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COUNTABLE EXTENSION OF TRIANGULAR NORMS AND THEIR APPLICATIONS TO THE FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

Olga Hadžić , Endre Pap and Mirko Budinčević

In this paper a fixed point theorem for a probabilistic q-contraction $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfies a grow condition, and T is a g-convergent t-norm (not necessarily $T \geq T_{\mathbf{L}}$) is proved. There is proved also a second fixed point theorem for mappings $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfy a weaker condition than in [13], and T belongs to some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms. An application to random operator equations is obtained.

1. INTRODUCTION

The origin of triangular norms was in the theory of probabilistic metric spaces, in the work K. Menger [9], see [4, 7, 14]. It turns out that t-norms and related t-conorms are crucial operations in several fields, e.g., in fuzzy sets, fuzzy logics (see [7]) and their applications, but also, among other fields, in the theory of generalized measures [7, 11, 17] and in nonlinear differential and difference equations [11].

We present in this paper some results on t-norms which are closely related to the fixed point theory in probabilistic metric spaces, see [4]. The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [15] for mappings $f : S \to S$, where $(S, \mathcal{F}, T_{\mathbf{M}})$ is a Menger space, where $T_{\mathbf{M}} = \min$. Further development of the fixed point theory in a more general Menger space $(S, \mathcal{F}, T)$ was connected with investigations of the structure of the t-norm T. Very soon the problem was in some sense completely solved. Namely, if we restrict ourselves to complete Menger spaces $(S, \mathcal{F}, T)$, where T is a continuous t-norm, then any probabilistic q-contraction $f : S \to S$ has a fixed point if and only if the t-norm T is of H-type, see [4].

We investigate in this paper the countable extension of t-norms and we introduce a new notion: the geometrically convergent (briefly g-convergent) t-norm, which is closely related to the fixed point property. We prove that t-norms of H-type and some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms are geometrically convergent. We prove also some practical criterions for the geometrically convergent t-norms.

A new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff's paper [16], where some additional growth conditions for the mapping $\mathcal{F} : S \times S \to \mathcal{D}^+$ are assumed, and $T \geq T_{\mathbf{L}}$. V. Radu [13] introduced a stronger growth condition for $\mathcal{F}$ than in Tardiff's paper (under the condition $T \geq T_{\mathbf{L}}$), which enables him to define a metric. By metric approach an estimation of the convergence with respect to the solution is obtained, see [4].

We prove in this paper a fixed point theorem for a probabilistic q-contraction $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfies Radu's condition, and T is a g-convergent t-norm (not necessarily $T \geq T_{\mathbf{L}}$). We prove a second fixed point theorem for mappings $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfy a weaker condition than in [13], and T belongs to some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms. An application to random operator equations is obtained.

Notions and notations can be found in [4, 7, 11, 14].

2. TRIANGULAR NORMS

A triangular norm (t-norm for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1]^2 \to [0, 1]$ which is commutative, associative, monotone and $T(x, 1) = x$. t-conorm $\mathbf{S}$ is defined by $\mathbf{S}(x, y) = 1 - T(1 - x, 1 - y)$.

If T is a t-norm, $x \in [0, 1]$ and $n \in \mathbb{N} \cup \{0\}$ then we shall write

$$x_T^{(n)} = \begin{cases} 1 & \text{if } n = 0, \\ T\left(x_T^{(n-1)}, x\right) & \text{otherwise.} \end{cases}$$

Definition 1. A t-norm T is of H-type if the family $(x_T^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$.

A trivial example of a t-norm of H-type is $T_{\mathbf{M}}$. There is a nontrivial example of a t-norm T such that $(x_T^{(n)})_{n \in \mathbb{N}}$ is an equicontinuous family at the point $x = 1$.

Example 2. Let $\bar{T}$ be a continuous t-norm and let for every $m \in \mathbb{N} \cup \{0\}$:

$$I_m = [1 - 2^{-m}, 1 - 2^{-m-1}].$$

If

$$T(x, y) = 1 - 2^{-m} + 2^{-m-1}\bar{T}(2^{m+1}(x - 1 + 2^{-m}),\ 2^{m+1}(y - 1 + 2^{-m}))$$

for $(x, y) \in I_m \times I_m$ and $T(x, y) = \min(x, y)$ for $(x, y) \notin \bigcup_{m \in \mathbb{N} \cup \{0\}} I_m \times I_m$ then the family $(x_T^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$, i.e., T is a t-norm of H-type.

Proposition 3. ([4]) If a continuous t-norm T is Archimedean than it can not be a t-norm of H-type.

A method of construction a new t-norm from a system of given t-norms is given in the following theorem, see [4, 7].

Theorem 3. Let $(T_k)_{k\in K}$ be a family of t-norms and let $((\alpha_k,\beta_k))_{k\in K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0,1]$ (i.e., K is an at most countable index set). Consider the linear transformations $\varphi_k : [\alpha_k,\beta_k] \to [0,1], k \in K$ given by

$$\varphi_k(u) = \frac{u-\alpha_k}{\beta_k-\alpha_k}.$$

Then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \varphi_k^{-1}(T_k(\varphi_k(x),\varphi_k(y))) & \text{if } (x,y) \in (\alpha_k,\beta_k)^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm, which is called the ordinal sum of $(T_k)_{k\in K}$ and will be denoted by $T = (< (\alpha_k,\beta_k), T_k >)_{k\in K}$.

The following proposition was proved in [12].

Proposition 5. A continuous t-norm T is of H-type if and only if $T = (< (\alpha_k,\beta_k), T_k >)_{k\in K}$ and $\sup \beta_k < 1$ or $\sup \alpha_k = 1$.

Remark 6. If $T = (< (\alpha_k,\beta_k), T_k >)_{k\in K}$ and $\sup \beta_k < 1$ or $\sup \alpha_k = 1$, then T is of H-type for any summands T_k (not only for continuous and Archimedean summands T_k, $k \in K$, see [12]). Hence , if

$$T = \big(< (1-2^{-k}, 1-2^{-k-1}),\ \bar{T} > \big)_{k\in\mathbb{N}\cup\{0\}}$$

we have $\sup \alpha_k = \sup(1-2^{-k}) = 1$ (cf. Example 2).

For an arbitrary t-norm of H-type we have by [4] the following characterization.

Theorem 7. Let T be a t-norm. Then (i) and (ii) hold, where:

(i) Suppose that there exists a strictly increasing sequence $(b_n)_{n\in\mathbb{N}}$ from the interval $[0,1)$ such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n,b_n) = b_n$. Then T is of H-type.

(ii) If T is continuous and of H-type, then there exists a sequence $(b_n)_{n\in\mathbb{N}}$ as in (i).

From the proof of the above theorem it follows that the condition of continuity of whole sequence $(x_T^{(n)})_{n\in\mathbb{N}}$ can be replaced by the condition that the function $\delta_T(x) = T(x,x)\,(x \in [0,1])$ is right-continuous on an interval $[b,1)$ for $b < 1$.

Theorem 8. Let T be a t-norm such that the function $\delta_T(x) = T(x,x)\,(x \in [0,1])$ is right-continuous on an interval $[b,1)$ for $b < 1$. Then T is a t-norm of H-type if and only if there exists a sequence $(b_n)_{n\in\mathbb{N}}$ from the interval $(0,1)$ of idempotents of T such that $\lim\limits_{n\to\infty} b_n = 1$.

In particular, for continuous t-norms the following characterization holds, [4].

Theorem 9. Let T be a continuous t-norm. Then the following are equivalent:

a) T is not of H-type.

b) There exist $a_T \in [0,1)$ and a continuous strictly increasing and surjective mapping $\varphi_{a_T} : [a_T, 1] \to [0,1]$ such that

$$T(x,y) = \varphi_{a_T}^{-1}(\varphi_{a_T}(x) \star \varphi_{a_T}(y)), \text{ for every } x, y \geq a_T,$$

where the operation $\star$ is either $T_{\mathbf{P}}$ or $T_{\mathbf{L}}$, where $T_{\mathbf{P}}(x,y) = xy$ and $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$.

3. COUNTABLE EXTENSION OF t-NORMS

An arbitrary t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking for $(x_1,\dots,x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the values $T(x_1,\dots,x_n)$ which is defined by

$$\mathop{\mathsf{T}}_{i=1}^{0} x_i = 1, \quad \mathop{\mathsf{T}}_{i=1}^{n} x_i = T\left(\mathop{\mathsf{T}}_{i=1}^{n-1} x_i, x_n\right) = T(x_1,\dots,x_n).$$

Specially, we have $T_{\mathbf{L}}(x_1,\dots,x_n) = \max\left(\sum\limits_{i=1}^{n} x_i - (n-1), 0\right)$ and $T_{\mathbf{M}}(x_1,\dots,x_n) = \min(x_1,\dots,x_n)$.

We can extend T to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,1]$ the values

$$\mathop{\mathsf{T}}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=1}^{n} x_i. \tag{1}$$

The limit on the right side of (1) exists since the sequence $(\mathop{\mathsf{T}}\limits_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Remark 10. An alternative approach to the infinitary extension of t-norms can be found in [10].

In the fixed point theory it is of interest to investigate the classes of t-norms T and sequences $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$, and

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=1}^{\infty} x_{n+i} = 1. \tag{2}$$

In the classical case $T = T_{\mathbf{P}}$ we have $(T_{\mathbf{P}})_{i=1}^n = \prod_{i=1}^n x_i$ and for every sequence $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,1]$ with $\sum_{i=1}^\infty (1-x_n) < \infty$ it follows that

$$\lim_{n\to\infty} (T_{\mathbf{P}})_{i=n}^\infty = \lim_{n\to\infty} \prod_{i=n}^\infty x_i = 1.$$

Namely, it is well known that

$$\prod_{i=1}^\infty x_i > 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \prod_{i=n}^\infty x_i = 1 \quad \Leftrightarrow \quad \sum_{i=1}^\infty (1-x_i) < \infty.$$

The equivalence

$$\sum_{i=1}^\infty (1-x_i) < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty} \mathop{\top}_{i=n}^\infty x_i = 1 \tag{3}$$

holds also for $T \geq T_{\mathbf{L}}$. Indeed

$$(T_{\mathbf{L}})_{i=1}^n x_i = \max\left(\sum_{i=1}^n x_i - (n-1), 0\right) = \max\left(\sum_{i=1}^n (x_i - 1) + 1, 0\right),$$

and therefore $\sum_{n=1}^\infty (1-x_n) < \infty$ holds if and only if

$$\lim_{n\to\infty} (T_{\mathbf{L}})_{i=n}^\infty x_i = \max\left(\lim_{n\to\infty} \sum_{i=n}^\infty (x_i - 1) + 1, 0\right) = 1.$$

For $T \geq T_{\mathbf{L}}$ we have $\mathop{\top}_{i=1}^n x_i \geq (T_{\mathbf{L}})_{i=1}^n x_i$ and therefore for such a t-norm T the implication

$$\sum_{i=1}^\infty (1-x_i) < \infty \quad \Rightarrow \quad \lim_{n\to\infty} \mathop{\top}_{i=n}^\infty x_i = 1$$

holds.

We shall need some families of t-norms given in the following example.

Example 11. (i) *The Dombi family* of t-norms $(T_\lambda^{\mathbf{D}})_{\lambda\in[0,\infty]}$ is defined by

$$T_\lambda^{\mathbf{D}}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda = 0, \\ T_{\mathbf{M}}(x,y) & \text{if } \lambda = \infty, \\ \left(1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}\right)^{-1} & \text{if } \lambda \in (0,\infty). \end{cases}$$

(ii) *The Schweizer–Sklar family* of t-norms $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$ is defined by

$$T_\lambda^{\mathbf{SS}}(x,y)=\begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda=-\infty,\\ (x^\lambda+y^\lambda-1)^{1/\lambda} & \text{if } \lambda\in(-\infty,0),\\ T_{\mathbf{P}}(x,y) & \text{if } \lambda=0,\\ (\max(x^\lambda+y^\lambda-1,0))^{1/\lambda} & \text{if } \lambda\in(0,\infty),\\ T_{\mathbf{D}}(x,y) & \text{if } \lambda=\infty. \end{cases}$$

(iii) *The Aczél–Alsina family* of t-norms $(T_\lambda^{\mathbf{AA}})_{\lambda\in[0,\infty]}$ is defined by

$$T_\lambda^{\mathbf{AA}}(x,y)=\begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda=0,\\ T_{\mathbf{M}}(x,y) & \text{if } \lambda=\infty,\\ e^{-(|\log x|^\lambda+|\log y|^\lambda)^{1/\lambda}} & \text{if } \lambda\in(0,\infty). \end{cases}$$

(iv) The family $(T_\lambda^{\mathbf{SW}})_{\lambda\in[-1,+\infty]}$ of *Sugeno–Weber t-norms* is given by

$$T_\lambda^{\mathbf{SW}}(x,y)=\begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda=-1,\\ T_{\mathbf{P}}(x,y) & \text{if } \lambda=\infty,\\ \max\left(0,\dfrac{x+y-1+\lambda xy}{1+\lambda}\right) & \text{otherwise}. \end{cases}$$

The condition $T\geq T_{\mathbf{L}}$ is fulfilled by the families: 1. $T_\lambda^{\mathbf{SS}}$ for $\lambda\in[-\infty,1]$; 2. $T_\lambda^{\mathbf{SW}}$ for $\lambda\in[0,\infty]$.

On the other side there exists a member of the family $(T_\lambda^{\mathbf{D}})_{\lambda\in(0,\infty)}$ which is incomparable with $T_{\mathbf{L}}$, and there exists a member of the family $(T_\lambda^{\mathbf{AA}})_{\lambda\in(0,\infty)}$ which is incomparable with $T_{\mathbf{L}}$.

We shall give some sufficient conditions for (2).

Proposition 12. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim\limits_{n\to\infty} x_n=1$ and t-norm T is of H-type. Then (2) holds.

P r o o f. Since t-norm T is of H-type for every $\lambda\in(0,1)$ there exists $\delta(\lambda)\in(0,1)$ such that

$$x\geq\delta(\lambda)\quad\Rightarrow\quad \mathop{\mathsf{T}}_{i=1}^{p} x>1-\lambda$$

for every $p\in\mathbb{N}$. Since $\lim\limits_{n\to\infty} x_n=1$ there exists $n_0(\lambda)\in\mathbb{N}$ such that $x_n\geq\delta(\lambda)$ for every $n\geq n_0(\lambda)$. Hence

$$\begin{aligned}\mathop{\mathsf{T}}_{i=1}^{p} x_{n+i} &\geq \mathop{\mathsf{T}}_{i=1}^{p}\delta(\lambda)\\ &> 1-\lambda,\end{aligned}$$

for every $n \geq n_0(\lambda)$ and every $p \in \mathbb{N}$. This means that (2) holds. □

Remark 13. If T is a t-norm such that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ from the interval $(0,1)$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} x_i = 1$, then T is continuous at the point $(1,1)$. Indeed, let $\lambda \in (0,1)$ be given. Then there exists $n_0(\lambda) \in \mathbb{N}$ such that

$$\mathop{\mathsf{T}}_{i=n_0(\lambda)}^{\infty} x_i > 1 - \lambda.$$

Since $T(x_{n_0(\lambda)}, x_{n_0(\lambda)+1}) \geq \mathop{\mathsf{T}}_{i=n_0(\lambda)}^{\infty} x_i > 1-\lambda$ we obtain that $x, y \geq \max(x_{n_0(\lambda)}, x_{n_0(\lambda)+1})$ implies $T(x,y) > 1 - \lambda$.

For some families of t-norms we shall characterize the sequences $(x_n)_{n\in\mathbb{N}}$ from $(0,1]$, which tend to 1 and for which (2) holds.

Lemma 14. Let T be a strict t-norm with an additive generator $\mathbf{t}$, and the corresponding multiplicative generator θ. Then we have

$$\mathop{\mathsf{T}}_{i=1}^{\infty} x_i = \mathbf{t}^{-1}\left(\sum_{i=1}^{\infty} \mathbf{t}(x_i)\right)$$

or

$$\mathop{\mathsf{T}}_{i=1}^{\infty} x_i = \theta^{-1}\left(\prod_{i=1}^{\infty} \theta(x_i)\right).$$

The preceding lemma and the continuity of the generators of strict t-norms imply the following proposition.

Proposition 15. Let T be a strict t-norm with an additive generator $\mathbf{t}$, and the corresponding multiplicative generator θ. For a sequence $(x_n)_{n\in\mathbb{N}}$ from the interval $(0,1)$ such that $\lim_{n\to\infty} x_n = 1$ the condition

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \mathbf{t}(x_i) = 0,$$

or the condition

$$\lim_{n\to\infty} \prod_{i=n}^{\infty} \theta(x_i) = 1,$$

holds if and only if (2) is satisfied.

Example 16. Let $(T_\lambda^{\mathbf{D}})_{\lambda\in(0,\infty)}$ be the Dombi family of t-norms and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from $(0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$. Then we have the following equivalence:

$$\sum_{i=1}^{\infty}\left(\frac{1-x_i}{x_i}\right)^{\lambda} < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty}(T_\lambda^{\mathbf{D}})_{i=n}^{\infty} x_i = 1.$$

For a t-norm $T_\lambda^{\mathbf{D}}, \lambda\in(0,\infty)$, the multiplicative generator $\theta_\lambda^{\mathbf{D}}$ is given by

$$\theta_\lambda^{\mathbf{D}}(x) = e^{-(\frac{1-x}{x})^{\lambda}}$$

and therefore with the property $\theta_\lambda^{\mathbf{D}}(1) = 1$. Hence

$$\begin{aligned}\prod_{i=n}^{\infty}\theta_\lambda^{\mathbf{D}}(x_i) &= \prod_{i=n}^{\infty} e^{-(\frac{1-x_i}{x_i})^{\lambda}}\\ &= e^{-\sum_{i=n}^{\infty}(\frac{1-x_i}{x_i})^{\lambda}},\end{aligned}$$

and therefore the above equivalence follows by Proposition 15. Since $\lim\limits_{n\to\infty} x_n = 1$, we have that

$$\left(\frac{1-x_n}{x_n}\right)^{\lambda} \sim (1-x_n)^{\lambda} \text{ as } n\to\infty.$$

Hence

$$\sum_{n=1}^{\infty}(1-x_n)^{\lambda} < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}\left(\frac{1-x_n}{x_n}\right)^{\lambda} < \infty,$$

which implies the equivalence

$$\sum_{n=1}^{\infty}(1-x_n)^{\lambda} < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty}(T_\lambda^{\mathbf{D}})_{i=n}^{\infty} x_i = 1.$$

Example 17. Let $(T_\lambda^{\mathbf{AA}})_{\lambda\in(0,\infty)}$ be the Aczél–Alsina family of t-norms given by

$$T_\lambda^{\mathbf{AA}}(x,y) = e^{-(|\log x|^{\lambda}+|\log y|^{\lambda})^{1/\lambda}}$$

and $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from $(0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$. Then we have the following equivalence

$$\sum_{i=1}^{\infty}(1-x_i)^{\lambda} < \infty \quad \Leftrightarrow \quad \lim_{n\to\infty}\left(T_\lambda^{\mathbf{AA}}\right)_{i=n}^{\infty} x_i = 1.$$

For a t-norm $T_\lambda^{\mathbf{AA}}, \lambda\in(0,\infty)$, the multiplicative generator $\theta_\lambda^{\mathbf{AA}}$ is given by

$$\theta_\lambda^{\mathbf{AA}}(x) = e^{-(-\log x)^{\lambda}}$$

and therefore with the property $\theta_\lambda^{\mathbf{AA}}(1)=1$. Hence

$$\begin{aligned}\prod_{i=n}^{\infty}\theta_\lambda^{\mathbf{AA}}(x_i) &= \prod_{i=n}^{\infty} e^{-(-\log x_i)^\lambda}\\ &= e^{-\sum_{i=n}^{\infty}(-\log x_i)^\lambda}.\end{aligned}$$

Since $\lim\limits_{i\to\infty} x_i = 1$ and $\log x_i \sim x_i - 1$ as $i\to\infty$ by Proposition 15. the above equivalence follows.

For t-norms $T_\lambda^{\mathbf{SW}}, \lambda\in(-1,\infty]$ we have the following proposition.

Proposition 18. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence from $(0,1)$ such that the series $\sum\limits_{n=1}^{\infty}(1-x_n)$ is convergent. Then for every $\lambda\in(-1,\infty]$

$$\lim_{n\to\infty}(T_\lambda^{\mathbf{SW}})_{i=n}^{\infty}x_i = 1.$$

P r o o f. An additive generator of $T_\lambda^{\mathbf{SW}}$ for $\lambda\in(-1,0)$ is given by

$$\mathbf{t}_\lambda^{\mathbf{SW}}(x) = -\log\left(\frac{1+\lambda x}{1+\lambda}\right)\cdot\frac{1}{\log(1+\lambda)}.$$

We shall prove that for some $n_1\in\mathbb{N}$ and every $p\in\mathbb{N}$

$$\prod_{i=1}^{p}\theta_\lambda^{\mathbf{SW}}(x_{n+i-1}) = \exp\left(\sum_{i=1}^{p}\log\left(\frac{1+\lambda x_{n+i-1}}{1+\lambda}\right)\cdot\frac{1}{\log(1+\lambda)}\right) > e^{-1} \tag{4}$$

for every $n\ge n_1$ since in this case

$$(T_\lambda^{\mathbf{SW}})_{i=1}^{p}x_{n+i-1} = (\theta_\lambda^{\mathbf{SW}})^{-1}\left(\prod_{i=1}^{p}\theta_\lambda^{\mathbf{SW}}(x_{n+i-1})\right). \tag{5}$$

We have to prove that for some $n_1\in\mathbb{N}$ and every $p\in\mathbb{N}$

$$-\frac{1}{\log(1+\lambda)}\sum_{i=0}^{p}\log\left(\frac{1+\lambda x_{n+i-1}}{1+\lambda}\right) < 1 \text{ for every } n>n_1, \tag{6}$$

since (6) implies (4). From $\lim\limits_{n\to\infty}(1-x_n)=0$ it follows that

$$\log\left(1+\frac{\lambda}{1+\lambda}(x_n-1)\right)\sim\frac{\lambda}{1+\lambda}(x_n-1)$$

and therefore the series

$$-\frac{1}{\log(1+\lambda)}\sum_{n=1}^{\infty}\log\left(1+\frac{\lambda}{1+\lambda}(x_n-1)\right)$$

is convergent. Hence it follows that there exists $n_1 \in \mathbb{N}$ such that (4) holds for every $n \geq n_1$ and every $p \in \mathbb{N}$, and this implies (5).

The above proposition holds also for $\lambda \geq 0$ since in this case $T_\lambda^{\mathbf{SW}} \geq T_{\mathbf{L}}$. □

It is of special interest for the fixed point theory in probabilistic metric spaces to investigate condition (2) for a special sequence $(1-q^n)_{n\in\mathbb{N}}$ for $q \in (0,1)$.

Proposition 19. If for a t-norm T there exists $q_0 \in (0,1)$ such that

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q_0^i) = 1, \tag{7}$$

then

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q^i) = 1,$$

for every $q \in (0,1)$.

P r o o f. If $q < q_0$ then $1-q^n > 1-q_0^n$ for every $n \in \mathbb{N}$ and therefore (7) implies

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q^i) \geq \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q_0^i) = 1.$$

Now suppose that $q > q_0$. First, we consider the special case when $q^2 = q_0$, i.e., $\sqrt{q_0} = q > q_0$. Then

$$\begin{aligned} \mathop{\mathsf{T}}_{i=2m}^{\infty} (1-q^i) &\geq T\left(\mathop{\mathsf{T}}_{i=m}^{\infty} (1-q^{2i}), \mathop{\mathsf{T}}_{i=m}^{\infty} (1-q^{2i+1})\right) \\ &\geq T\left(\mathop{\mathsf{T}}_{i=m}^{\infty} (1-q_0^i), \mathop{\mathsf{T}}_{i=m}^{\infty} (1-q_0^i)\right) \end{aligned}$$

and since T by Remark 13 is continuous at $(1,1)$ it follows that

$$\lim_{m\to\infty} \mathop{\mathsf{T}}_{i=2m}^{\infty} (1-q^i) \geq T(1,1) = 1.$$

Therefore

$$\lim_{m\to\infty} \mathop{\mathsf{T}}_{i=2m+1}^{\infty} (1-q^i) \geq \lim_{m\to\infty} \mathop{\mathsf{T}}_{i=2m}^{\infty} (1-q^i) = 1.$$

Now we consider an arbitrary $q > q_0$ from the interval $(0,1)$. Since for $q > q_0$ there exists $m \in \mathbb{N}$ such that $q_0^{2^{-m}} > q$ we reduce this situation on the case of the m-iterations of the preceding procedure. □

Definition 20. We say that a t-norm T is geometrically convergent (briefly g-convergent, in [4] called q-convergent for some $q \in (0,1)$) if

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q^i) = 1.$$

for every $q \in (0,1)$.

Since $\lim\limits_{n\to\infty}(1-q^n) = 1$ and $\sum\limits_{n=1}^{\infty}(1-(1-q^n))^s < \infty$ for every $s > 0$ it follows that all t-norms from the family

$$\bigcup_{\lambda\in(0,\infty)} \{T_\lambda^{\mathbf{D}}\} \bigcup \bigcup_{\lambda\in(0,\infty)} \{T_\lambda^{\mathbf{AA}}\} \bigcup \mathcal{T}^H \bigcup_{\lambda\in(-1,\infty]} \{T_\lambda^{\mathbf{SW}}\}$$

are g-convergent, where $\mathcal{T}^H$ is the class of all t-norms of H-type.

The following example shows that not every strict t-norm is g-convergent.

Example 21. Let T be the strict t-norm with an additive generator $\mathbf{t}(x) = -\frac{1}{\log(1-x)}$. In this case the series $\sum\limits_{i=1}^{\infty} \mathbf{t}(1-q^i)$ for any $q \in (0,1)$ is not convergent since

$$\sum_{i=1}^{\infty} \mathbf{t}(1-q^i) = -\sum_{i=1}^{\infty} \frac{1}{\log(q^i)} = -\sum_{i=1}^{\infty} \frac{1}{i \log q}.$$

In the following two propositions we shall give sufficient conditions for a t-norm T to be g-convergent.

Proposition 22. Let T and T_1 be strict t-norms and $\mathbf{t}$ and $\mathbf{t}_1$ their additive generators, respectively, and there exists $b \in (0,1)$ such that $\mathbf{t}(x) \leq \mathbf{t}_1(x)$ for every $x \in (b,1]$. If T_1 is g-convergent, then T is g-convergent.

P r o o f. Since T_1 is g-convergent we have $\lim\limits_{n\to\infty}(T_1)_{i=n}^{\infty}(1-q^i) = 1$. Therefore

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \mathbf{t}_1(1-q^i) = 0. \tag{8}$$

Since there exists $n_0 \in \mathbb{N}$ such that $1 - q^{n_0} \in (b,1]$ we have by the condition of the proposition that

$$\mathbf{t}(1-q^n) \leq \mathbf{t}_1(1-q^n) \text{ for every } n \geq n_0.$$

Therefore, by (8) $\lim\limits_{n\to\infty} \sum\limits_{i=n}^{\infty} \mathbf{t}(1-q^i) = 0$, i.e., T is g-convergent. □

Proposition 23. Let T be a strict t-norm with a generator $\mathbf{t}$ which has a bounded derivative on an interval $(b, 1)$ for some $b \in (0, 1)$. Then T is g-convergent.

Proof. By the Lagrange mean value theorem we have for every $x \in (b, 1)$ that

$$\mathbf{t}(x) - \mathbf{t}(1) = \mathbf{t}(x) = \mathbf{t}'(\xi)(x - 1)$$

for some $\xi \in (x, 1)$, and therefore

$$\sum_{i=i_0}^{\infty} \mathbf{t}(1 - q^i) \leq M \sum_{i=i_0}^{\infty} q^i,$$

where $M = \sup_{x \in (b,1)} |\mathbf{t}'(x)|$, and $1 - q^{i_0} \in (b, 1)$. □

Proposition 24. Let T be a t-norm and $\psi : (0, 1] \to [0, \infty)$. If for some $\delta \in (0, 1)$ and every $x \in [0, 1],\ y \in [1 - \delta, 1]$

$$|T(x, y) - T(x, 1)| \leq \psi(y) \tag{9}$$

then for every sequence $(x_n)_{n \in \mathbb{N}}$ from the interval $[0, 1]$ such that $\lim\limits_{n \to \infty} x_n = 1$ and $\sum\limits_{n=1}^{\infty} \psi(x_n) < \infty$, relation (2) holds.

For the proof see [4].

Corollary 25. Let T and ψ be as in Proposition 25. If for some $q \in (0, 1)$,

$$\sum_{n=1}^{\infty} \psi(1 - q^n) < \infty$$

then T is g-convergent.

Proof. Since $\lim\limits_{n \to \infty} (1 - q^n) = 1$ by Proposition 25 we obtain that
$\lim\limits_{n \to \infty} \mathop{\mathsf{T}}\limits_{i=n}^{\infty} (1 - q^n) = 1.$ □

Example 26. Let $\alpha > 0,\ p > 1$ and $z_{\alpha,p} : (0, 1] \times [0, 1] \to [0, \infty)$ be defined in the following way:

$$z_{\alpha,p}(x, y) = \begin{cases} y - \dfrac{\alpha}{|\ln(1 - x)|^p} & \text{if} \quad (x, y) \in (0, 1) \times [0, 1], \\ \qquad y & \text{if} \quad (x, y) \in \{1\} \times [0, 1]. \end{cases}$$

In this case the function $z_{\alpha,p}$ is equal zero on the curve which connects the points $(1, 0)$ and $(1 - e^{-\alpha^{1/p}}, 1)$, where $1 - e^{-\alpha^{1/p}} < 1$.

Let T be a t-norm such that $T(x,y) \geq z_{\alpha,p}(x,y)$ for every $(x,y) \in [1-\delta,1] \times [0,1]$. Then for every $(x,y) \in [0,1] \times [1-\delta,1)$

$$\begin{aligned} |T(x,y) - T(x,1)| &= |T(y,x) - T(1,x)| \\ &\leq |z_{\alpha,p}(y,x) - z_{\alpha,p}(1,x)| \\ &\leq \frac{\alpha}{|\ln(1-y)|^p}, \end{aligned}$$

i.e., (9) holds for

$$\psi(y) = \begin{cases} \dfrac{\alpha}{|\ln(1-y)|^p} & \text{if} \quad y \in [1-\delta,1), \\ 0 & \text{if} \quad y = 1. \end{cases}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \psi(1-q^n) &= \sum_{n=1}^{\infty} \frac{\alpha}{|\ln(q^n)|^p} \\ &= \sum_{n=1}^{\infty} \frac{\alpha}{n^p |\ln(q)|^p} < \infty, \end{aligned}$$

T is g-convergent.

4. FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

Let Δ^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0,1]$ such that $\sup_{x \in \mathbb{R}} F(x) = 1$).

The ordered pair $(S, \mathcal{F})$ is said to be *a probabilistic metric space* if S is a nonempty set and $\mathcal{F} : S \times S \to \Delta^+$ ($\mathcal{F}(p,q)$ is written by $F_{p,q}$ for every $(p,q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \Rightarrow u = v$ $(u, v \in S)$.
2. $F_{u,v} = F_{v,u}$ for every $u, v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbb{R}_+ = [0, \infty)$.

A *Menger space* is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space, T is a t-norm and the following inequality holds

$$F_{u,v}(x+y) \geq T(F_{u,w}(x), F_{w,v}(y)) \text{ for every } u, v, w \in S \text{ and every } x > 0, y > 0.$$

The (ε, λ)-topology in S is introduced by the family of neighbourhoods

$$\mathcal{U} = \{U_v(\varepsilon, \lambda)\}_{(v,\varepsilon,\lambda) \in S \times \mathbb{R}_+ \times (0,1)},$$

where

$$U_v(\varepsilon, \lambda) = \{u \mid u \in S,\ F_{u,v}(\varepsilon) > 1 - \lambda\}.$$

4.1. Probabilistic q-contraction and g-convergent t-norms

Definition 27. ([15]) Let $(S, \mathcal{F})$ be a probabilistic metric space. A mapping $f : S \to S$ is a probabilistic q-contraction ($q \in (0,1)$) if

$$F_{fp_1, fp_2}(x) \geq F_{p_1, p_2}\left(\frac{x}{q}\right) \tag{10}$$

for every $p_1, p_2 \in S$ and every $x \in \mathbb{R}$.

By Remark 13 each g-convergent t-norm T satisfies the condition $\sup_{x<1} T(x,x) = 1$, which ensures the metrizability of the (ε, λ)-topology.

Theorem 28. Let $(S, \mathcal{F}, T)$ be a complete Menger space and $f : S \to S$ a probabilistic q-contraction such that for some $p \in S$ and $k > 0$

$$\sup_{x>0} x^k (1 - F_{p,fp}(x)) < \infty. \tag{11}$$

If t-norm T is g-convergent, then there exists a unique fixed point z of the mapping f and $z = \lim_{n \to \infty} f^n p$.

P r o o f. Let $\mu \in (q, 1)$ and $\delta = q/\mu < 1$. We shall prove that $(f^n p)_{n \in \mathbb{N}}$ is a Cauchy sequence. Choose $\varepsilon > 0$ and $\lambda \in (0,1)$ and prove that there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that

$$F_{f^n p, f^{n+m} p}(\varepsilon) > 1 - \lambda \quad \text{for every } n \geq n_0(\varepsilon, \lambda) \text{ and every } m \in \mathbb{N}.$$

Since the series $\sum_{i=1}^{\infty} \delta^i$ is convergent, there exists $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that $\sum_{i=n_1}^{\infty} \delta^i \leq \varepsilon$. Let $n > n_1$. Then we have

$$\begin{aligned}
F_{f^n p, f^{n+m} p}(\varepsilon) &\geq F_{f^n p, f^{n+m} p}\left(\sum_{i=n}^{\infty} \delta^i\right) \\
&\geq F_{f^n p, f^{n+m} p}\left(\sum_{i=n}^{n+m-1} \delta^i\right) \\
&\geq \underbrace{T\Big(T\Big(\cdots\Big(T}_{(m-1)\text{-times}}\Big(F_{f^n p, f^{n+1} p}(\delta^n), F_{f^{n+1} p, f^{n+2} p}(\delta^{n+1})\Big), \\
&\qquad \ldots, F_{f^{n+m-1} p, f^{n+m} p}(\delta^{n+m-1})\Big) \\
&\geq \underbrace{T\Big(T\Big(\cdots\Big(T}_{(m-1)\text{-times}}\Big(F_{p,fp}\Big(\frac{1}{\mu^n}\Big), F_{p,fp}\Big(\frac{1}{\mu^{n+1}}\Big)\Big), \ldots, F_{p,fp}\Big(\frac{1}{\mu^{n+m-1}}\Big)\Big).
\end{aligned}$$

Let $M > 0$ be such that

$$x^k (1 - F_{p,fp}(x)) \leq M \text{ for every } x > 0. \tag{12}$$

Suppose that n_2 is such that

$$1 - M(\mu^k)^n \in [0, 1) \text{ for every } n \geq n_2. \tag{13}$$

From (12) it follows that

$$F_{p,fp}\left(\frac{1}{\mu^n}\right) > 1 - M(\mu^k)^n \text{ for every } n \in \mathbb{N}$$

and by (13) for $n \geq \max(n_1, n_2)$

$$F_{f^n p, f^{n+m} p}(\varepsilon) \geq \underbrace{T\Big(T\Big(\cdots\Big(T}_{(m-1)\text{-times}}\Big(1 - M(\mu^k)^n, 1 - M(\mu^k)^{n+1}\Big), \ldots, 1 - M(\mu^k)^{n+m-1}\Big).$$

Let s_0 be such that $M(\mu^k)^{s_0} < \mu^k$. Then for every $n \in \mathbb{N}$

$$1 - M(\mu^k)^{n+s_0} \geq 1 - (\mu^k)^{n+1}$$

and therefore for $n \geq \max(n_1, n_2)$ and $m \in \mathbb{N}$

$$\begin{aligned} F_{f^{n+s_0} p, f^{n+s_0+m} p}(\varepsilon) &\geq \underbrace{T\Big(T\Big(\cdots\Big(T}_{(m-1)\text{-times}}\Big(1 - M(\mu^k)^{n+s_0}, 1 - M(\mu^k)^{n+s_0+1}\Big), \\ &\qquad \ldots, 1 - M(\mu^k)^{n+s_0+m-1}\Big) \\ &\geq \mathop{\mathsf{T}}_{i=n+1}^{\infty} (1 - (\mu^k)^i). \end{aligned}$$

Since T is g-convergent we conclude that $(f^n p)_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $z = \lim_{n \to \infty} f^n p$. By the continuity of the mapping f it follows that $fz = z$. □

Corollary 29. Let $(S, \mathcal{F}, T)$ be a complete Menger space such that T is a strict t-norm with a multiplicative generator θ, and $f : S \to S$ a probabilistic q-contraction such that for some $k > 0$ and $p \in S$ (11) holds. If there exists $\mu \in (0, 1)$ such that

$$\lim_{n \to \infty} \prod_{i=n}^{\infty} \theta(1 - \mu^i) = 1,$$

then there exists a unique fixed point x of the mapping f and $x = \lim_{n \to \infty} f^n p$.

Let

$$\mathcal{T} = \bigcup_{\lambda \in (0,\infty)} \{T_\lambda^{\mathbf{D}}\} \bigcup \bigcup_{\lambda \in (0,\infty)} \{T_\lambda^{\mathbf{AA}}\}.$$

Corollary 30. Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geq T_1$ for some $T_1 \in \mathcal{T}$ and $f : S \to S$ a probabilistic q-contraction such that for some $k > 0$ and $p \in S$ (11) holds. Then there exists a unique fixed point x of the mapping f and $x = \lim_{n\to\infty} f^n p$.

From the proof of Theorem 28 it follows that $f : S \to S$ has a unique fixed point if (11) and the condition that T is g-convergent is replaced by the condition

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} F_{p,fp}\left(\frac{1}{\mu^i}\right) = 1 \quad (\mu \in (0,1)). \tag{14}$$

Using Examples 16 and 17 and Proposition 18 we obtain a fixed point theorem, where the condition (11) is replaced by the condition

$$\sup_{x>1} \ln^k x(1 - F_{p,fp}(x)) < \infty, \tag{15}$$

for some $k > 0$, which under some additional conditions implies (14).

Theorem 31. Let $(S, \mathcal{F}, T)$ be a complete Menger space and $f : S \to S$ a probabilistic q-contraction. Suppose that one of the following two conditions is satisfied:
(i) $T \in \{T_\lambda^{\mathbf{D}}, T_\lambda^{\mathbf{AA}}\}$ for some $\lambda > 0$ and there exists $p \in S$ such that (15) holds, where $k\lambda > 1$.
(ii) $T = T_\lambda^{\mathbf{SW}}$ for some $\lambda \in (-1, \infty]$ and there exists $p \in S$ such that (15) holds, where $k > 1$.

Then there exists a unique fixed point z of the mapping f and $z = \lim_{n\to\infty} f^n p$.

P r o o f. (i) Suppose that $\sup_{x>1} \ln^k x(1 - F_{p,fp}(x)) < \infty$, i.e., that there exists $M > 0$ such that

$$\ln^k x(1 - F_{p,fp}(x)) < M \text{ for every } x > 1. \tag{16}$$

Relation (16) implies that

$$\begin{aligned} F_{p,fp}\left(\frac{1}{\mu^n}\right) &\geq 1 - \frac{M}{\ln^k\left(\frac{1}{\mu^n}\right)} \\ &= 1 - \frac{M}{n^k |\ln \mu|^k} \quad (\mu \in (0,1)). \end{aligned}$$

Suppose that $1 - \frac{M}{n^k|\ln\mu|^k} > 0$ for every $n \geq n_0$. Then

$$\mathop{\mathsf{T}}_{i=n}^{\infty} F_{p,fp}\left(\frac{1}{\mu^i}\right) \geq \mathop{\mathsf{T}}_{i=n}^{\infty} \left(1 - \frac{M}{n^k|\ln\mu|^k}\right) \text{ for every } n \geq n_0.$$

By Examples 16 and 17

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} \left(1 - \frac{M}{n^k|\ln\mu|^k}\right) = 1$$

since for $k\lambda > 1$

$$\sum_{i=1}^{\infty} \frac{M^{\lambda}}{i^{k\lambda} |\ln \mu|^{k\lambda}} < \infty.$$

Hence (14) holds.

(ii) If $T = T_{\lambda}^{\mathbf{SW}}$ for some $\lambda \in (-1, \infty]$ and (16) holds for some $k > 1$ then (14) holds, since by Proposition 18, $\sum_{i=1}^{\infty} \frac{M}{i^k |\ln \mu|^k} < \infty$ implies (14). □

Remark. It is obvious by Proposition 18 that in the case (ii) the condition (15) can be replaced by the Tardiff's condition (see [16])

$$\int_1^{\infty} \ln u \, \mathrm{d}F_{p,fp}(u) < \infty.$$

4.2. An application to random operator equations

Special non-additive measures, so called decomposable measures, see [11], generate a probabilistic metric space ([4]) on which Theorem 28 implies a random fixed point theorem.

Definition 32. Let $\mathbf{S}$ be a t-conorm. An $\mathbf{S}$-decomposable measure m is a set function $m : \mathcal{A} \to [0,1]$ such that $m(\emptyset) = 0$ and

$$m(A \cup B) = \mathbf{S}(m(A), m(B))$$

whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

Example 33. Taking $\mathbf{S_L}$ t-conorm, $\Omega = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and $m(E) = \min(|E|/N, 1)$ for a fixed natural number N, where $|E|$ is the cardinal number of E, we obtain that m is $\mathbf{S_L}$-decomposable measure.

Definition 34. Let $\mathbf{S}$ be a left-continuous t-conorm. A set function $m : \mathcal{A} \to [0,1]$ is σ-$\mathbf{S}$-decomposable measure if $m(\emptyset) = 0$ and

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathop{\mathbf{S}}_{i=1}^{\infty} m(A_i)$$

for every sequence $(A_i)_{i \in \mathbb{N}}$ from $\mathcal{A}$ whose elements are pairwise disjoint set.

The set function considered in Example 33. is σ-$\mathbf{S_L}$-decomposable.

An $\mathbf{S}$-decomposable measure m is monotone, which means that $A, B \in \mathcal{A},\ A \subseteq B$ implies $m(A) \le m(B)$. A measure m is of (NSA)-*type* (see [17]) if and only if $\mathbf{s} \circ m$ is a finite additive measure, where $\mathbf{s}$ is an additive generator of the t-conorm $\mathbf{S}$ (see [17]), which is continuous, non-strict, and Archimedean, and with respect to which m is decomposable ($\mathbf{s}(1) = 1$). If $(\Omega, \mathcal{A}, m)$ is a measure space and (M, d) is a separable metric space, by S we shall denote the set of all the equivalence classes of measurable mappings $X : \Omega \to M$. An element from S will be denoted by $\widehat{X}$ if $\{X(\omega)\} \in \widehat{X}$. The following proposition is proved in [14].

Proposition 35. Let $(\Omega, \mathcal{A}, m)$ be a measure space, where m is a continuous $\mathbf{S}$-decomposable measure of (NSA)-type with monotone increasing generator $\mathbf{s}$. Then $(S, \mathcal{F}, T)$ is a Menger space, where $\mathcal{F}$ and t-norm T are given in the following way $(\mathcal{F}(\widehat{X}, \widehat{Y}) = F_{\widehat{X},\widehat{Y}})$:

$$F_{\widehat{X},\widehat{Y}}(u) = m(\{\omega \mid \omega \in \Omega,\ d(X(\omega), Y(\omega)) < u\}) = m(\{d(X,Y) < u\})$$

(for every $\widehat{X}, \widehat{Y} \in S, u \in \mathbb{R}$),

$$T(x,y) = \mathbf{s}^{-1}(\max(0, \mathbf{s}(x) + \mathbf{s}(y) - 1)),\ \text{for every } x, y \in [0,1].$$

Let $f : \Omega \times M \to M$ be a continuous random operator. Then for every measurable mapping $X : \Omega \to M$, the mapping $\omega \mapsto f(\omega, X(\omega))(\omega \in \Omega)$ is measurable. If $X : \Omega \to M$ is a measurable mapping let $(\hat{f}\widehat{X})(\omega) = f(\omega, X(\omega)), \omega \in \Omega, X \in \widehat{X}$. Hence $\hat{f} : S \to S$.

Corollary 36. Let $(\Omega, \mathcal{A}, m)$ be a measure space, where m is a continuous $\mathbf{S}$-decomposable measure of (NSA)-type , $\mathbf{s}$ is a monotone increasing additive generator of $\mathbf{S}$, (M, d) a complete separable metric space and $f : \Omega \times M \to M$ a continuous random operator such that for some $q \in (0,1)$

$$m(\{\omega \mid \omega \in \Omega, d((\hat{f}\widehat{X})(\omega), (\hat{f}\widehat{Y})(\omega)) < u\})$$

$$\geq m\left(\{\omega \mid \omega \in \Omega, d(X(\omega), Y(\omega)) < \frac{u}{q}\}\right) \tag{17}$$

for every measurable mappings $X, Y : \Omega \to M$ and every $u > 0$. If there exists a measurable mapping $U : \Omega \to M$ such that for some $k > 0$

$$\sup_{x>0} x^k (1 - m(\{d(\widehat{U}, \hat{f}\widehat{U}) < x\})) < \infty$$

and t-norm T defined by

$$T(x,y) = \mathbf{s}^{-1}(\max(0, \mathbf{s}(x) + \mathbf{s}(y) - 1), x, y \in [0,1],$$

is g-convergent, then there exists a random fixed point of the operator f.

Corollary 37. Let $(\Omega, \mathcal{A}, m)$ be a measure space, where m is a continuous $\mathbf{S}_{\lambda}^{\mathbf{SW}}$-decomposable measure of (NSA)-type for some $\lambda \in (-1, \infty]$, (M, d) a complete separable metric space and $f : \Omega \times M \to M$ a continuous random operator such that for some $q \in (0,1)$ (17)holds for every measurable mappings $X, Y : \Omega \to M$ and every $u > 0$. If there exists a measurable mapping $U : \Omega \to M$ such that for some $k > 1$

$$\sup_{x>1} \ln^k x (1 - m(\{d(\widehat{U}, \hat{f}\widehat{U}) < x\})) < \infty,$$

then there exists a random fixed point of the operator f.

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REFERENCES

[1] J. Aczél: Lectures on Functional Equations and their Applications. Academic Press, New York 1969.

[2] O. Hadžić and E. Pap: On some classes of t-norms important in the fixed point theory. Bull. Acad. Serbe Sci. Art. Sci. Math. *25* (2000), 15–28.

[3] O. Hadžić and E. Pap: A fixed point theorem for multivalued mappings in probabilistic metric spaces and an application in fuzzy metric spaces. Fuzzy Sets and Systems *127* (2002), 333–344.

[4] O. Hadžić and E. Pap: Fixed Point Theory in Probabilistic Metric Spaces. Kluwer Academic Publishers, Dordrecht 2001.

[5] T. L. Hicks: Fixed point theory in probabilistic metric spaces. Univ. u Novom Sadu, Zb. Rad. Prirod.–Mat. Fak. Ser. Mat. *13* (1983), 63–72.

[6] O. Kaleva and S. Seikalla: On fuzzy metric spaces. Fuzzy Sets and Systems *12* (1984), 215–229.

[7] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. (Trends in Logic 8.) Kluwer Academic Publishers, Dordrecht 2000.

[8] E. P. Klement, R. Mesiar, and E. Pap: Uniform approximation of associative copulas by strict and non-strict copulas. Illinois J. Math. *45* (2001), 4, 1393–1400.

[9] K. Menger: Statistical metric. Proc. Nat. Acad. Sci. U. S. A. *28* (1942), 535–537.

[10] R. Mesiar and H. Thiele: On T-quantifiers and S-quantifiers: Discovering the World with Fuzzy Logic (V. Novák and I. Perfilieva, eds., Studies in Fuzziness and Soft Computing vol. 57), Physica–Verlag, Heidelberg 2000, pp. 310–326.

[11] E. Pap: Null–Additive Set Functions. Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava 1995.

[12] E. Pap, O. Hadžić, and R. Mesiar: A fixed point theorem in probabilistic metric spaces and applications in fuzzy set theory. J. Math. Anal. Appl. *202* (1996), 433–449.

[13] V. Radu: Lectures on probabilistic analysis. Surveys. (Lectures Notes and Monographs Series on Probability, Statistics & Applied Mathematics 2), Universitatea de Vest din Timişoara 1994.

[14] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. Elsevier North–Holland, New York 1983.

[15] V. M. Sehgal and A. T. Bharucha–Reid: Fixed points of contraction mappings on probabilistic metric spaces. Math. Systems Theory *6* (1972), 97–102.

[16] R. M. Tardiff: Contraction maps on probabilistic metric spaces. J. Math. Anal. Appl. *165* (1992), 517–523.

[17] S. Weber: $\perp$-decomposable measures and integrals for Archimedean t-conorm $\perp$. J. Math. Anal. Appl. *101* (1984), 114–138.

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