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ON SOME SUFFICIENT OPTIMALITY CONDITIONS IN MULTIOBJECTIVE DIFFERENTIABLE PROGRAMMING

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Some results on sufficient optimality conditions in multiobjective differentiable programming are estabilished under generalized F-convexity assumptions. Various levels of convexity on the component of the functions involved are imposed, and the equality constraints are not necessarily linear. In the nonlinear case scalarization of the objective function is used.

1. INTRODUCTION

Optimal solutions of multiobjective programming problems were studied by several authors (cf., e.g. [4–7], [9–14]). Following Marusciac [7], Singh [11] established Frith John type optimality criteria in the differentiable case. Necessary conditions are based on the idea of convergence of a vector at point with respect to a set (cf. [5], [7]). Singh [11] invoked a weaker constraint qualification than Marusciac [7] (see Takayama [14] for a basic treatment of multiobjective programming along the lines of classical nonlinear programming, and Stadler [13] for a survey of multiobjective optimization.)

In [7], Marusciac stated two sufficiency criteria (cf. [7], Theorems 3.2, 3.3) for multiobjective differentiable programming involving inequality and equality constraints with various levels of convexity on the component of the function involved, and the linear equality constraints. Singh [11] removed the last restriction and established a new sufficienty criterion (cf. [11], Theorem 3.4) using a scalarization of the objective function. In his paper Tucker's theorem of the alternative (cf. [15]), rather than Motzkin's theorem applied in Marusciac [7], was invoked.

In this note we use a type of generalized convexity (cf. [3], [8]) to obtain a generalization of sufficient optimality conditions for multiobjective differentiable programming given by Lin [5], Marusciac [7] and Singh [11].

In Hanson and Mond [3], and in Mond [8] generalized convexity was defined using sublinear functionals satisfying certain convexity type conditions. It was shown in [3] and [8] that Wolfe's duality holds under assumption that a sublinear functional exists such that the Lagrangian satisfied some generalized convexity conditions.

2. SOME PRELIMINARIES

For convenience, we first recall the following notations. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be vectors in the Euclidean space $\mathbb{R}^n$. By $x \leq y$, we mean $x_i \leq y_i$ for all i; by $x < y$, we mean $x_i \leq y_i$ for all i and $x_j < y_j$ for at least one j $(1 \leq j \leq n)$; and by $x \ll y$, we mean $x_i < y_i$ for all i.

Consider the following vector minimization problem:

$$\text{subject to} \quad \begin{array}{l} \text{minimize } f(x) = (f_1(x), \ldots, f_m(x)) \\ g(x) \leq 0, \ h(x) = 0, \ x \in X \end{array} . \tag{VP}$$

where $X \subseteq \mathbb{R}^n$, $f : X \longrightarrow \mathbb{R}^m$, $g : X \longrightarrow \mathbb{R}^p$, $h : X \longrightarrow \mathbb{R}^q$, $f = (f_1, \ldots, f_m)$, $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_q)$.

We will denote by P the set $P = \{1, 2, \ldots, p\}$ and by X_0 the set $X_0 = \{x \in X \mid g(x) \leq 0, h(x) = 0\}$.

Definition 2.1. We say that $x^0 \in X_0$ is a Pareto minimal point of problem (VP) if and only if there exists no $x \in X_0$ such that $f(x) < f(x^0)$.

Definition 2.2. We say that $x^0 \in X_0$ is a weak Pareto minimal point of (VP) if and only if there exists no $x \in X_0$ such that $f(x) \ll f(x^0)$.

Definition 2.3. A functional $F : X_0 \times X_0 \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is sublinear (in θ) if for any x, $x^0 \in X_0$,

$$F\left(x, x^0;\ \theta_1 + \theta_2\right) \leq F\left(x, x^0; \theta_1\right) + F\left(x, x^0; \theta_2\right), \qquad \text{for all } \theta_1, \theta_2;$$

and

$$F\left(x, x^0; \alpha\,\theta\right) = \alpha\, F\left(x, x^0; \theta\right) \qquad \text{for any } \alpha \in \mathbb{R},\ \alpha \gg 0, \text{ and } \theta.$$

Now we give some examples of sublinear functionals. In the first three examples we consider some sublinear functionals which are independent of the vectorial problem (VP).

Example 1. $F(x, x^0; y) = (x - x^0)^{\mathrm{T}} y$, where the symbol $^{\mathrm{T}}$ denotes the transpose.

Example 2. Let $\eta : X_0 \times X_0 \longrightarrow \mathbb{R}^n$. Define $F(x, x^0; y) = y^{\mathrm{T}} \eta(x, x^0)$.

Example 3. $F(x, x^0; y) = \|x - x^0\| \cdot \|y\|$.

The next sublinear functionals are dependent by (VP).

Example 4. $F(x, x^0; y) = \|y\| \cdot \|f(x) - f(x^0)\|$.

Example 5. $F(x, x^0; y) = \sum_{i=1}^{n} |y_i| \cdot \left| \sum_{i=1}^{m} \left(f_i(x) - f_i(x^0)\right) + \sum_{j=1}^{p} g_j(x) \right|$.

For some discussion about sublinear functionals see Hanson and Mond [3] and Mond [8].

Let us consider a sublinear functional F and the function $\varphi : X_0 \longrightarrow \mathbb{R}$. We suppose that φ is a differentiable function at x^0, an interior point of X.

Definition 2.4. The function φ is said to be F-convex in x^0 if for all $x \in X_0$ we have:

$$\varphi(x) \geq \varphi(x^0) + F\left(x, x^0; \nabla \varphi(x^0)\right).$$

Definition 2.5. The function φ is F-quasiconvex in x^0 if for all $x \in X_0$ such that $\varphi(x) \leq \varphi(x^0)$ we have: $F\left(x, x^0; \nabla \varphi(x^0)\right) \leq 0$.

Definition 2.6. The function φ is F-pseudoconvex in x^0 if for all $x \in X_0$ such that $F\left(x,\, x^0;\, \nabla\, \varphi(x^0)\right) \geq 0$ it results $\varphi(x) \geq \varphi(x^0)$.

From the above definitions it is clear that an F-convex function is both F-quasiconvex and F-pseudoconvex.

Remark 1. In the case of Example 1, Definition 2.4 − 2.6 boil down definition of convexity, quasiconvexity and pseudoconvexity, respectively.

Remark 2. In the case of Example 2, Definitions 2.4− 2.6 reduce to definitions of invexity, quasiinvexity and pseudoinvexity (Hanson [2], Ben-Israel and Mond [1]) respectively.

3. SUFFICIENCY CONDITIONS

Let F be a sublinear functional as in Definition 2.3 and $x^0 \in X_0$ an interior point of X, $J_0 = \left\{ j \mid g_j(x^0) = 0 \right\}$, $g^0 = (g_j)_{j \in J_0}$, $s =$ card J_0. We have:

Theorem 3.1. Suppose that

(i) f, g^0, h are differentiable at x^0;

(ii) $f_i,\ 1 \leq i \leq m,\ g_j,\ j \in J_0,\ h_k,\ 1 \leq k \leq q$, are F-convex at x^0;

(iii) there exists $u^0 \gg 0,\ u^0 \in \mathbb{R}^m,\ v^0 \in \mathbb{R}^s_+,\ w^0 \in \mathbb{R}^q_+$ such that for any $x \in X_0$,

$$F\left(x,\, x^0;\, \sum_{i=1}^{m} u_i^0\, \nabla\, f_i(x^0) + \sum_{j \in J_0} v_j^0\, \nabla\, g_j(x^0) + \sum_{k=1}^{q} w_k^0\, \nabla\, h_k(x^0)\right) \geq 0.$$

Then, x^0 is a (weak) Pareto minimal point of Problem (VP).

P r o o f. Suppose that x^0 is not a (weak) Pareto minimal of Problem (VP). Then, there exists an $\bar{x} \in X_0$ such that

$$f(\bar{x}) - f(x^0) < 0 \tag{1}$$
$$g(\bar{x}) \leq 0 \tag{2}$$
$$h(\bar{x}) = 0. \tag{3}$$

Then, in view of (1)− (3) and $x^0 \in X_0$, we have

$$g_j(\bar{x}) - g_j(x^0) \leq 0, \qquad j \in J_0; \tag{4}$$
$$h(\bar{x}) - h(x^0) = 0. \tag{5}$$

Now, by hypothesis (ii) and (1), (4), (5), we have

$$F\left(\bar{x},\, x^0;\, \nabla\, f_i(x^0)\right) \leq 0, \qquad \text{for any } i, \tag{6}$$

and there exists at least one index i such that we have a strict inequality;

$$F\left(\bar{x},\, x^0;\, \nabla\, g_j(x^0)\right) \leq 0, \qquad \text{for any } j \in J_0, \tag{7}$$
$$F\left(\bar{x},\, x^0;\, \nabla\, h_k(x^0)\right) \leq 0, \qquad \text{for any } k. \tag{8}$$

Then, for any $u \gg 0,\ u \in \mathbb{R}^m,\ v \geq 0,\ v \in \mathbb{R}^s,\ w \geq 0,\ w \in \mathbb{R}^q$, from (6) − (8) and the sublinearity of F, we obtain

$$F\left(\bar{x},\, x^0;\, \sum_{i=1}^{m} u_i\, \nabla\, f_i(x^0) + \sum_{j \in J_0} v_j\, \nabla\, g_j(x^0) + \sum_{k=1}^{q} w_k\, \nabla\, h_k(x^0)\right) \ll 0$$

that contradicts (iii). Hence, x^0 is a Pareto (weak) minimal point of Problem (VP), and the proof is complete. □

Theorem 3.2. Assume that

(i) $f,\, g^0,\, h$ are differentiable at x^0,

(ii) there exist $u^0 \gg 0,\; u^0 \in \mathbb{R}^m,\; v^0 \in \mathbb{R}^s_+,\; w^0 \in \mathbb{R}^q$ such that

a) $\omega(x) := \sum_{i=1}^{m} u_i^0 f_i(x) + \sum_{j\in J_0} v_j^0 g_j(x) + \sum_{k=1}^{q} w_k^0 h_k(x)$ is F-pseudoconvex at x^0;

b) $F\left(x,\, x^0;\; \sum_{i=1}^{m} u_i^0 \nabla f_i(x^0) + \sum_{j\in J_0} v_j^0 \nabla g_j(x^0) + \sum_{k=1}^{q} w_k^0 \nabla h_k(x^0)\right) \geq 0$

for any $x \in X_0$.

Then, x^0 is a Pareto (weak) minimal point of Problem (VP).

Proof. Assume that x^0 is not a Pareto (weak) minimal point of Problem (VP). Then, there exists $\bar{x} \in X_0$ such that

$$f(\bar{x}) - f(x^0) < 0. \tag{9}$$

But, $x^0,\, \bar{x} \in X_0$, and then

$$g^0(\bar{x}) - g^0(x^0) \leq 0, \tag{10}$$
$$h(\bar{x}) - h(x^0) = 0. \tag{11}$$

Because $u^0 \gg 0,\; v^0 \geq 0$, from (9)$-$(11) we obtain: $\omega(\bar{x}) \ll \omega(x^0)$ where ω is defined by (ii-a). Since ω is F-pseudoconvex at x^0, we have

$$F\left(\bar{x},\, x^0;\; \sum_{i=1}^{m} u_i^0 \nabla f_i(x^0) + \sum_{j\in J_0} v_j^0 \nabla g_j(x^0) + \sum_{k=1}^{q} w_k^0 \nabla h_k(x^0)\right) \ll 0. \tag{12}$$

But condition (12) violates hypothesis (ii-b). Thus, this contradiction leads to the conclusion that x^0 is a Pareto minimal point for Problem (VP). □

Theorem 3.3. We assume that

(i) $f,\, g,\, h$ are differentiable at x^0,

(ii) there exist $u^0 \gg 0,\; u^0 \in \mathbb{R}^m,\; v^0 \in \mathbb{R}^p_+,\; w^0 \in \mathbb{R}^q$, such that

a) $\sum_{i=1}^{m} u_i^0 f_i$ is F-pseudoconvex at x^0;

b) $\sum_{j\in J:0} v_j^0 g_j$ is F-quasiconvex at x^0;

c) $\sum_{k=1}^{q} w_k^0 h_k$ is F-quasiconvex at x^0;

d) $F\left(x,\, x^0;\; \sum_{i=1}^{m} u_i^0 \nabla f_i(x^0) + \sum_{j\in J_0} v_j^0 \nabla g_j(x^0) + \sum_{k=1}^{q} w_k^0 \nabla h_k(x^0)\right) \geq 0$
for all $x \in X_0$;

e) $\sum_{j=1}^{p} v_j^0 g_j(x^0) = 0$.

Then, x^0 is a Pareto minimal point of problem (VP).

P r o o f. Because,

$$\sum_{j=1}^{p} v_j^0 \, g_j(x^0) = 0, \qquad g(x^0) \leq 0, \qquad v^0 \geq 0,$$

it follows that: $v_j^0 \, g_j(x^0) = 0$, for all j. Hence, $\sum_{j \in J_0} v_j^0 \, g_j(x^0) = 0$. Also, for any $x \in X_0$, $\sum_{j \in J_0} v_j^0 \, g_j(x) \leq 0$. Therefore

$$\sum_{j \in J_0} v_j^0 \, g_j(x) \leq 0 = \sum_{j \in J_0} v_j^0 \, g_j(x^0). \tag{13}$$

By (13) and hypothesis (ii-b), we have

$$F\left(x,\, x^0;\, \sum_{j \in J_0} v_j^0 \, \nabla g_j(x^0)\right) \leq 0, \qquad \text{for any } x \in X_0. \tag{14}$$

From $v_j^0 \, g_j(x^0) = 0$, for all $j \in J_0$, it results that $v_j^0 = 0$ for all $j \in P \setminus J_0$. Hence,

$$\sum_{j \notin J_0} v_j^0 \, \nabla g_j(x^0) = 0. \tag{15}$$

From (15) and the fact $F(x,\, x^0;\, 0) = 0$ for all $x \in X_0$,

$$F\left(x,\, x^0;\, \sum_{j \notin J_0} v_j^0 \, \nabla g_j(x^0)\right) = 0. \tag{16}$$

Combining (14) and (16) with sublinearity of F we obtain

$$F\left(x,\, x^0;\, \sum_{j=1}^{p} v_j^0 \, \nabla g_j(x^0)\right) \leq 0, \qquad \text{for all } x \in X_0. \tag{17}$$

Similarly, since

$$w_k^0 \, h_k(x^0) = 0 = w_k^0 \, h_k(x), \qquad x \in X_0,\ 1 \leq k \leq q,$$

it follows

$$\sum_{k=1}^{q} w_k^0 \, h_k(x^0) = \sum_{k=1}^{q} w_k^0 \, h_k(x), \qquad x \in X_0. \tag{18}$$

By (18) and hypothesis (ii-c), we have

$$F\left(x,\, x^0;\, \sum_{k=1}^{q} w_k^0 \, \nabla h_k(x^0)\right) \leq 0, \qquad \text{for any } x \in X_0. \tag{19}$$

Because F is a sublinear functional, from (17) and (19) we have

$$F\left(x,\, x^0;\, \sum_{j=1}^{p} v_j^0 \, \nabla g_j(x^0) + \sum_{k=1}^{q} w_k^0 \, \nabla h_k(x^0)\right) \leq 0, \qquad x \in X_0. \tag{20}$$

By using (20), (ii-d) and sublinearity of F we obtain

$$F\left(x,\, x^0;\, \sum_{i=1}^{m} u_i^0 \, \nabla f_i(x^0)\right) \geq 0, \qquad x \in X_0.$$

Therefore, by hypothesis (ii-a), it follows

$$\sum_{i=1}^{m} u_i^0 f_i(x) \geq \sum_{i=1}^{m} u_i^0 f_i(x^0), \qquad \text{for all } x \in X_0. \tag{21}$$

Hence, by (21) and Theorem 6.1 in [2], x^0 is a Pareto minimal point of Problem (VP). The proof is now complete. □

Now we consider some remarks.

Remark 3. If $F\left(x, x^0; y\right) = (x - x^0)^{\mathrm{T}} y$, then we obtain Theorems 3.3, 3.4 from Singh [11]. Assumption (ii-b) by Theorem 3.2 reduces to assumption (ii) (a) from Singh [11, Thm. 3.3] and assumption (ii-d) by Theorem 3.3 reduces to (ii) (d) from Singh [11, Thm. 3.4]. When assumption (2.6) (a Kuhn–Tucker condition) from Marusiac [7, Thms 3.1, 3.2] holds, then these assumptions are valid. Also, in this case our generalized F-convexity conditions become the convexity, quasiconvexity or pseudoconvexity conditions.

Remark 4. Generally, scalarization means the replacement of a vector optimization problem by a suitable scalar optimization problem which is an optimization problem with a real valued objective functional. Since the scalar optimization theory is widely developed, scalarization turns out to be of great importance for the vector optimization theory. Solutions of multiobjective optimization problems can be characterized and computed as solutions of appropriate scalar optimization problems. For scalarization in multiobjective programming see, for example Jahn [4], Luc [6], Singh [10] and Pascoletti and Serafini [9]. Our scalarization given in the last two theorems is on the lines of Singh [11] and Lin [5].

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