplished through the use of the dual lattice structures introduced by Basile and Marro [4] which is one of the key features of the geometric approach to linear MIMO systems. The strength of this approach lies in the great simplicity of the conditions solving the problem and in the great easiness of their checkability by using algorithms developed for the MATLAB platform.

We will also determine necessary and sufficient conditions for solving the problem of disturbance decoupling with stability with algebraic output feedback. The same conditions will be also derived by using the concept of invariant zeros first introduced by Rosenbrock [9], referring to the Smith form and the system matrix, and then revised under a geometric light by Anderson [1]. Unfortunately these conditions are not constructive.

Finally some numerical examples will be presented. A solution will be determined, when possible, through specific algorithms developed on the MATLAB platform.

The following notation is used. $\boldsymbol{R}$ stands for the field of real numbers. Sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{X}$, $\mathcal{I}$, $\mathcal{V}$, etc.; since most of the geometric theory of dynamic system herein presented is developed in the vector space $\boldsymbol{R}^n$, we reserve the symbol $\mathcal{X}$ for the full space, i. e., we assume $\mathcal{X} := \boldsymbol{R}^n$. The orthogonal complement of any subspace $\mathcal{Y} \subseteq \mathcal{X}$ is denoted by $\mathcal{Y}^\perp$, matrices and linear maps by slanted capitals like A, B, etc., the image and the null space of the generic matrix or linear transformation A by $\mathrm{im}A$ and $\ker A$ respectively, the transpose of the generic real matrix A by A^T, the spectrum of A by $\sigma(A)$, the $n\times n$ identity matrix by I_n. The restriction of map A to the A-invariant subspace $\mathcal{L}$ is denoted by $A|_{\mathcal{L}}$. Given two A-invariants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1 \subseteq \mathcal{L}_2$, the map induced by A on the quotient space $\mathcal{L}_1/\mathcal{L}_2$ is denoted by $A|_{\mathcal{L}_1/\mathcal{L}_2}$. Notation $\mathcal{Z}_1 - \mathcal{Z}_2$ will be used for the difference of sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ with repetition count.

2. GENERAL BACKGROUND AND STATEMENT OF THE PROBLEM

Let us consider a system described by a five-map system (A, B, C, D, E), modeled by

$$\dot{x}(t) = A\,x(t) + B\,u(t) + D\,d(t)\,, \quad x(0) = x_0 \tag{1}$$

$$y(t) = C\,x(t) \tag{2}$$

$$e(t) = E\,x(t) \tag{3}$$

where $x \in \mathcal{X}$ $(= \boldsymbol{R}^n)$, $u \in \mathcal{U}$ $(= \boldsymbol{R}^p)$ and $y \in \mathcal{Y}$ $(= \boldsymbol{R}^q)$ denote respectively the state, input and output. In the following the short notations $\mathcal{B} := \mathrm{im}\,B$, $\mathcal{C} := \ker C$, $\mathcal{D} := \mathrm{im}\,D$ and $\mathcal{E} := \ker E$ will be used.

The problem of simple disturbance decoupling by means of output algebraic feedback is stated as follows:

Problem 1. Given system (1) – (3) determine, if possible, a feedback matrix K (having p rows and q columns) such that:

i) $e(t) = 0$, $t \geq 0$, for all admissible $d(\cdot)$ and for $x(0) = 0$.

The problem of disturbance decoupling with stability by means of output algebraic feedback is stated as follows:

Problem 2. Assume for system (1) – (3) that (A, B) is stabilizable and (A, C) detectable. Determine, if possible, a feedback matrix K (having p rows and q columns) such that:

i) Problem 1 is solvable;

ii) $\lim_{t\to\infty} x(t) = 0$ for all $x(0)$ with $d(\cdot) = 0$.

Conditions (i) and (ii) are called respectively the *structure requirement* and the *stability requirement*.

The following theorem introduced by Basile and Marro [4] (p. 256) is basic to solve Problem 1 in the geometric approach framework:

Theorem 1. Refer to triple (A, B, C). There exists a matrix K such that a given subspace $\mathcal{V}$ is an $(A + BKC)$-invariant if and only if $\mathcal{V}$ is both an $(A, \mathcal{B})$-controlled invariant and an $(A, \mathcal{C})$-conditioned invariant.

This theorem is constructive, meaning that given such a subspace $\mathcal{V}$ there exists a procedure to determine matrix K.

Let us recall now the definitions of lattices $\phi(\mathcal{B} + \mathcal{D}, \mathcal{E})$ and $\psi(\mathcal{E} \cap \mathcal{C}, \mathcal{D})$, i. e. the dual lattice structures, on which the next considerations will be based:

$$\phi(\mathcal{B} + \mathcal{D}, \mathcal{E}) := \{\mathcal{V} \mid A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} + \mathcal{D},\ \mathcal{V} \subseteq \mathcal{E},\ \mathcal{V}^* \cap (\mathcal{B} + \mathcal{D}) \subseteq \mathcal{V}\} \tag{4}$$

is the lattice of all $(A, \mathcal{B} + \mathcal{D})$-controlled invariants self bounded with respect to $\mathcal{E}$, and its supremum and infimum are given by

$$\mathcal{V}^* := \max \mathcal{V}(A, \mathcal{B} + \mathcal{D}, \mathcal{E}) \tag{5}$$
$$\mathcal{V}_m := \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{E}, \mathcal{B} + \mathcal{D}) \tag{6}$$

respectively, while

$$\psi(\mathcal{C} \cap \mathcal{E}, \mathcal{D}) := \{\mathcal{S} \mid A(\mathcal{S} \cap \mathcal{C} \cap \mathcal{E}) \subseteq \mathcal{S},\ \mathcal{D} \subseteq \mathcal{S},\ \mathcal{S} \subseteq \mathcal{S}^* + (\mathcal{C} \cap \mathcal{E})\} \tag{7}$$

is the lattice of all $(A, \mathcal{C} \cap \mathcal{E})$-conditioned invariants self hidden with respect to $\mathcal{D}$, with infimum and supremum given by

$$\mathcal{S}^* := \min \mathcal{S}(A, \mathcal{C} \cap \mathcal{E}, \mathcal{D}) \tag{8}$$
$$\mathcal{S}_M := \mathcal{S}^* + \max \mathcal{V}(A, \mathcal{D}, \mathcal{C} \cap \mathcal{E}). \tag{9}$$

All of the above subspaces are easily determined through the standard geometric approach algorithms.

Finally let us recall the definitions of left and right invertibility. Under the assumption that B and C have maximum rank, the triple (A, B, C) is said to be:

left-invertible if and only if $\mathcal{V}_0^* \cap \mathcal{B} = \emptyset$, with $\mathcal{V}_0^* := \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$;

right-invertible if and only if $\mathcal{S}_0^* + \mathcal{C} = \boldsymbol{R}^n$, with $\mathcal{S}_0^* := \min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$.

3. STRUCTURAL CONDITIONS

Let us consider Problem 1. Clearly the problem admits solution if and only if the reachable set by d, i.e. the minimal $(A+BKC)$-invariant containing $\mathcal{D}$ is contained in $\mathcal{E}$. By Theorem 1, Problem 1 is solvable if and only if there exists a subspace $\mathcal{V}$ such that:

$$\text{i)} \qquad \mathcal{D} \subseteq \mathcal{V} \subseteq \mathcal{E} \tag{10}$$

$$\text{ii)} \qquad \mathcal{V} \text{ is an } (A,\mathcal{B})\text{−controlled invariant} \tag{11}$$

$$\text{iii)} \qquad \mathcal{V} \text{ is an } (A,\mathcal{C})\text{−conditioned invariant.} \tag{12}$$

Necessary but not sufficient conditions for the existence of such a $\mathcal{V}$ are:

$$\mathcal{D} \subseteq \mathcal{V}^* \tag{13}$$

$$\mathcal{S}^* \subseteq \mathcal{E} \tag{14}$$

$$\mathcal{S}^* \subseteq \mathcal{V}^*. \tag{15}$$

The proof of the above conditions is trivial. Conditions (13) and (14) derive from (10) while (15) derives from (11) – (12).

Under assumptions (13) – (15) some very interesting properties regarding lattices (4) and (7) can be determined. Under assumption (13) it can be proved that

i) every subspace of lattice (4) contains $\mathcal{D}$,

ii) $\mathcal{V}^* = \max \mathcal{V}(A,\mathcal{B},\mathcal{E})$,

while under (14)

i) every subspace of lattice (7) is contained in $\mathcal{E}$,

ii) $\mathcal{S}^* = \min \mathcal{S}(A,\mathcal{C},\mathcal{D})$,

as seen in Basile–Marro [4] (pp. 225–226).

Two very useful sublattices of (4) and (7), introduced by Basile–Marro [4] (p. 271) are:

$$\phi_R := \{\mathcal{V} \mid \mathcal{V} \in \phi(\mathcal{B}+\mathcal{D},\mathcal{E}),\ \mathcal{V}_m \subseteq \mathcal{V} \subseteq \mathcal{V}_M\} \tag{16}$$

$$\psi_R := \{\mathcal{S} \mid \mathcal{S} \in \psi(\mathcal{C}\cap\mathcal{E},\mathcal{D}),\ \mathcal{S}_m \subseteq \mathcal{S} \subseteq \mathcal{S}_M\}. \tag{17}$$

where $\mathcal{V}_m$ and S_M are given by (6) and (9) and, under assumptions (13) – (15),

$$\mathcal{V}_M := \mathcal{V}_m + \mathcal{S}_M\,, \tag{18}$$

$$\mathcal{S}_m := \mathcal{V}_m \cap \mathcal{S}_M. \tag{19}$$

These sublattices are the core of the dual-lattice structures theory. Such sublattices are lattices themselves but, most importantly, it is possible to state a one-to-one corrispondence between their elements through the following relations:

$$\mathcal{V} = \mathcal{V}_m + \mathcal{S}\,, \tag{20}$$

$$\mathcal{S} = \mathcal{V} \cap \mathcal{S}_M \tag{21}$$

where $\mathcal{V} \in \psi_R$ and $\mathcal{S} \in \phi_R$.

Conditions (13) – (15) will be considered automatically satisfied from now on.

Using these sublattices we are able to introduce our first main result:

Theorem 2. Referring to (1) – (3), Problem 1 is solvable if relation

$$\mathcal{V}_m \subseteq \mathcal{S}_M \tag{22}$$

holds.

P r o o f . Subspaces $\mathcal{V}_m$ and $\mathcal{S}_M$ are both solutions of the problem under assumption (22). In fact $\mathcal{V}_m$ satisfies (10) – (11), being the infimum of $\phi(\mathcal{B}+\mathcal{D}, \mathcal{E})$, and (12) since

$$\mathcal{S}_m := \mathcal{V}_m \cap \mathcal{S}_M = \mathcal{V}_m \Rightarrow \mathcal{V}_m \in \psi(\mathcal{C} \cap \mathcal{E}, \mathcal{D}). \tag{23}$$

Dually S_M satisfies (10) and (12), being the supremum of $\psi(\mathcal{C} \cap \mathcal{E}, \mathcal{D})$, and (11) since

$$\mathcal{V}_M := \mathcal{V}_m + \mathcal{S}_M = \mathcal{S}_M \Rightarrow \mathcal{S}_M \in \phi(\mathcal{B}+\mathcal{D}, \mathcal{E}). \tag{24}$$

□

This very interesting result has been derived without any assumption on the system's invertibility. Unfortunately condition (22) is only sufficient. It becomes both necessary and sufficient if the system in hand is both left and right-invertible, as stated in the following:

Theorem 3. Let the given system be both left-invertible with respect to u and right-invertible with respect to y. Problem 1 is solvable if and only if relation (22) holds.

P r o o f . *(Only if)* Being the system left and right-invertible all $(A, \mathcal{B})$-controlled invariant subspaces are also self bounded with respect to $\mathcal{E}$ and all $(A, \mathcal{C})$-conditioned invariant subspaces are also self hidden with respect to $\mathcal{D}$. This means that any subspace solving the problem, i. e. satisfying conditions (10) – (12), must be an element of both $\phi(\mathcal{B}+\mathcal{D}, \mathcal{E})$ and $\psi(\mathcal{C} \cap \mathcal{E}, \mathcal{D})$. Clearly if relation (22) is not satisfied then lattices (4) and (7) have no intersection and so the problem has no solution.

(If) Implied by Theorem 2. □

It is important to note that hardly the systems in hand are both left and right-invertible. If one of these assumption fails relation (22) is not necessary anymore. Anyway we are able to state new necessary and sufficient conditions for the solvability of the problem:

Theorem 4. Let the given system be left-invertible with respect to input u. Problem 1 is solvable if and only if $\mathcal{V}_m$ is an $(A,\mathcal{C})$-conditioned invariant.

P r o o f. *(Only if)* By the left invertibility assumption $\mathcal{V}^*\cap\mathcal{B}=\emptyset$. This means that every $(A,\mathcal{B})$-controlled invariant is self bounded with respect to $\mathcal{E}$. This means that a subspace $\mathcal{V}$ satisfying properties (10) – (12), has to be searched for in $\phi(\mathcal{B}+\mathcal{D},\mathcal{E})$. We want to show that if such a $\mathcal{V}$ exists then $\mathcal{V}_m$ is an $(A,\mathcal{C})$-conditioned invariant. Being $\mathcal{V}$ an element of $\phi(\mathcal{B}+\mathcal{D},\mathcal{E})$, hence $\mathcal{V}_m\subseteq\mathcal{V}$, it follows that

$$(\mathcal{V}_m\cap\mathcal{C})\subseteq(\mathcal{V}\cap\mathcal{C})\Rightarrow A\,(\mathcal{V}_m\cap\mathcal{C})\subseteq A\,(\mathcal{V}\cap\mathcal{C}) \tag{25}$$

and, being $\mathcal{V}$ an $(A,\mathcal{C})$-conditioned invariant,

$$A\,(\mathcal{V}_m\cap\mathcal{C})\subseteq A\,(\mathcal{V}\cap\mathcal{C})\subseteq\mathcal{V}. \tag{26}$$

On the other hand, $\mathcal{V}_m$ being an $(A,\mathcal{B})$-controlled invariant implies that

$$A\,\mathcal{V}_m\subseteq\mathcal{V}_m+\mathcal{B}\Rightarrow A\,(\mathcal{V}_m\cap\mathcal{C})\subseteq\mathcal{V}_m+\mathcal{B}. \tag{27}$$

The two relations above show that $A\,(\mathcal{V}_m\cap\mathcal{C})$ is included in the intersection of subspaces $\mathcal{V}$ and $\mathcal{V}_m+\mathcal{B}$, hence

$$A\,(\mathcal{V}_m\cap\mathcal{C})\subseteq(\mathcal{V}_m+\mathcal{B})\cap\mathcal{V}=(\mathcal{V}\cap\mathcal{V}_m)+(\mathcal{B}\cap\mathcal{V})=\mathcal{V}_m \tag{28}$$

since $\mathcal{V}_m\subseteq\mathcal{V}$ and since $\mathcal{V}^*\cap\mathcal{B}=\emptyset\Rightarrow\mathcal{V}\cap\mathcal{B}=\emptyset$.

(If) Obvious by virtue of (10) – (12). □

Corollary 1. Let the given system be right-invertible with respect to output y. The disturbance decoupling problem with algebraic output feedback is solvable if and only if $\mathcal{S}_M$ is an $(A,\mathcal{B})$-controlled invariant.

P r o o f. Dual to proof of Theorem 4. □

All of these conditions are easily checkable through appropriate algorithms and are constructive, meaning that, when a subspace satisfying Theorem 1 has been determined, it is easy to obtain a matrix K solving the problem.

The following decomposition, which can be applied to a system satisfying statements (13) – (15), is very useful to prove the next statements.

Decomposition 1. Consider the similarity transformation $T:=[T_1\,T_2\,T_3\,T_4\,T_5\,T_6]$, with $\mathrm{im}T_1=S^*:=\min\mathcal{S}(A,\mathcal{C},\mathcal{D})$, $\mathrm{im}\,[T_1T_2]=\mathcal{S}_M\cap\mathcal{V}_m$, $\mathrm{im}\,[T_1\ T_2\ T_3]=\mathcal{S}_M$, $\mathrm{im}[T_1\ T_2\ T_4]=\mathcal{V}_m$, $\mathrm{im}\,[T_1\ T_2\ T_3\ T_4\ T_5]=\mathcal{V}^*=\max\mathcal{V}(A,\mathcal{B},\mathcal{E})$ and T_6 such that T is nonsingular. In the new basis matrices A,B,C,D and E are expressed by

$$A'=\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14} & A'_{15} & A'_{16}\\ A'_{21} & A'_{22} & A'_{23} & A'_{24} & A'_{25} & A'_{26}\\ O & O & A'_{33} & O & A'_{53} & A'_{56}\\ A'_{41} & O & O & A'_{44} & A'_{45} & A'_{46}\\ O & O & O & O & A'_{55} & A'_{56}\\ A'_{61} & O & O & A'_{64} & A'_{65} & A'_{66} \end{bmatrix}, \tag{29}$$

$$B' = \begin{bmatrix} B'_1 \\ B'_2 \\ O \\ B'_4 \\ O \\ B'_6 \end{bmatrix}, D' = \begin{bmatrix} D'_1 \\ O \\ O \\ O \\ O \\ O \end{bmatrix} \tag{30}$$

$$C' = \begin{bmatrix} C'_1 & O & O & C'_4 & C'_5 & C'_6 \end{bmatrix}, \tag{31}$$

$$E' = \begin{bmatrix} O & O & O & O & O & E'_6 \end{bmatrix}. \tag{32}$$

This representation has been obtained without any assumptions on either left or right invertibility for the system.

Proof. It is always possible to choose T_2 and T_3 so that $\mathrm{im}[T_2\ T_3] \subseteq \ker C$ since $\mathcal{S}_M$ is self hidden with respect to $\mathcal{D}$. It is always possible to choose T_1, T_2, T_4 and T_6 so that $\mathrm{im}[T_1\ T_2\ T_4\ T_6] \subseteq \mathrm{im}\, B$ since $\mathcal{V}_m$ is self bounded with respect to $\mathcal{E}$. The particular form of matrices B' and C' are due to these particular choices. The particular form of matrix D' is due to relation $\mathcal{D} \subseteq \mathcal{S}^*$. The particular form of matrix E' is due to relation $\mathcal{V}^* \subseteq \mathcal{E}$. The particular form of matrix A' is due to $\mathcal{V}_m$ being an $(A, \mathcal{B})$-controlled invariant and $\mathcal{S}_M$ being an $(A, \mathcal{C})$-conditioned invariant. □

This decomposition becomes even more simple if the system is left or right-invertible. In the former case matrices B'_1, B'_2 and B'_4 are zero and so we obtain

$$B' = \begin{bmatrix} O \\ O \\ O \\ O \\ O \\ B'_6 \end{bmatrix}, \tag{33}$$

while in the latter matrices C'_4, C'_5 and C'_6 are zero giving us matrix

$$C' = \begin{bmatrix} C'_1 & O & O & O & O & O \end{bmatrix}. \tag{34}$$

It is also important to note the following:

Corollary 2. Let the given system be left-invertible. If subspace $\mathcal{V}_m$ is an $(A, \mathcal{C})$-conditioned invariant, solving Problem 1, subspace $\mathcal{V}_M$ also solves the problem.

Proof. The given problem admits a solution, as seen in Theorem 2, if and only if $\mathcal{V}_m$ is a solution, i.e. a matrix K exists such that $\mathcal{V}_m$ is an $(A+BKC)$-invariant. Given the particular structure of matrix B', a matrix K which solves the problem is such that $A'_{61} + B'_6 K C'_1 = 0$ and $A'_{64} + B'_6 K C'_4 = 0$, and obviously in that case

$\mathcal{V}_m + \mathcal{S}_M$ is also an $(A + BKC)$-invariant. This implies, using Theorem 1, that $\mathcal{V}_m + \mathcal{S}_M$ is both an $(A, \mathcal{B})$-controlled invariant and an $(A, \mathcal{C})$-conditioned invariant. □

Corollary 3. Let the given system be right-invertible. If subspace $\mathcal{S}_M$ is an $(A, \mathcal{B})$-controlled invariant solving Problem 1, subspace $\mathcal{S}_m$ is both an $(A, \mathcal{C})$-conditioned invariant and a $(A, \mathcal{B})$-controlled invariant.

Proof. Dual to proof of Corollary 2. □

Corollary 4. Let the system be SISO and either left or right-invertible. Problem 1 has in general no solution.

Proof. Matrix K is a real number for the SISO nature of the system. Suppose the system left-invertible. It appears obvious that generally it is impossible to choose K such that $A'_{61} + B'_6 K C'_1 = 0$ and $A'_{64} + B'_6 K C'_4 = 0$ both hold and so the problem doesn't admit a solution. □

This is a clear example of how regulators using algebraic output feedback work. For the disturbance decoupling to have a solution using this kind of feedback it is obviously necessary that the same problem has a solution through state feedback, but this is not sufficient. In fact it is also necessary to use "enough" outputs to evaluate "enough" state variables, since through algebraic output feedback we have no information on x. SISO systems, in general, have too few outputs to solve the problem using this kind of regulator. Clearly, extending the rank of matrix C, which on a practical level means using more sensors, increases knowledge of state x and consequently the chances that Problem 1 is solvable.

In the most general case of a system being neither left nor right-invertible we are only able to state a constructive sufficient condition as seen in Theorem 2. Anyway if relation (22) does not hold, as it often happens, the following result is very useful for the search of a resolvent:

Property 1. Problem 1 admits a solution if and only if

i) $\mathcal{V}_m$ is an $(A, \mathcal{C})$-conditioned invariant, or

ii) $\mathcal{S}_M$ is an $(A, \mathcal{B})$-controlled invariant, or

iii) a subspace $\mathcal{V}$ being both an $(A, \mathcal{B})$-controlled invariant and an $(A, \mathcal{C})$-conditioned invariant exists such that $\mathcal{S}_m \subset \mathcal{V} \subset \mathcal{V}_m$ or $\mathcal{S}_M \subset \mathcal{V} \subset \mathcal{V}_M$.

Proof. *(Only if)* Suppose that a solution $\mathcal{V}$ exists such that $\mathcal{S}^* \subseteq \mathcal{V} \subseteq \mathcal{S}_M \cap \mathcal{V}_m$. In such case it is possible to extend Decomposition 1 by choosing $T_1 = [T_{11}\ T_{12}]$ with $\mathrm{im}(T_{11}) = S^*$ and $\mathrm{im}\,(T_1) = \mathcal{V}$. For $\mathcal{V}$ to be a solution of Problem 1 there must exist a K such that $A'_{41_1} + B'_4 K C'_{1_1} = 0$, $A'_{61_1} + B'_6 K C'_{1_1} = 0$, $A'_{41_2} + B'_4 K C'_{1_2} = 0$

and $A'_{61_2} + B'_6 K C'_{1_2} = 0$. So matrix K solves the problem also for S_M. The same considerations can be repeated if we suppose that a solution $\mathcal{V}$ exists such that $\mathcal{S}_m \subseteq \mathcal{V} \subseteq \mathcal{S}_M$. Dual considerations can be made if the supposed solution $\mathcal{V}$ is such that $\mathcal{V}_m \subseteq \mathcal{V} \subseteq \mathcal{V}^*$.

(If) Obvious. □

If neither $\mathcal{V}_m$ nor S_M are a solution for the problem, the subspace that solves it has to be looked for in a "narrower" space, but we can't state if that subspace actually exists or not and we have no procedure to determine it. Anyway in many practical cases it has been shown that subspace $\mathcal{V}_m$ solves the problem. An example of a system following Property 1 is here presented:

Example 1. Given system (A, B, C, D, E)

$$A' = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 1 & 4 & 7 & 2 & -2 \\ 0 & 0 & -1 & 0 & 0 & 2 & 7 \\ 1 & 0 & 0 & -3 & -2 & 1 & 1 \\ 4 & 0 & 0 & -3 & 5 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 7 & 0 & 0 & 2 & -6 & 1 & 1 \end{bmatrix}, \tag{35}$$

$$B' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad D' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{36}$$

$$C' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{37}$$

$$E' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

It is easy to compute

$$\mathcal{V}_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{39}$$

Both $\mathcal{V}_m$ and $\mathcal{S}_M$ do not solve the problem. Anyway subspace

$$\mathcal{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

for which $\mathcal{S}_m \subseteq \mathcal{V} \subseteq \mathcal{V}_m$ is a solution of the problem being both an $(A, \mathcal{B})$-controlled invariant and an $(A, \mathcal{C})$-conditioned invariant.

4. DISTURBANCE DECOUPLING WITH STABILITY

Let us consider now Problem 2. The following holds:

Theorem 5. Let the given system be left-invertible with respect to input u and and the pair (A, B) be stabilizable. Referring to Decomposition 1, Problem 2 is solvable if and only if:

i) $\mathcal{V}_m$ is an $(A, \mathcal{C})$-conditioned invariant;

ii) subspace $\mathcal{V}_M$ is internally stabilizable;

iii) $\exists\, F \mid (A + BF)\mathcal{V}_M \subseteq \mathcal{V}_M\,, \quad (A + BF)_{\mathcal{X}/\mathcal{V}_M}$ is stable$\,, \quad \ker C \subseteq \ker F$.

P r o o f. Let us consider Decomposition 1 with B' having the particular structure seen for left-invertible systems. Obviously the first condition is still necessary but not sufficient anymore. This is due to the fact that we are looking for a matrix K such that $\mathcal{V}_m$ is an $(A + BKC)$-invariant and so that matrix $(A + BKC)$ has all eigenvalues stable. Clearly if the former condition is verified then matrix $(A+BKC)$ has a triangular structure so that

$$\sigma(A + BKC) = \sigma(A_1) \cup \sigma(A_2)$$

where

$$A_1 := \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A_{14} \\ A'_{21} & A'_{22} & A'_{23} & A'_{24} \\ O & O & A'_{33} & O \\ A'_{41} & O & O & A'_{44} \end{bmatrix} \tag{41}$$

$$A_2 := \begin{bmatrix} A'_{55} & A'_{56} \\ A'_{65} + B'_6 K C'_5 & A'_{66} + B'_6 K C'_6 \end{bmatrix}. \tag{42}$$

The first set of eigenvalues is stable if and only if relation (ii) holds while the second one is stable if and only if:

1. the system is stabilizable so that $\mathcal{V}_M$ is externally stabilizable as an $(A,\mathcal{B})$-controlled invariant i. e. $\exists\, F \,|\, (A+BF)\mathcal{V}_M \subseteq \mathcal{V}_M\,,\ (A+BF)_{\mathcal{X}/\mathcal{V}_M}$ is stable;
2. there exists a matrix K of output feedback which results perfectly mappable with one of the state feedback matrices satisfying condition 1 just mentioned, i. e. given one of the above F, there exists K such that
   $F = KC \Leftrightarrow C^T K^T = F^T \Leftrightarrow \operatorname{im} F^T \subseteq \operatorname{im} C^T \Leftrightarrow \ker C \subseteq \ker F.$

i. e. (iii) holds. □

Corollary 6. Let the given system be right-invertible with respect to output y and the pair (A,C) be detectable. Referring to Decomposition 1, Problem 2 is solvable if and only if:

i) $\mathcal{S}_M$ is an $(A,\mathcal{B})$-controlled invariant.

ii) subspace $\mathcal{S}_m$ is externally stabilizable

iii) $\exists\, G \,|\, (A+CG)\mathcal{S}_m \subseteq \mathcal{S}_m\,,\quad (A+CG)_{\mathcal{S}_m}$ is stable$\,,\quad \operatorname{im} G \subseteq \operatorname{im} B.$

P r o o f . Dual of proof to Theorem 5. □

The same conditions can be stated in terms of invariant zeros. The following holds:

Theorem 6. Let the given system be left-invertible with respect to input u and and the pair (A,B) be stabilizable. Referring to Decomposition 1, Problem 2 is solvable if and only if:

i) $\mathcal{V}_m$ is an $(A,\mathcal{C})$-conditioned invariant;

ii) $\mathcal{Z}(u;e) - \mathcal{Z}(u,d;e)$ are all stable

iii) $\mathcal{Z}(u,d;e) \cap \mathcal{Z}(d;y,e)$ are all stable

iv) $\exists\, F \,|\, (A+BF)\mathcal{V}_M \subseteq \mathcal{V}_M\,,\quad (A+BF)_{\mathcal{X}/\mathcal{V}_M}$ is stable$\,,\quad \ker C \subseteq \ker F.$

P r o o f . We want to show the equivalence of these conditions with the ones stated in Theorem 5. The first and last conditions are the same for both theorems, so we just need to show that condition (ii) of Theorem 5 holds if and only if conditions (ii) – (iii) hold for Theorem 6. Condition (ii) of Theorem 5 can be divided in two parts:

1. $\mathcal{V}_m$ is internally stabilizable,
2. submatrix A'_{33} has stable eigenvalues.

It has been shown in the past (by Piazzi and Marro [8]) that $\mathcal{V}_m$ is internally stabilizable if and only if $\mathcal{Z}(u;e)-\mathcal{Z}(u,d;e)$ are all stable. It is easy to see that submatrix A'_{33} has stable eigenvalues if and only if $\mathcal{Z}(u,d;e)\cap\mathcal{Z}(d;y,e)$ are all stable since

$$\mathcal{Z}(u,d;e)=\sigma(A'_{33})\cup\sigma(A'_{55}) \tag{43}$$
$$\mathcal{Z}(d;y,e)=\sigma(A'_{22})\cup\sigma(A'_{33}) \tag{44}$$

and so the equivalence is proved. □

Corollary 7. Let the given system be right-invertible with respect to input y and and the pair (A,C) be detectable. Referring to Decomposition 1, Problem 2 is solvable if and only if:

i) $\mathcal{S}_M$ is an $(A,\mathcal{B})$-controlled invariant;

ii) $\mathcal{Z}(d;y)-\mathcal{Z}(d;y,e)$ are all stable

iii) $\mathcal{Z}(u,d;e)\cap\mathcal{Z}(d;y,e)$ are all stable

iv) $\exists\, G \mid (A+CG)\mathcal{S}_m\subseteq\mathcal{S}_m\,,\quad (A+CG)_{\mathcal{S}_m}$ is stable$\,,\quad \operatorname{im} G\subseteq\operatorname{im} B$.

Proof. Dual to proof of Theorem 6. □

5. EXAMPLE

Let us consider now a numerical example:

Example 2. Given system (A,B,C,D,E)

$$A'=\begin{bmatrix} -1 & 5 & 3 & 4 & 5 & 6\\ -1 & -2 & -1 & 4 & 7 & 2\\ 0 & 0 & -3 & 0 & 2 & 7\\ -1 & 0 & 0 & -3 & -1 & 1\\ 0 & 0 & 0 & 0 & -4 & -1\\ -2 & 0 & 0 & 2 & 1 & 5 \end{bmatrix}, \tag{45}$$

$$B=\begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix},\quad D=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \tag{46}$$

$$C=\begin{bmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0 \end{bmatrix}, \tag{47}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

which is obviously left invertible with respect to u, it is easy to compute subspace

$$\mathcal{V}_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{49}$$

which is a $(A, \mathcal{C})$-conditioned invariant, i. e. solves the structural conditions given by Theorem 4. Morover subspace $\mathcal{V}_M$ is internally stabilizable. Using the constructive algorithm given by Theorem 1. it is easy to obtain an output feedback matrix solving the problem in hand given by

$$K = \begin{bmatrix} -2 & 0 & 2 \end{bmatrix}. \tag{50}$$

The complete system which makes use of such algebraic feedback unit is stable since its eigenvalues are given by the set $\{-0.9754 + 3.3391i, -0.9754 - 3.3391i, -4.0493, -3, -4.5000 + 0.8660i, -4.5000 - 0.8660i\}$.

6. CONCLUSIONS

A solution for the problem of disturbance decoupling using algebraic output feedback has been considered. The necessary and sufficient conditions for the structural problem (without stability) are easily checkable and constructive. For the problem with stability requirement the conditions are not constructive anymore: the solution has to be searched among the output-to-input matrices solving the structural problem. The structural part of the problem may have no solution, or only one solution, in which case we have no freedom on choosing matrix K so that the final system is stable, or more solutions, giving us a chance to look for a matrix K solving the problem with stability.

Clearly, this new approach to the problem of disturbance decoupling has many advantages in the use of algebraic output feedback, giving us a chance to build simple and robust regulators which do not use a state observer.

The problem of disturbance decoupling for non-invertible systems still remains partially open.

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