KYBERNETIKA — VOLUME 28 (1992), NUMBER 3, PAGES 169 – 180

ON LEAST SQUARES ESTIMATION IN CONTINUOUS TIME LINEAR STOCHASTIC SYSTEMS [1]

TYRONE E. DUNCAN, PETR MANDL AND BOŻENNA PASIK–DUNCAN

The sufficient conditions for the convergence of a family of least squares estimates of some unknown parameters are given. The unknown parameters appear affinely in the linear transformations of the state and the control in a linear stochastic system. If the noise in the stochastic system is colored then the family of least squares estimates does not converge to the value and the bias is given explicitly.

1. INTRODUCTION

This paper considers stochastic systems whose trajectories satisfy the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= f(\alpha)\,X(t)\,\mathrm{d}t + g(\alpha)\,U(t)\,\mathrm{d}t + \mathrm{d}B(t), \\ X(0) &= x \end{aligned} \tag{1}$$

where $t \geq 0,\ B = \{B(t),\, t \geq 0\}$ is, unless stated otherwise, an n-dimensional Wiener process with incremental covariance matrix $h \geq 0$, that is formally,

$$\mathrm{d}B(t)\,\mathrm{d}B'(t) = h\,\mathrm{d}t$$

and $U = \{U(t),\, t \geq 0\}$ is an m-dimensional stochastic process that is nonanticipative with respect to B. The matrices $f(\alpha)$ and $g(\alpha)$ depend on α as

$$f(\alpha) = f_0 + \alpha^1\, f_1 + \cdots + \alpha^p\, f_p \tag{2}$$

and

$$g(\alpha) = g_0 + \alpha^1\, g_1 + \cdots + \alpha^s\, g_s \tag{3}$$

where $0 \leq p \leq s$ are integers and $\alpha = \left(\alpha^1, \ldots, \alpha^s\right)$ is a parameter vector whose value is estimated from the observation of X and U by a least squares method. The matrices $f_0, \ldots, f_p,\ g_0, \ldots, g_s$ are fixed and known. If $p \leq s$ is not mentioned then it is assumed that $p = s$.

Let ℓ be a symmetric, nonnegative definite matrix. The least squares estimate of the unknown vector α based on $\{X(t),\, 0 \leq t \leq T\}$ and $\{U(t),\, 0 \leq t \leq T\}$ denoted $\alpha^\star(T)$, is the minimizer of the formal functional

$$\begin{aligned} L(T;\, a) \;=\; \int_0^T \Big[&\Big(\dot{X}(t) - f(a)\,X(t) - g(a)\,U(t)\Big)'\, \ell\, \Big(\dot{X}(t) \\ &- f(a)\,X(t) - g(a)\,U(t)\Big) - \dot{X}'(t)\,\ell \dot{X}(t)\Big]\,\mathrm{d}t. \end{aligned} \tag{4}$$

[1] This research was partially supported by the U. S. National Science Foundation Grant ECS-8718026.

In (4), $\dot{X}(t)\,\mathrm{d}t = \mathrm{d}X(t)$ and the undefined product $\dot{X}(t)\,\ell\dot{X}(t)$ is eliminated by subtraction because it does not depend on a. Equating the gradient with respect to a of $L(T;\,a)$ to zero yields the following family of linear equations

$$\sum_{j=1}^{s}\int_0^T (f_i\,X(t)+g_i\,U(t))'\,\ell\,(f_j\,X(t)+g_j\,U(t))\,\mathrm{d}t\,\alpha^{\star j}(T) \tag{5}$$
$$= \int_0^T (f_i\,X(t)+g_i\,U(t))'\,\ell\,(d\,X(t)-f_0\,X(t)\,\mathrm{d}t-g_0\,U(t)\,\mathrm{d}t)$$

for $i=1,2,\ldots,s$ and $\alpha^\star(T)=\left(\alpha^{\star 1}(T),\ldots,\alpha^{\star s}(T)\right)$. Using (1), (5) can be rewritten as

$$\sum_{j=1}^{s}\int_0^T (f_iX(t)+g_iU(t))'\,\ell\,(f_jX(t)+g_jU(t))\,\mathrm{d}t\,\left(\alpha^{\star j}(T)-\alpha^j\right) \tag{6}$$
$$= \int_0^T (f_iX(t)+g_iU(t))'\,\ell\,\mathrm{d}B(t)$$

for $1,2,\ldots,s$.

The family of estimates $(\alpha^\star(T),\,T>0)$ is consistent (resp. strongly consistent) if $\alpha^\star(T)\to\alpha$ in probability (resp. almost surely) as $T\to\infty$. The consistency of the family of estimates is the basis of the concepts of identification and self-tuning.

The results of this paper are presented in the following two sections. In Section 2, the process U does not depend on X, that is, U is not feedback control. It is assumed that the empirical covariance function of U converges to a nonrandom limit as $T\to\infty$. The asymptotic distribution of $\alpha^*(T)$ is obtained and, in the case that the noise in (1) is colored, explicit formulas for the asymptotic error in $\alpha^*(T)$ are presented. However the independence hypothesis on X and U is not satisfied in important cases, e. g. when U is a self-tuning control and $\alpha^*(T)$ is used to adjust the feedback gain.

In Section 3 the methods of [3] are extended to the case where the gain factor acting on U also contains unknown parameters. A sufficient condition for strong consistency of $\alpha^*(T)$ is presented. It involves the hypotheses guaranteeing the identifiability of α under linear feedback controls, stability conditions on X and U, and an excitation condition on U needed to estimate the parameters occurring solely in the gain factor. Another approach to strong consistency is given in [2]. Systems with a drift depending on unknown parameters are treated in [1]. The relaxation of the stability conditions for estimation is considered in [4].

Example. For illustration of the subsequent results consider the system described in Diagram 1.

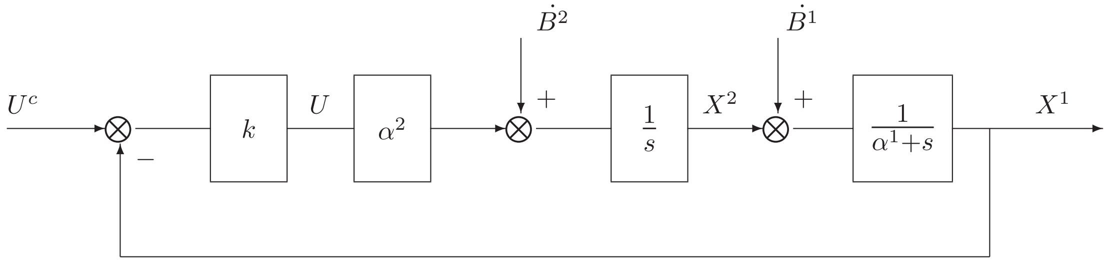

Diagram 1.

Let U^c be the input signal so the system is described by the stochastic differential equation

$$\mathrm{d}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix} = \begin{pmatrix} -\alpha^1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ k\,\alpha^2 \end{pmatrix} U^c(t)\,\mathrm{d}t + \mathrm{d}\begin{pmatrix} B^1(t) \\ B^2(t) \end{pmatrix}.$$

The results of Section 2 can be applied if B is independent of U^c.

Let U^c be described by

$$\mathrm{d}\,U^c(t) = -\alpha^3\,U^c(t)\,\mathrm{d}t + \mathrm{d}\,B^3(t)$$

and set $X^3 = U^c$ so that

$$\mathrm{d}\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} = \begin{pmatrix} -\alpha^1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha^3 \end{pmatrix}\begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ k\,\alpha^2 \\ 0 \end{pmatrix} U(t)\,\mathrm{d}t + \mathrm{d}\begin{pmatrix} B^1(t) \\ B^2(t) \\ B^3(t) \end{pmatrix}.$$

If the gain k has a desirable $k(\alpha)$ that depends nontrivially on the parameter vector then the results of Section 3 can be used to investigate the self-tuning property of the family of gains $(k(\alpha^\star(t)),\ t > 0)$ where

$$U(t) = k\,(\alpha^\star(t))\,\left(X^3(t) - X^1(t)\right)$$

for $t \geq 0$.

2. INDEPENDENT INPUT AND NOISE PROCESSES

For two matrices of the same type the dot product is $M \cdot N = \operatorname{trace}(MN')$. The terms p-lim and l.i.m. mean the limit in probability and the limit in quadratic mean, respectively. An ergodic property for U is assumed to obtain the following asymptotic distribution of $\alpha^\star(T)$ as $T \longrightarrow \infty$.

Proposition 1. Let $f = f(\alpha)$ in (1) be a stable matrix and let U be independent of B such that

$$\sup_{t\geq 0} \mathsf{E}|U(t)|^2 \leq \text{const}\,. \tag{7}$$

$$\operatorname*{l.i.m.}_{T\to\infty} \frac{1}{T}\int_0^T U(s)\,U(s+t)'\,\mathrm{d}s = R(t) \tag{8}$$

where $t \geq 0$ and $R(t)$ is not random. Let V be the solution of the Lyapunov equation

$$fV + Vf' + Qg' + gQ' + h = 0 \tag{9}$$

where $g = g(\alpha)$ from (1) and

$$Q = \int_0^\infty \exp(sf)\, gR(s)\,\mathrm{d}s \tag{10}$$

and let θ and Δ be the matrices given by

$$\begin{aligned}\theta &= \left(f_i'\ell f_j \cdot V + \left(f_i'\ell g_j + f_j'\ell g_i\right)\cdot Q + g_i'\ell g_j \cdot R\right)\\ \Delta &= \left(f_i'\ell h\ell f_j \cdot V + \left(f_i'\ell h\ell g_j + f_j'\ell h\ell g_i\right)\cdot Q + g_i'\ell h\ell g_j \cdot R\right)\end{aligned}$$

where $R = R(0)$ and $i,\, j \in \{1,\ldots,s\}$.

If θ is nonsingular, then $(\alpha^\star(T)-\alpha)\,\sqrt{T}$ has asymptotically the normal distribution $N\left(0,\, \theta^{-1}\,\Delta\,\theta^{-1}\right)$ as $T\to\infty$.

Proof. Let F be defined by

$$F(s) = \exp(sf).$$

From (1) it follows immediately that

$$X(t) = \int_0^T F(t-s)\, gU(s)\,\mathrm{d}s + \int_0^T F(t-s)\,\mathrm{d}B(s) + F(t)\,x.$$

It can be verified using (7, 8) that

$$\begin{aligned}&\operatorname*{p\text{-}lim}_{T\to\infty}\ \frac{1}{T}\int_0^T X(t)\,X'(t)\,\mathrm{d}t \qquad (11)\\ &= \int_0^\infty F(s)\left[\int_0^\infty \left(gR(t)\,g'F'(t) + F(t)\,gR(t)\,g'\right)\,\mathrm{d}t + h\right] F'(s)\,\mathrm{d}s = V.\end{aligned}$$

Using (10) we obtain

$$V = \int_0^\infty F(s)\ [gQ' + Qg' + h]\ F'(s)\,\mathrm{d}s.$$

Thus V is the unique, symmetric solution of (9). Similar to the verification of (11) it follows that

$$\operatorname*{p\text{-}lim}_{T\to\infty}\ \frac{1}{T}\int_0^T X(t)\,U'(t)\,\mathrm{d}t = \int_0^\infty F(s)\,gR(s)\,\mathrm{d}s = Q. \tag{12}$$

From (7), (8), (11), (12) it follows by passage to the limit in (6) that

$$\operatorname*{p\text{-}lim}_{T\to\infty}\ \frac{1}{T}\int_0^T \left(f_i\,X(t) + g_i\,U(t)\right)'\,\ell\,\left(f_j\,X(t)+g_j\,U(t)\right)\,\mathrm{d}t = \theta \tag{13}$$

for $j,\, j\in\{1,\ldots,s\}$. For the quadratic variation of the stochastic integral in (6) we have

$$\operatorname*{p\text{-}lim}_{T\to\infty}\ \frac{1}{T}\int_0^T \left(f_i\,X(t) + g_i\,U(t)\right)'\,\ell h\ell\,\left(f_j\,X(t)+g_j\,U(t)\right)\,\mathrm{d}t = \Delta.$$

Thus the family of random variables

$$\frac{1}{\sqrt{T}}\int_0^T \left(f_i\,X(t)+g_i\,U(t)\right)'\,\ell\,\mathrm{d}W(t)$$

for $i = 1,2,\ldots,s$ and $T > 0$ has asymptotically the joint $N(0,\Delta)$ distribution. This conclusion follows easily by representing the family of stochastic integrals as a time changed Wiener process. From (6), (13) it follows that $(\alpha^\star(T)-\alpha)\,\sqrt{T}$ has asymptotically a $N\left(0,\,\theta^{-1}\,\Delta\,\theta^{-1}\right)$ distribution as $T\to\infty$. □

Remark. Let U satisfy

$$\mathrm{d}\, U(t) = c\, U(t)\, \mathrm{d}t + \mathrm{d}^0 W(t)$$

where $T \geq 0$, c is a stable matrix and ${}^0W$ is a Wiener process with incremental covariance matrix r. In this case $R = R(0)$ and Q in (10) satisfy the following equations

$$\begin{aligned} cR + Rc' + r &= 0 \\ fQ + Qc' + gR &= 0. \end{aligned}$$

The effect of the correlation in the noise process in (1) can be seen by assuming that B satisfies the stochastic differential equation

$$\mathrm{d}B(t) = bB(t)\,\mathrm{d}t + \mathrm{d}W(t) \tag{14}$$

where $t \geq 0$, b is a stable matrix and W is a Wiener process such that $\mathrm{d}W(t)\,\mathrm{d}W'(t) = h\,\mathrm{d}t$. The consistency of the family $(\alpha^\star(T),\, T \geq 0)$ is no longer valid. The following proposition describes explicitly the asymptotic bias of the family of estimators.

Proposition 2. Assume that the hypotheses of Proposition 1 are satisfied and that B in (1) satisfies (14). Let R_B, Q_B and V_B be the solutions of the following equations

$$\begin{aligned} bR_B + R_B b' + h &= 0 \\ fQ_B + Q_B b' + bR_B &= 0 \\ fV_B + V_B f' + Q_B b' + bQ_B' &= 0 \end{aligned}$$

and define the matrix θ_B and the vector β by the following equations

$$\theta_B = (f_i' \ell f_j \cdot V_B)$$

where $i,\, j \in \{1, \ldots, s\}$ and

$$\beta = \left(f_1' \ell b \cdot Q_B, \ldots, f_s' \ell b \cdot Q_B\right)'.$$

If $\theta + \theta_B$ is nonsingular then

$$\operatorname*{p\text{-}lim}_{T\to\infty} \alpha^\star(T) = \alpha \left(\theta + \theta_B\right)^{-1} \beta. \tag{15}$$

Proof. Rewrite (1) as

$$\mathrm{d}X(t) = fX(t)\,\mathrm{d}t + (g,b) \begin{pmatrix} U(t) \\ B(t) \end{pmatrix} \mathrm{d}t + \mathrm{d}W(t). \tag{16}$$

Let $R_B(\cdot)$, $\mathbb{R}(\cdot)$ and $\mathbb{Q}$ be defined by the following equations

$$\begin{aligned} R_B(\tau) &= R_B \exp(\tau b) \\ \mathbb{R}(\tau) &= \begin{pmatrix} R(\tau) & 0 \\ 0 & R_B(\tau) \end{pmatrix} \\ \mathbb{Q} &= \int_0^\infty F(s)\,(g,b)\,\mathbb{R}(s)\,\mathrm{d}s = (Q, Q_B). \end{aligned}$$

Applying the method of the proof of Proposition 1 to (16) it follows that

$$\operatorname*{p-lim}_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,X'(t)\,\mathrm{d}t = V + V_B$$

$$\operatorname*{p-lim}_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,U'(t)\,\mathrm{d}t = Q$$

$$\operatorname*{p-lim}_{T\to\infty} \frac{1}{T}\int_0^T X(t)\,B'(t)\,\mathrm{d}t = Q_B.$$

The verification of these equations is simplified by expressing the solution of (1) as a sum of the responses to U and B. Using this same method if equation (6) is divided by T and $T \to \infty$, then (15) is obtained. □

3. NONANTICIPATIVE INPUT

In this section it is assumed that B in (1) is a Wiener process and it is allowed that $p \le s$ in (2), (3). To achieve the consistency of the family of estimates $(\alpha^\star(T),\, T > 0)$, the possible values of α must be distinguishable using the weight matrix ℓ for any feedback control $U(t) = k\,X(t)$. If $p < s$ then there are components of α that cannot be estimated if $u \equiv 0$ or even if u does not vary sufficiently. This requirement is reflected in the hypotheses of the following proposition. $\sqrt{\ell}$ denotes the symmetric square root of ℓ.

Proposition 3. Assume that the following conditions are satisfied:

i) The matrices $g = g(\alpha)$ and h have full rank.

ii) For any gain matrix $k \in L(\mathbb{R}^n, \mathbb{R}^m)$ the family of linear transformations $\left(\sqrt{\ell}\,(f_i + g_i k)\,,\ i = 1, 2, \dots p\right)$ is linearly independent and

$$\operatorname{span}\left(\sqrt{\ell}\,(f_i + g_i k)\,,\ i = 1, 2, \dots, p\right) \cap \operatorname{span}\left(\sqrt{\ell}\,g_{p+j},\ j = 1, 2, \dots, s - p\right) = \{0\}.$$

iii) The family of linear transformations $\left(\sqrt{\ell}\,g_{p+j},\ j = 1, \dots, s - p\right)$ is linearly independent.

iv)

$$\frac{1}{T}\int_0^T \left(|X(t)|^2 + |U(t)|^2\right)\,\mathrm{d}t$$

for $T > 0$ is bounded a. s. and

$$\lim_{T\to\infty} \frac{|X(t)|^2}{T} = 0 \qquad \text{a. s.}$$

v) If $p < s$, then for each nonzero $y \in \mathbb{R}^m$

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \left(y'U(t)\right)^2\,\mathrm{d}t > 0 \qquad \text{a. s.}$$

Then

$$\lim_{T\to\infty} \alpha^\star(T) = \alpha \qquad \text{a. s.} \tag{17}$$

where $\alpha^\star(T)$ gives the minimum of (6).

To verify Proposition 3 a positivity (a. s.) of a quadratic form is verified. This type of property is often called persistent excitation. By slightly modifying (6) we have

$$\sum_{i,j=1}^{s} \frac{1}{T} \int_0^T (f_i\, X(t) + g_i\, U(t))'\, \ell\, (f_j\, X(t) + g_j\, U(t))\; \mathrm{d}t\, \left(\alpha^{\star i}(T) - \alpha^i\right) \left(\alpha^{\star j}(T) - \alpha^j\right)$$

$$= \sum_{i=1}^{s} \frac{1}{T} \int_0^T (f_i\, X(t) + g_i\, U(t))'\, \ell\, \mathrm{d}B(t)\, \left(\alpha^{\star i}(T) - \alpha^i\right). \tag{18}$$

If the quadratic form on the left hand side of (18) is asymptotically positive and the coefficients of the components of the vector $\alpha^\star(T) - \alpha$ on the right hand side tend to zero then we obtain the consistency of the family of estimates $(\alpha^\star(T),\, T > 0)$. The positivity of the quadratic form is described in the following lemma.

Lemma 1. Assume the hypotheses in Proposition 3. For each nonzero $\mu \in \mathbb{R}^s$

$$\liminf_{T\to\infty} \sum_{i,j=1}^{s} \frac{1}{T} \int_0^T (f_i\, X(t) + g_i\, U(t))'\, \ell\, (f_j\, X(t) + g_j\, U(t))\; \mathrm{d}t\, \mu^i\, \mu^j > 0 \qquad \text{a.s.} \tag{19}$$

Proof. Fix a nonzero $\mu \in \mathbb{R}^s$. Define the matrices r and q by the following equations

$$r = \sqrt{\ell} \sum_{i=1}^{p} \mu^i\, f_i$$

$$q = \sqrt{\ell} \sum_{i=1}^{s} \mu^i\, g_i.$$

The sum in (19) can be expressed as

$$\frac{1}{T} \int_0^T |r\, X(t) + q\, U(t)|^2\; \mathrm{d}t. \tag{20}$$

If

$$\mu^1 = \mu^2 = \cdots = \mu^p = 0 \tag{21}$$

then by (iii) of Proposition 3, $q \neq 0$ and (19) follows from (v) of Proposition 3.

Now assume that (21) is not satisfied. The condition (ii) of Proposition 3 implies that

$$r + q\, k \neq 0 \tag{22}$$

for all $k \in L(\mathbb{R}^n, \mathbb{R}^m)$. By coordinate transformations of X and U it can be assumed that g satisfies

$$g = \begin{pmatrix} I_m \\ 0 \end{pmatrix}. \tag{23}$$

Now the range of U is enlarged to $\mathbb{R}^n$ and the input is denoted $\mathbb{U}$. Equation (1) is modified as

$$\mathrm{d}\, X(t) = f\, X(t)\, \mathrm{d}t + \mathbb{U}(t)\, \mathrm{d}t + \mathrm{d}\, B(t). \tag{24}$$

By coordinate transformation on the two summands $\mathbb{R}^m \oplus \mathbb{R}^{n-m}$ it can be assumed that $q \in L\left(\mathbb{R}^{\tilde{m}}, \mathbb{R}^n\right)$ where $\tilde{m} \leq m$ has full rank. In these coordinates the control is again denoted $\mathbb{U}$ by abuse of notation. Define $Q \in L(\mathbb{R}^n, \mathbb{R}^n)$ as $Q = (q, 0)$. A control problem for (24) is to minimize (almost surely) the asymptotic average cost $\frac{C(T)}{T}$ as $T \to \infty$ where

$$C(T) = \int_0^T \left(|r\, X(t) + Q\, \mathbb{U}(t)|^2 + c\, |X(t)|^2 + c\, |\mathbb{U}(t)|^2\right) \mathrm{d}t, \tag{25}$$

$(X(t),\, t \geq 0)$ satisfies (24) and $c > 0$ is a small parameter.

The algebraic Riccati equation for this control problem is

$$vf + f'v + r'r + cI - (v + r'Q)\ (Q'Q + cI)^{-1}\ (Q'r + v) = 0 \tag{26}$$

and its solution v satisfies the stationary Hamilton–Jacobi equation

$$\inf_{u \in \mathbb{R}^n} \left[2x'vfx + 2x'vu + |rx + Qu|^2 + c|x|^2 + c|u|^2\right] = 0. \tag{27}$$

Let U satisfy (iv) and (v) of Proposition 3 and let $\mathbb{U} = (U, 0)$. Apply the Itô formula to $X'(T)\, v(c)\, X(t)$ and use (26), (27) to obtain the inequality

$$\begin{aligned} \frac{1}{T}\int_0^T \left(|r\, X(t) + q\, U(t)|^2\right) \mathrm{d}t \geq&\ \mathrm{trace}\,(v(c)\, h) \\ -\ \frac{c}{T}\int_0^T \left(|X(t)|^2 + |U(t)|^2\right) +&\ \frac{2}{T}\int_0^T X'(t)\, v(c)\, \mathrm{d}B(t) \\ -\ \frac{1}{T} X'(t)\, v(c)\, X(T) +&\ \frac{1}{T} X'(0)\, v(c)\, X(0). \end{aligned} \tag{28}$$

The last three terms on the right hand side converge (almost surely) to zero by the assumptions. We shall show that

$$\limsup_{c \downarrow 0}\ \mathrm{trace}\, \frac{(v(c)\, h)}{\sqrt{c}} > 0. \tag{29}$$

This will verify (19).

Partition the matrix v, that is the solution of (26), into blocks

$$v(c) = \begin{bmatrix} v_{11}(c) & v_{12}(c) \\ v_{21}(c) & v_{22}(c) \end{bmatrix}$$

where v_{11} is an $\tilde{m} \times \tilde{m}$ matrix and v_{22} is an $(n - \tilde{m}) \times (n - \tilde{m})$ matrix. It follows easily from the Riccati equation and the fact that $v(c)$ determines the optimal cost that

$$v_{11}(c) = u_{11} + w_{11}(c) \tag{30}$$

where $u_{11} \geq 0$ does not depend on c and

$$\lim_{c \downarrow 0} w_{11}(c) = 0.$$

Furthermore

$$\lim_{c \downarrow 0} v_{ij}(c) = 0$$

for $(i, j) \neq (1, 1)$. The matrix u_{11} satisfies

$$u_{11}\, F_{11} + F'_{11}\, u_{11} + R_{11} - u_{11}(q'q)^{-1}\, u_{11} = 0$$

where

$$\begin{aligned} F &= f - \begin{pmatrix} (q'q)^{-1}\, q'r \\ 0 \end{pmatrix} \\ F &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \\ R &= r' \left(I - q(q'q)^{-1} q'\right)\, r \\ R &= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \end{aligned}$$

If $R_{11} \neq 0$ then $u_{11} \neq 0$.

Now it is shown that $R \neq 0$. The nonzero columns of q are eigenvectors of $q(q'q)^{-1}q'$ with eigenvalue 1. The vectors that are a basis for the orthogonal complement of the subspace spanned by the eigenvectors with eigenvalue 1 are eigenvectors of $q(q'q)^{-1}q'$ with eigenvalue 0. If $R = 0$ then the columns of r are in the span of the columns of q which would contradict (22).

If $u_{11} \neq 0$ then from (30) it follows that

$$\lim_{c \downarrow 0} \operatorname{trace}(v(c)\,h) \geq \operatorname{trace}\left(\sqrt{h}\,u_{11}\,\sqrt{h}\right) > 0$$

and (29) is satisfied.

If $u_{11} = 0$ then $R_{11} = 0$. Since R is a symmetric, nonnegative definite matrix

$$R = \begin{bmatrix} 0 & 0 \\ 0 & R_{22} \end{bmatrix}$$

and $R_{22} \neq 0$ because $R \neq 0$. The equation for the (2,2) block elements of (26) is

$$v_{21}\,\tilde{f}_{12} + v_{22}\,f_{22} + \tilde{f}'_{12}\,v_{12} + f'_{22}\,v_{22} + R_{22} + c\,I_{n-m} - c^{-1}\,v_{22}\,v_{22} = 0 \tag{31}$$

where

$$\tilde{f}_{12} = f_{12} - \left[(Q'Q + c\,I)^{-1}\,Qr\right]_{12}$$

and

$$\left[r'r - r'Q(Q'Q + c\,I)^{-1}Q'r\right]_{22} = R_{22}.$$

Note that $\tilde{f}_{12}$ and R_{22} do not depend on c because $Q = (q, 0)$.

Let $\overline{v}(c)$ be defined by the equation

$$\overline{v}(c) = c^{-1}\,v(c).$$

Then (31) can be rewritten in terms of $\overline{v}$ as

$$\overline{v}_{12}\,\tilde{f}_{12} + \overline{v}_{22}\,f_{22} + \tilde{f}'_{12}\,\overline{v}_{12} + f'_{22}\,\overline{v}_{22} + c^{-1}\,R_{22} + I_{n-\tilde{m}} - \overline{v}_{22}\,\overline{v}_{22} = 0. \tag{32}$$

Since $|a-b|^2 \leq 2\left(|a|^2 + |b|^2\right)$ we have

$$(\overline{v}_{22}(c) - f_{22})'\,(\overline{v}_{22}(c) - f_{22}) \leq 2\,(\overline{v}_{22}(c)\,\overline{v}_{22}(c) + f'_{22}\,f_{22})\,. \tag{33}$$

Using (32) in (33) we have the inequality

$$2\,\overline{v}_{22}(c)\,\overline{v}_{22}(c) \geq c^{-1}\,R_{22} + I_{n-m} + \overline{v}_{21}(c)\,\tilde{f}_{12} + \tilde{f}'_{12}\,\overline{v}_{12}(c) - f'_{22}\,f_{22}.$$

Using a property of the trace of a symmetric, nonnegative definite linear transformation it follows that

$$\begin{aligned} (\operatorname{tr}\overline{v}(c))^2 &\geq (\operatorname{tr}\,(\overline{v}_{22}(c)))^2 \geq \sum_j \langle \overline{v}_{22}(c)\,e_j,\, \overline{v}_{22}(c)\,e_j \rangle \\ &\geq \frac{c^{-1}}{2} \sum_j \langle R_{22}e_j,\, ej \rangle + \sum_j \langle \overline{v}_{21}(c)\,f_{12}e_j,\, e_j \rangle \\ &\quad + \sum_j \langle e_j,\, e_j \rangle - \frac{1}{2} \sum_j \langle f_{22}e_j,\, f_{22}e_j \rangle \end{aligned} \tag{34}$$

where the index of summation $j = 1, \ldots, n - \tilde{m}$ and $(e_j,\ j = 1, \ldots, n - \tilde{m})$ is an orthonormal basis of $\mathbb{R}^{n-\tilde{m}}$. Since $\overline{v}_{21}(c) = c^{-1}\,v_{21}(c)$ and $\lim_{c\downarrow 0} v_{21}(c) = 0$ it follows that the right hand side of (34) is bounded below by $c^{-1}k$ where $k > 0$ and c is sufficiently small. Thus (29) is satisfied. □

Proof of Proposition 3.
Denote the quadratic form in (19) as

$$\frac{1}{T}\sum_{i,j} Q^{ij}(T)\,\mu^i\,\mu^j. \tag{35}$$

Assumption (iv) ensures that for $\mu,\,\lambda \in \mathbb{R}^s$

$$\left|\frac{1}{T}\sum_{i,j} Q^{ij}(T)\,\mu^i\,\mu^j - \frac{1}{T}\sum_{i,j} Q^{ij}(T)\,\lambda^i\,\lambda^j\right| \le K|\mu-\lambda|$$

for all $T > 0$ where K is a real-valued random variable. From Lemma 1 it follows using continuity, compactness of the unit sphere in $\mathbb{R}^s$ and homogeneity of (19) that

$$\liminf_{T\to\infty}\frac{1}{T}\sum_{i,j} Q^{ij}(T)\,\mu^i\,\mu^j \ge \eta|\mu|^2 \qquad \text{a. s.} \tag{36}$$

for all $\mu \in \mathbb{R}^s$ where η is a positive random variable. Rewrite (18) using (35) as

$$\frac{1}{T}\sum_{i,j} Q^{ij}(T)\,\left(\alpha^{\star i}(T)-\alpha^i\right)\,\left(\alpha^{\star j}(T)-\alpha^j\right) = \frac{1}{T}\sum_{i} L^i(T)\,\left(\alpha^{\star i}(T)-\alpha^i\right) \tag{37}$$

where

$$L^i(T) = \int_0^T (f_i\,X(t)+g_i\,U(t))'\,\ell\,\mathrm{d}W(t). \tag{38}$$

The stochastic integral (38) can be expressed as a random time change of a Wiener process so the Strong Law of Large Numbers for a Wiener process and (iv) imply that

$$\lim_{T\to\infty}\frac{1}{T}L^i(T) = 0 \qquad \text{a. s.} \tag{39}$$

for $i = 1, 2, \ldots, s$. Using (36), (37) it follows that there is a random time τ such that for $T > \tau$

$$\frac{\eta}{2}\,|\alpha^\star(T)-\alpha|^2 \le \frac{1}{T}\sum_i L^i(T)\,\left(\alpha^{\star i}(T)-\alpha^i\right) \qquad \text{a. s.}$$

This inequality and (39) verify (17). □

(Received June 21, 1991.)

REFERENCES

[1] M. Boschková: Self-tuning control of stochastic systems in presence of drift. Kybernetika *24* (1988), 5, 347–362.
[2] T. E. Duncan and B. Pasik–Duncan: Adaptive control of continuous-time linear stochastic systems. Math. Control Signals Systems *3* (1990), 45–60.
[3] P. Mandl, T. E. Duncan and B. Pasik–Duncan: On the consistency of a least squares identification procedure. Kybernetika *24* (1988), 5, 340–346.
[4] P. Mandl: Consistency of estimators in controlled systems. In: Stochastic Differential Systems (N. Christopeit, K. Helmes and M. Kohlmann, eds.), Springer-Verlag, Berlin – Heidelberg 1989, pp. 227–234.

RNDr. Petr Mandl, DrSc., matematicko-fyzikální fakulta University Karlovy (Faculty of Mathematics and Physics – Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.

Professor Tyrone E. Duncan, Ph.D., Professor Dr. hab. Bożenna Pasik–Duncan, Department of Mathematics, The University of Kansas, Lawrence, KS 66 045. U.S.A.