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CHI-SQUARED GOODNESS-OF-FIT TEST FOR THE FAMILY OF LOGISTIC DISTRIBUTIONS

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Chi-squared goodness-of-fit test for the family of logistic distributions is proposed. Different methods of estimation of the unknown parameters θ of the family are compared. The problem of homogeneity is considered.

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be independent identically distributed random variables and suppose that according to the hypothesis H_0

$$\mathsf{P}\{X_i \le x\} = F(x; \boldsymbol{\theta}), \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)^{\mathrm{T}} \in \Theta \subset \mathbb{R}^s, \quad x \in \mathbb{R}^1, \tag{1}$$

where Θ is an open set. We divide the real line into k intervals $I_1, \ldots, I_k$:

$$I_1 \cup \ldots \cup I_k = \mathbb{R}^1, \quad I_i \cap I_j = \emptyset, \quad i \neq j.$$

We shall suppose that

$$p_i(\boldsymbol{\theta}) = \mathsf{P}\{X_1 \in I_i \mid H_0\} > 0, \qquad i = 1, \ldots, k. \tag{2}$$

Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^{\mathrm{T}}$ be the vector of frequencies arising as a result of grouping the random variables $X_1, \ldots, X_n$ into the classes $I_1, \ldots, I_k$. We denote

$$\mathbf{X}_n^2(\boldsymbol{\theta}) = \mathbf{X}_n^{\mathrm{T}}(\boldsymbol{\theta}) \mathbf{X}_n(\boldsymbol{\theta}) = \sum_{i=1}^{k} \frac{(\nu_i - np_i(\boldsymbol{\theta}))^2}{np_i(\boldsymbol{\theta})}, \tag{3}$$

where

$$\mathbf{X}_n = \left(\frac{\nu_1 - np_1(\boldsymbol{\theta})}{\sqrt{np_1(\boldsymbol{\theta})}}, \ldots, \frac{\nu_k - np_k(\boldsymbol{\theta})}{\sqrt{np_k(\boldsymbol{\theta})}} \right)^{\mathrm{T}}. \tag{4}$$

Following Cramer [5] we suppose that

1. $p_i(\boldsymbol{\theta}) > c > 0$, $i = 1, \ldots, k$ $(k \ge s + 2)$;

2. $\dfrac{\partial^2 p_i(\boldsymbol{\theta})}{\partial\theta_j \partial\theta_\ell}$ are continuous functions;

3. the information matrix of Fisher

$$\mathbf{J} = \mathbf{J}(\boldsymbol{\theta}) = \left[\sum_{\ell=1}^{k} \frac{1}{p_\ell(\boldsymbol{\theta})} \frac{\partial p_\ell(\boldsymbol{\theta})}{\partial\theta_i} \frac{\partial p_\ell(\boldsymbol{\theta})}{\partial\theta_j}\right]_{s\times s} = \mathbf{B}^{\mathrm{T}}(\boldsymbol{\theta})\,\mathbf{B}(\boldsymbol{\theta}) \tag{5}$$

exists, and $\operatorname{rank}\mathbf{J} = s$, where

$$\mathbf{B}(\boldsymbol{\theta}) = \left[\frac{1}{\sqrt{p_\ell(\boldsymbol{\theta})}} \frac{\partial p_\ell(\boldsymbol{\theta})}{\partial\theta_j}\right]_{k\times s}. \tag{6}$$

In this case $n\mathbf{J}$ is the information matrix of Fisher of the statistic $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)^{\mathrm{T}}$.

Let $\tilde{\boldsymbol{\theta}}_n$ be the minimum chi-squared estimator for $\boldsymbol{\theta}$,

$$\mathbf{X}_n^2(\tilde{\boldsymbol{\theta}}_n) = \min_{\boldsymbol{\theta}\in\,\Theta} \mathbf{X}_n^2(\boldsymbol{\theta}), \tag{7}$$

or an estimator asymptotically equivalent to it.

Theorem (Fisher [11], Cramer [5]). If the regularity conditions of Cramer hold then

$$\lim_{n\to\infty} \mathsf{P}\{\mathbf{X}_n^2(\tilde{\boldsymbol{\theta}}_n) \geq x \mid H_0\} = \mathsf{P}\{\chi^2_{k-s-1} \geq x\}. \tag{8}$$

The limit distribution χ^2_{k-s-1} *can only be used* if $\tilde{\boldsymbol{\theta}}_n$ is the minimum chi-squared estimator or an asymptotically equivalent estimator. Thus, see Cramer [5], one can use the root of the system:

$$\sum_{i=1}^{k} \frac{\nu_i}{np_i(\boldsymbol{\theta})} \frac{\partial p_i(\boldsymbol{\theta})}{\partial\theta_j} = 0, \quad j = 1, \dots, s. \tag{9}$$

The problem of finding the root of (9) is usually difficult, so as an approximation to the value of $\boldsymbol{\theta}$ one often uses the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_n$, calculated to the non-grouped data $X_1, X_2, \dots, X_n$. *It is important to remark that when $\boldsymbol{\theta}$ is unknown and we have to estimate it, the limit distribution of the Pearson's statistic $\mathbf{X}_n^2(\boldsymbol{\theta})_n^*$ changes in accordance with asymptotical properties of an estimator $\boldsymbol{\theta}_n^*$.*

For example, if $\boldsymbol{\theta}_n^* = \hat{\boldsymbol{\theta}}_n$, then under certain regularity conditions we have the next

Theorem (Chernoff and Lehmann [4]).

$$\lim_{n\to\infty} \mathsf{P}\{\mathbf{X}_n^2(\hat{\boldsymbol{\theta}}_n) \geq x \mid H_0\} = \mathsf{P}\{\chi^2_{k-s-1} + \sum_{i=1}^{s} \lambda_i \xi_i^2 \geq x\},$$

where χ^2_{r-s-1}, $\xi_1, \ldots, \xi_s$ are independent, $\xi_i \sim N(0,1)$ and $\lambda_i = \lambda_i(\boldsymbol{\theta})$, $0 < \lambda_i(\boldsymbol{\theta}) <$ < 1, $i = 1, 2, \ldots, s$, are the roots of the equation

$$\mid (1-\lambda)\mathbf{I}(\boldsymbol{\theta}) - \mathbf{J}(\boldsymbol{\theta}) \mid = 0,$$

$\mathbf{I}(\boldsymbol{\theta})$ – the information matrix of Fisher, corresponding to one observation X_i.

Remark 1. We note here that in continuous case $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^{\mathrm{T}}$ is not sufficient statistic, and hence the matrix $\mathbf{I}(\boldsymbol{\theta}) - \mathbf{J}(\boldsymbol{\theta})$ is positively definite.

Remark 2. Let us consider the density family

$$f(x;\boldsymbol{\theta}) = h(x)\exp\{\sum_{m=1}^{s} \theta_m x^m + v(\boldsymbol{\theta})\}, \quad x \in \mathcal{X} \subseteq \mathbb{R}^1,$$

$\mathcal{X}$ is open in $\mathbb{R}^1$, $\mathcal{X} = \{x : f(x;\boldsymbol{\theta}) > 0\}$, $\boldsymbol{\theta} \in \Theta$.

This family is very rich: it contains Poisson, normal distributions etc. It is evident that

$$\mathbf{U}_n = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2, \ldots, \sum_{i=1}^{n} X_i^s\right)^{\mathrm{T}}$$

is the complete minimal sufficient statistic for $\boldsymbol{\theta}$.

We suppose that

1. the support $\mathcal{X}$ does not depend on $\boldsymbol{\theta}$;
2. the matrix of Hessen

$$\mathbf{H}_v(\boldsymbol{\theta}) = -\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} v(\boldsymbol{\theta})\right]_{s\times s}$$

   of the function $v(\boldsymbol{\theta})$ is positively definite;
3. the moments $a_s(\boldsymbol{\theta}) = \mathsf{E}_{\boldsymbol{\theta}} X_1^s$ exist.

In this case, using the results of Zacks [22], it is not difficult to show (see, for example, [7] – [10]) that the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_n = \hat{\boldsymbol{\theta}}_n(\mathbf{U}_n)$ and the method of moments estimator $\bar{\boldsymbol{\theta}}_n = \bar{\boldsymbol{\theta}}_n(\mathbf{U}_n)$ of $\boldsymbol{\theta}$ coincide, i. e. $\bar{\boldsymbol{\theta}}_n = \hat{\boldsymbol{\theta}}_n$. Let

$$\mathbf{a}(\boldsymbol{\theta}) = (a_1(\boldsymbol{\theta}), \ldots, a_s(\boldsymbol{\theta}))^{\mathrm{T}} \quad \text{and} \quad \mathbf{T}_n = \frac{1}{n}\mathbf{U}_n.$$

One can verify that

$$\mathbf{a}(\boldsymbol{\theta}) = -\frac{\partial}{\partial\boldsymbol{\theta}} v(\boldsymbol{\theta}),$$

and hence the likelihood equation is $\mathbf{T}_n = \mathbf{a}(\boldsymbol{\theta})$, i.e. $\hat{\boldsymbol{\theta}}_n$ is a root of this equation. On the other hand we have $\mathsf{E}_{\boldsymbol{\theta}}\mathbf{T}_n \equiv \mathbf{a}(\boldsymbol{\theta})$, and hence from the properties of the statistic $\mathbf{U}_n$ it follows that $\mathbf{T}_n$ is the MVUE of $\mathbf{a}(\boldsymbol{\theta})$, and $\bar{\boldsymbol{\theta}}_n$ is the root of the same equation $\mathbf{T}_n = \mathbf{a}(\boldsymbol{\theta})$, which we used to find $\hat{\boldsymbol{\theta}}_n$. Hence $\hat{\boldsymbol{\theta}}_n = \bar{\boldsymbol{\theta}}_n$. We remark that in general *an estimator based on the method of moments is not asymptotically efficient, and hence does not satisfy the Chernoff–Lehmann theorem.* In "Handbook of the Logistic Distribution" (edited by N. Balakrishnan [2]), Chap. 13, it is reported that the Dahiya–Gurland [6] extension of the Chernoff–Lehmann theorem is applied by Massaro and d'Agostino to construct the chi-squared test of Pearson with random intervals for the family of the logistic distributions using $\bar{\boldsymbol{\theta}}_n = (\bar{\mathbf{X}}_n, s_n^2)^{\mathrm{T}}$ (the moment method estimator of $\boldsymbol{\theta} = (\mathsf{E}X_1, \mathsf{Var}X_1)^{\mathrm{T}}$), as it was done by Dahiya and Gurland [6] for testing the normality (we note that in the normal case the method of moments and the maximum likelihood method are equivalent). But $\bar{\boldsymbol{\theta}}_n$ is not efficient and even not asymptotically efficient for the logistic family, since this family does not belong to the exponential family and $(\bar{\mathbf{X}}_n, s_n^2)^{\mathrm{T}}$ is not sufficient statistic in this situation. Hence, the tables of critical points, proposed by Massaro et d'Agostino in Section 13.9 are not valid. For this reason it is necessary to have a statistic which limit distribution is well known when we apply the maximum likelihood estimator or anyone BAN estimator. In the papers of Nikulin [15, 16, 17] (see also, for example, Rao and Robson [20], Moore and Spruill [14]), is exposed how to construct a chi-squared test for a continuous distribution, based on the statistic $\mathbf{Y}_n^2(\boldsymbol{\theta}_n^*)$, we shall define it in Section 3. We note that the technique of chi-squared tests for the exponential family of distributions of rank one, $s = 1$, and some applications of MVUE's were exposed by Nikulin and Voinov [18], Voinov and Nikulin [21].

2. LOGISTIC DISTRIBUTION AND THE CHI–SQUARED GOODNESS–OF–FIT TEST

Let $\mathbb{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ be a random sample, i.e. $X_1, \ldots, X_n$ are independent identically distributed random variables. In this section we consider the problem of testing the hypothesis H_0 that the distribution function of X_1 belongs to the family of logistic distributions $G\left(\frac{x-\mu}{\sigma}\right)$ depending on the shift parameter μ and the scale parameter σ:

$$\mathsf{P}\{X_1 \le x \mid H_0\} = G\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{1+\exp\{-\frac{\pi}{\sqrt{3}}\left(\frac{x-\mu}{\sigma}\right)\}}, \quad x \in \mathbb{R}^1, \quad (10)$$

$$\mu = \mathsf{E}\{X_1 \mid H_0\}, \quad |\,\mu\,| < \infty, \quad \sigma^2 = \mathsf{Var}X_1, \quad \sigma > 0.$$

Under H_0 the density function of X_i is

$$\frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right) = G'\left(\frac{x-\mu}{\sigma}\right) = \frac{\pi}{\sqrt{3}\sigma} \frac{\exp\left(-\frac{\pi}{\sqrt{3}}\frac{x-\mu}{\sigma}\right)}{\left[1+\exp\left(-\frac{\pi}{\sqrt{3}}\frac{x-\mu}{\sigma}\right)\right]^2}, \quad x \in \mathbb{R}^1. \quad (11)$$

We point out that "Handbook of the Logistic Distribution" [1], was published recently about the theory, the methodology and some applications of the family of logistic distributions.

We denote $\boldsymbol{\theta} = (\mu, \sigma)^{\mathrm{T}}$, and let $\hat{\boldsymbol{\theta}}_n = (\hat{\mu}_n, \hat{\sigma}_n)^{\mathrm{T}}$ be the maximum of likelihood estimator of $\boldsymbol{\theta}$. Since there is no any other sufficient statistic for $\boldsymbol{\theta}$ than the trivial one $\mathbb{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$, the maximum likelihood equation has no explicit root. Balakrishnan and Cohen [2] proposed an approximate solution of the maximum likelihood equations based on a "type II censored sample" of Harter and Moore [12]. They proved that this approximate solution gives an asymptotically efficient estimator, i. e. asymptotically equivalent to $\hat{\boldsymbol{\theta}}_n$.

Let $\hat{\boldsymbol{\theta}}_n$ be such an estimator. The limit covariance matrix of the random vector $\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ will be $\mathbf{I}^{-1}$, where

$$\mathbf{I} = \frac{1}{\sigma^2}\,[I_{ij}]_{2\times 2} = \frac{1}{9\sigma^2}\begin{bmatrix} \pi^2 & 0 \\ 0 & \pi^2+3 \end{bmatrix}, \tag{12}$$

$$I_{11} = \int_{-\infty}^{+\infty} \left[\tfrac{g'(x)}{g(x)}\right]^2 g(x))\,\mathrm{d}x = \frac{\pi^2}{9},$$

$$I_{22} = \int_{-\infty}^{+\infty} x^2 \left[\tfrac{g'(x)}{g(x)}\right]^2 g(x)\,\mathrm{d}x - 1 = \frac{\pi^2+3}{9},$$

and since $g(x)$ is symmetric

$$I_{12} = I_{21} = \int_{-\infty}^{+\infty} x \left[\tfrac{g'(x)}{g(x)}\right]^2 g(x)\,\mathrm{d}x = 0.$$

Let us fix the vector $\mathbf{p} = (p_1, p_2, \ldots, p_k)^{\mathrm{T}}$ of positive probabilities such that

$$p_1 = \ldots = p_k = 1/k, \tag{13}$$

and let $y_0 = -\infty,\ y_k = +\infty$,

$$y_i = G^{-1}(p_1 + \ldots + p_i) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{i}{k-i}\right), \qquad i = 1, \ldots, k-1. \tag{14}$$

Further, let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_k)^{\mathrm{T}}$ be the frequency vector arising from grouping $X_1, \ldots, X_n$ over the intervals with random ends

$$(-\infty, z_1], (z_1, z_2], \ldots, (z_{k-1}, +\infty), \qquad \text{where} \qquad z_i = z_i(\hat{\boldsymbol{\theta}}_n) = \hat{\mu}_n + \hat{\sigma}_n y_i, \tag{15}$$

and let

$$\mathbf{a} = (a_1, \ldots, a_k)^{\mathrm{T}}, \quad \mathbf{b} = (b_1, \ldots, b_k)^{\mathrm{T}}, \quad \mathbf{W}^{\mathrm{T}} = -\frac{1}{\sigma}\left[\mathbf{a} \vdots \mathbf{b}\right], \tag{16}$$

where for $i = 1, 2, \ldots, k$

$$
\begin{aligned}
a_i &= g(y_i) - g(y_{i-1}) = \frac{\pi}{k^2\sqrt{3}}(k - 2i + 1), \\
b_i &= y_i g(y_i) - y_{i-1} g(y_{i-1}) = \\
&= \frac{1}{k^2}\left[(i-1)(k-i+1)\ln\frac{k-i+1}{i-1} - i(k-i)\ln\frac{k-i}{i}\right], \\
\alpha(\boldsymbol{\nu}) &= k\sum_{i=1}^{k} a_i\nu_i = \frac{\pi}{\sqrt{3}k}\left[(k+1)n - 2\sum_{i=1}^{k} i\nu_i\right], \qquad (17) \\
\beta(\boldsymbol{\nu}) &= k\sum_{i=1}^{k} b_i\nu_i = \frac{1}{k}\sum_{i=1}^{k-1}(\nu_{i+1} - \nu_i)i(k-i)\ln\frac{k-i}{i}, \qquad (18) \\
\lambda_1 &= I_{11} - k\sum_{i=1}^{k} a_i^2 = \frac{\pi^2}{9k^2}, \qquad \lambda_2 = I_{22} - k\sum_{i=1}^{k} b_i^2. \qquad (19)
\end{aligned}
$$

Since g is symmetric we have $a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k = 0$. Let

$$\mathbf{B} = \mathbf{D} - \mathbf{p}\mathbf{p}^{\mathrm{T}} - \mathbf{W}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{W}, \qquad (20)$$

where $\mathbf{D}$ is the diagonal matrix with the elements $1/k$ on the main diagonal. The matrix $\mathbf{B}$ does not depend on $\boldsymbol{\theta}$, and $\operatorname{rank}\mathbf{B} = k - 1$, i. e. the matrix $\mathbf{B}$ is singular, while the matrix $\tilde{\mathbf{B}}$, obtained as a result of deleting the last row and column in $\mathbf{B}$, has an inverse

$$\tilde{\mathbf{B}}^{-1} = \mathbf{A} + \mathbf{A}\tilde{\mathbf{W}}^{\mathrm{T}}(\mathbf{I} - \tilde{\mathbf{W}}\mathbf{A}\tilde{\mathbf{W}}^{\mathrm{T}})^{-1}\tilde{\mathbf{W}}\mathbf{A}, \qquad (21)$$

where $\mathbf{A} = \tilde{\mathbf{D}}^{-1} + \mathbf{1}\mathbf{1}^{\mathrm{T}}/p_k$, $\tilde{\mathbf{D}}^{-1}$ is a diagonal matrix with elements $\frac{1}{p_1}, \ldots, \frac{1}{p_{k-1}}$ on the main diagonal, $\mathbf{1} = \mathbf{1}_{k-1}$ is the vector of dimension $(k-1)$, all elements of which are equal to 1, $\tilde{\mathbf{W}}$ is a matrix obtained from $\mathbf{W}$ by deleting the last column. Since the vector $\tilde{\boldsymbol{\nu}} = (\nu_1, \ldots, \nu_{k-1})^{\mathrm{T}}$ is asymptotically normally distributed with parameters

$$\mathsf{E}\tilde{\boldsymbol{\nu}} = n\tilde{\mathbf{p}} + O(\sqrt{n}\mathbf{1}_s) \quad \text{and} \quad \mathsf{E}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}})^{\mathrm{T}}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}}) = n\tilde{\mathbf{B}} + O(\sqrt{n}\mathbf{1}_{s\times s}), \qquad (22)$$

where $\tilde{\mathbf{p}} = (p_1, \ldots, p_{k-1})^{\mathrm{T}}$, we obtain the next result

Theorem 1. The statistic

$$\mathbf{Y}_n^2 = \frac{1}{n}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}})^{\mathrm{T}}\tilde{\mathbf{B}}^{-1}(\tilde{\boldsymbol{\nu}} - n\tilde{\mathbf{p}}) = \mathbf{X}_n^2 + \frac{\lambda_1\beta^2(\boldsymbol{\nu}) + \lambda_2\alpha^2(\boldsymbol{\nu})}{n\lambda_1\lambda_2} \qquad (23)$$

has, as $n \to \infty$, chi-squared limit distribution with $(k-1)$ degrees of freedom, where

$$\mathbf{X}_n^2 = \sum_{i=1}^{k}\frac{(\nu_i - np_i)^2}{np_i} = \frac{k}{n}\sum_{i=1}^{k}\nu_i^2 - n. \qquad (24)$$

Remark. We consider the hypothesis H_η, according to which X_i follows $G(\frac{x-\mu}{\sigma}, \eta)$, where $G(x,\eta)$ is continuous, $| x |< \infty$, $\eta \in \mathbf{H} \subset \mathbb{R}^1$, $G(x,0) = G(x)$, and $\eta = 0$ is a limit point of $\mathbf{H}$. Let us assume also, that

$$\frac{\partial}{\partial x} G(x,\eta) = g(x,\eta) \quad \text{and} \quad \frac{\partial}{\partial \eta} g(x,\eta) \mid_{\eta=0} = \psi(x) \tag{25}$$

exist, where $g(x,0) = g(x) = G'(x)$. In this case if $\frac{\partial^2 g(x,\eta)}{\partial \eta^2}$ exists and is continuous for all x in the neighbourhood of the $\eta = 0$, then for $z_i = y_i\sigma + \mu$ we have

$$\mathsf{P}\{z_{i-1} < X_i \le z_i \mid H_\eta\} = p_i + \eta c_i + o(\eta), \tag{26}$$

where

$$c_i = \int_{y_{i-1}}^{y_i} \Psi(x)\,\mathrm{d}x, \qquad i = 1,\dots,k, \tag{27}$$

and finally, in the limit, as $n \to \infty$, the statistic $\mathbf{Y}_n^2$ has noncentral chi-squared distribution with $(k-1)$ degrees of freedom and with non-centrality parameter λ:

$$\lim_{n\to\infty} \mathsf{P}\{\mathbf{Y}_n^2 \ge x \mid H_\eta\} = \mathsf{P}\{\chi^2_{k-1}(\lambda) \ge x\}, \tag{28}$$

where

$$\lambda = \sum_{i=1}^{k} \frac{c_i^2}{p_i} + \frac{\lambda_2\alpha^2(\mathbf{c}) + \lambda_1\beta^2(\mathbf{c})}{\lambda_1\lambda_2}, \qquad \mathbf{c} = (c_1, c_2, \dots, c_k)^t, \tag{29}$$

$\mathbf{p}$, $\alpha(\mathbf{c})$, $\beta(\mathbf{c})$, λ_1, λ_2 are given by (13), (17), (18), (19) respectively.

3. HOMOGENEITY TEST

Let us consider the problem of homogeneity of two samples in the case of the family of logistic distributions, following the paper of Bolshev and Nikulin [3].

Let us suppose that $\mathbb{X}_1 = (X_{11},\dots,X_{1n_1})^{\mathrm{T}}$ and $\mathbb{X}_2 = (X_{21},\dots,X_{2n_2})$ such that

$$\mathsf{P}\{X_{1i} \le x\} = G\left(\frac{x-\mu_1}{\sigma_1}\right) \quad \text{and} \quad \mathsf{P}\{X_{2i} \le x\} = G\left(\frac{x-\mu_2}{\sigma_2}\right), \qquad x \in \mathbb{R}^1, \tag{30}$$

where $| \mu_i |< \infty$, $\sigma_i > 0$, μ_i, σ_i are unknown, $i = 1, 2$; $G\left(\frac{x-\mu}{\sigma}\right)$ is given by (35). We wish to test the hypothesis H_0 according to which $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$, i.e. under H_0

$$X_{ij} \sim G\left(\frac{x-\mu}{\sigma}\right),$$

for some μ and σ. Under H_0 we can find the maximum likelihood estimator $\hat{\boldsymbol{\theta}}_N = (\hat\mu_N, \hat\sigma_N^2)^T$ of $\boldsymbol{\theta} = (\mu,\sigma^2)^{\mathrm{T}}$ obtained from all $N = n_1+n_2$ observations $X_{11},\dots,X_{1n_1}$, $X_{21},\dots,X_{2n_2}$.

Further, let $\mathbf{p} = (p_1,\dots,p_k)^{\mathrm{T}}$, $p_i = \frac{1}{k}$, $i = 1,\dots,k$ and $\boldsymbol{\nu}_i = (\nu_{i1},\dots,\nu_{ik})^{\mathrm{T}}$ be the vector of frequencies obtained by grouping the sample $\mathbb{X}_i$, $(i = 1, 2)$ using the intervals $[z_{j-1}(\hat{\boldsymbol{\theta}}_N), z_j(\hat{\boldsymbol{\theta}}_N)]$, as in (15), where $z_0 = -\infty$, $z_k = +\infty$, $z_j = \hat\sigma_N y_j + \hat\mu_N$,

$y_j = G^{-1}\left(\frac{j}{k}\right)$, and let $\boldsymbol{\nu} = \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$, $\mathbf{a}, \mathbf{b}, \mathbf{W}$, $\mathbf{B}_i$, $\mathbf{D}, \mathbf{I}$, $\alpha(\boldsymbol{\nu})$, $\beta(\boldsymbol{\nu})$, λ_1, λ_2, $\tilde{\boldsymbol{\nu}}$, $\tilde{\boldsymbol{\nu}}_1$, $\tilde{\boldsymbol{\nu}}_2$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{W}}$ as in (12)–(22),

$$\tilde{\mathbf{B}}_i = \tilde{\mathbf{D}} - \tilde{\mathbf{p}}\tilde{\mathbf{p}}^{\mathrm{T}} - \frac{n_i}{N}\tilde{\mathbf{W}}^{\mathrm{T}}\mathbf{I}^{-1}\tilde{\mathbf{W}}, \qquad \boldsymbol{\Delta}_i = \frac{1}{\sqrt{n_i}}(\tilde{\boldsymbol{\nu}}_i - n_i\tilde{\mathbf{p}}), \qquad i = 1, 2.$$

Theorem 2. Under H_0 the vector $\boldsymbol{\Delta} = (\boldsymbol{\Delta}_1^{\mathrm{T}}, \boldsymbol{\Delta}_2^{\mathrm{T}})^{\mathrm{T}}$ is asymptotically normally distributed when $\min(n_1, n_2) \to \infty$ with $\mathsf{E}\boldsymbol{\Delta} = \mathbf{0}_{2s}$ and covariance matrix

$$\mathbf{U} = \begin{bmatrix} \tilde{\mathbf{B}}_1 & -\frac{\sqrt{n_1 n_2}}{N}\tilde{\mathbf{W}}^{\mathrm{T}}\mathbf{I}^{-1}\tilde{\mathbf{W}} \\ -\frac{\sqrt{n_1 n_2}}{N}\tilde{\mathbf{W}}^{\mathrm{T}}\mathbf{I}^{-1}\tilde{\mathbf{W}} & \tilde{\mathbf{B}}_2 \end{bmatrix}.$$

Theorem 3. Under H_0 the statistic

$$\mathbf{Y}_n^2 = \boldsymbol{\Delta}^{\mathrm{T}}\mathbf{U}^{-1}\boldsymbol{\Delta} =$$

$$\sum_{i=1}^{k}\frac{(\nu_{1i} - n_1 p_i)^2}{n_1 p_i} + \sum_{i=1}^{k}\frac{(\nu_{2i} - n_2 p_i)^2}{n_2 p_i} + \frac{1}{N\lambda_1\lambda_2}\{\lambda_1\beta^2(\boldsymbol{\nu}) + \lambda_2\alpha^2(\boldsymbol{\nu})\}.$$

has, in the limit as $\min(n_1, n_2) \to \infty$, a chi-squared distribution with $2(k-1)$ degrees of freedom:

$$\lim_{n\to\infty} \mathsf{P}\{\mathbf{Y}_n^2 \geq x \mid H_0\} = \mathsf{P}\{\chi^2_{2(k-1)} \geq x\}.$$

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