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PROTO–METRIZABLE FUZZY TOPOLOGICAL SPACES

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In this paper we define for fuzzy topological spaces a notion corresponding to proto-metrizable topological spaces. We obtain some properties of these fuzzy topological spaces, particularly we give relations with non-archimedean, and metrizable fuzzy topological spaces.

1. INTRODUCTION AND DEFINITIONS

Proto-metrizable topological spaces were introduced by P. J. Nyikos in [11]. A topological space X is proto-metrizable if it is paracompact and has an ortho-base $\mathcal{B}$ (i. e. $\mathcal{B}$ is a base for X, if $\mathcal{B}' \subset \mathcal{B}$ then either $\bigcap_{B\in\mathcal{B}'} B$ is open or $\bigcap_{B\in\mathcal{B}'} B$ is a point x, and $\mathcal{B}'$ is a base for the neighborhoods of x). This class of spaces contains all non-archimedean topological spaces and all metrizable topological space. Many of the metrization theorems for non-archimedean topological spaces are also true for proto-metrizable spaces.

G. Gruenhage and P. Zenor [6] have found a remarkable characterization of proto-metrizable topological spaces.

Definition 1.1. [6] Let X be a topological space. A pair-base for X is a collection of ordered pairs of open sets (B_1, B_2) such that $B_1 \subset B_2$ and such that if x is contained in a open set U, there exists (B_1, B_2) such that $x \in B_1 \subset B_2 \subset U$. A pair-base is of rank 1 if for any two members (B_1, B_2) and (B_1', B_2') whose smaller sets meet $B_1 \cap B_1' \neq \emptyset$, either $B_1 \subset B_2'$ or $B_1' \subset B_2$.

The theorem of Gruenhage and Zenor establish that topological spaces with rank 1 pair-bases coincide with proto-metrizable topological spaces.

The study on proto-metrizable topological spaces has been continued by Nyikos and Fuller [5, 11, 12].

We introduce, in this paper, the notion of proto-metrizable fuzzy topological spaces. We obtain some properties of these fuzzy topological spaces, particularly we give relations with non-archimedean and metrizable fuzzy topological spaces.

We use the category of the fuzzy topological spaces in the Chang's sense.

Next, we list some definitions which we will use in this paper.

Definition 1.2. [8] Let $r \in (0,1]$. We say that a fuzzy topological space (X,T) is r-paracompact (resp. r^*-paracompact) if for each r-open Q -cover of X there exists an open refinement of it which is both locally finite (resp. $*$-locally finite) in X and an r-Q-cover of X. The fuzzy topological space (X,T) is called S-paracompact (resp. S^* -paracompact) if for every $r \in (0,1]$, X is r-paracompact (resp. r^*-paracompact).

Definition 1.3. [4] Let μ be a fuzzy set in a fuzzy topological space (X,T). We say that μ is fuzzy paracompact (resp. $*$-fuzzy paracompact) if for each open in the Lowen's sense, $\mathcal{U}$ of μ, and for each $\xi \in (0,1]$ there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is both locally finite (resp. $*$-locally finite) in μ and cover of $\mu - \xi$ in the Lowen's sense. We say that a fuzzy topological space (X,T) is fuzzy paracompact (resp. $*$-fuzzy paracompact) if each constant fuzzy set in X is fuzzy paracompact (resp. $*$-fuzzy paracompact).

Definition 1.4. [9] A base $\mathcal{B}$ for a fuzzy topological space (X,T) such that every subfamily $\mathcal{B}$' satisfies either $\bigwedge_{B\in\mathcal{B}'} B$ is open fuzzy, or $\bigwedge_{B\in\mathcal{B}'}$ is a fuzzy point and $\mathcal{B}'$ is a base for the Q-neighborghoods of it, is called a fuzzy ortho-base for (X,T).

Definition 1.5. [9] A fuzzy topological space such that every open fuzzy covering of it has a locally finite refinement by clopen fuzzy sets, is called an ultraparacompact fuzzy topological space.

Definition 1.6. [9] A fuzzy topological space (X,T) is said to be non-archimedean fuzzy if there exists a base $\mathcal{B}$ for T by clopen crisp fuzzy sets such that for any pair B_1, B_2 of quasi-coincident members of $\mathcal{B}$, we have either $B_1 \subset B_2$ or $B_2 \subset B_1$.

Definition 1.7. [10] A fuzzy topological space (X,T) is weakly induced by a topological space (X,T_0) if:

(a) $T_0 = [T]$, where $[T] = \{A \subset X | \chi_A \in T\}$ and

(b) every $\mu \in T$ is a lower semicontinuous function from (X,T_0) into $[0,1]$.

Definition 1.8. [7] A fuzzy extension of a topological property is said to be good, when it is possessed by (X,T) if, and only if, the original property is possessed by $(X,[T])$.

2. PROTO–METRIZABLE FUZZY TOPOLOGICAL SPACES

Definition 2.1. Let (X,T) be a fuzzy topological space. We will say that (X,T) is proto-metrizable fuzzy if it is paracompact fuzzy and there exists an ortho-base fuzzy for it.

Proposition 2.1. Every ultraparacompact fuzzy topological space is paracompact fuzzy.

P r o o f. For each $\alpha \in (0,1]$ and for each family $\mathcal{U}_\alpha$ of open fuzzy sets such that $\alpha 1_X \leq \bigvee_{U\in\mathcal{U}_\alpha} U$, by the hypothesis, there exists a locally finite refinement $\mathcal{V}_1$ by clopen fuzzy sets which verifies $1_X \leq \bigvee_{V\in\mathcal{V}_1} V$ (and also $1_X - \xi \leq \bigvee_{V\in\mathcal{V}_1} V$ for all $\xi \in (0,1]$).

If $\alpha \in (0,1)$, for each $U \in \mathcal{U}_\alpha$, let $U^*(x) = \min\{\frac{1}{\alpha}U(x), 1\}$, then $1_X \leq \bigvee_{U\in\mathcal{U}_\alpha} U^*$. For $r \in [0,1)$, we have that

$$\begin{aligned} U^{*-1}((r,1]) &= \left\{x\in X|\min\{\frac{1}{\alpha}U(x),1\}>r\right\} = \left\{x\in X|\frac{1}{\alpha}U(x)>r,\ \text{if}\ \frac{1}{\alpha}U(x)\leq 1\right\} \\ &= \begin{cases} U^{-1}((\alpha r,1]), & \text{when } \alpha r < 1 \\ X, & \text{when } \alpha r \geq 1 \text{ (because if } \frac{1}{\alpha}U(x)\leq 1, \text{ then} \\ & U(x) > \alpha r \geq 1 \text{ and this is contradictory).} \end{cases} \end{aligned}$$

Then U^* is open fuzzy for each $U \in \mathcal{U}_\alpha$. By the hypothesis, there exists a locally finite refinement $\mathcal{V}_\alpha$ by clopen fuzzy sets, such that $1_X \leq \bigvee_{V\in\mathcal{V}_\alpha} V$ (and also, $\alpha 1_X \leq \bigvee_{V\in\mathcal{V}_\alpha}(\alpha V)$ and $\alpha 1_X - \xi \leq \bigvee_{V\in\mathcal{V}_\alpha} \alpha V$ for all $\xi \in (0,1]\}$). Finally, $V \leq U^* \leq \frac{1}{\alpha}U$ implies that $\alpha V \leq U$. □

Corollary 2.2. Every non-archimedean fuzzy topological space is paracompact fuzzy.

P r o o f. It follows from the above proposition and [9, Proposition 3.9]. □

Proposition 2.3. Let (X,T) be a proto-metrizable fuzzy topological space, then $(X,[T])$ is proto-metrizable.

P r o o f. Since (X,T) is paracompact fuzzy and has an ortho-base fuzzy, we have that $(X,[T])$ is paracompact, [2, 4] and it has an ortho-base because, if $\mathcal{B}$ is ortho-base fuzzy for (X,T), thus $\mathcal{B}^* = \{A \subset X|\chi_A \in \mathcal{B}\}$ is a base for $(X,[T])$ and, if $\mathcal{B}' \subset \mathcal{B}^*$ then $\{\chi_B|B \in \mathcal{B}'\} \subset \mathcal{B}$ and either $\bigwedge_{B\in\mathcal{B}'}\chi_B = \chi_{\bigcap_{B\in\mathcal{B}'}B}$ is open fuzzy ($\Leftrightarrow \bigcap_{B\in\mathcal{B}'} B$ is open in $[T]$), on $\bigwedge_{B\in\mathcal{B}'}\chi_B = \chi_{\bigcap_{B\in\mathcal{B}'}B}$ is a fuzzy point x_1 and $\{\chi_B|B \in \mathcal{B}'\}$ is a base for the Q-neighborhoods of it ($\Leftrightarrow \bigcap_{B\in\mathcal{B}'} B$ is a point and $\mathcal{B}'$ is a base for the neighborhoods of it, because, $x \in A$ open in $[T] \Leftrightarrow x_1 \in \chi_A$ open fuzzy, if there exists χ_B such that $B \in \mathcal{B}'$ and $x_1 q \chi_B \subset \chi_A$, then $x \in B \subset A$). □

Corollary 2.4. Let (X,T) be a fuzzy topological space. If (X,T) is proto-metrizable fuzzy and c-connected, then it is metrizable fuzzy.

P r o o f. The c-connectedness is good extension of the connectedness, [1], then $(X,[T])$ is proto-metrizable (from the above proposition) and connected, thus, by [11, Theorem 3.2], $(X,[T])$ is metrizable and (X,T) is metrizable fuzzy [3]. □

Corollary 2.5. Let (X, T) be a fuzzy topological space. If (X, T) is proto-metrizable fuzzy and verifies some good extension of compactness, then it is metrizable fuzzy.

Proof. From Proposition 2.3, $(X, [T])$ is proto-metrizable, also it is compact, thus, by [11], $(X, [T])$ is metrizable and (X, T) is metrizable fuzzy [3]. $\square$

Definition 2.2. Let (X, T) be a fuzzy topological space. We will say fuzzy pair-base to a collection $\mathcal{B} = \{(\chi_{B_1}, \chi_{B_2}) | \chi_{B_1}$ open fuzzy, $\chi_{B_1} \le \chi_{B_2}$ and such that for every open fuzzy μ and for every crisp fuzzy point $x_1 \le \mu$, there exist (χ_{B_1}, χ_{B_2}) such that $x_1 \le \chi_{B_1} \le \chi_{B_2} \le \mu\}$.

A fuzzy pair-base is of rank 1 if for any two members (χ_{B_1}, χ_{B_2}), $(\chi_{B'_1}, \chi_{B'_2}) \in \mathcal{B}$ such that $\chi_{B_1} q \chi_{B_2}$ either $\chi_{B_1} \le \chi_{B'_2}$ or $\chi_{B'_1} \le \chi_{B_2}$.

Proposition 2.6. Let (X, T) be a fuzzy topological space with a fuzzy pair-base of rank 1, then $(X, [T])$ is a topological space with a pair-base of rank 1.

Proof. Let $\mathcal{B}$ be a fuzzy pair-base of rank 1 for (X, T), this $\mathcal{B}^* = \{(B_1, B_2) | (\chi_{B_1}, \chi_{B_2}) \in \mathcal{B}\}$ is pair-base of rank 1 for $(X, [T])$ because, for each $A \in [T]$ and each $x \in A$, we have χ_A open fuzzy and $x_1 \le \chi_A$, then there exists $(\chi_{B_1}, \chi_{B_2}) \in \mathcal{B}$ such that $x_1 \le \chi_{B_1} \le \chi_{B_2} \le \chi_A$, then $x \in B_1 \subset B_2 \subset A$. And, also, if $(B_1, B_2), (B'_1, B'_2) \in \mathcal{B}^*$ verify $B_1 \cap B'_1 \ne \emptyset$, then $\chi_{B_1} q \chi_{B'_1}$, thus, either $\chi_{B_1} \le \chi_{B'_2} (\Leftrightarrow B_1 \subset B'_2)$ or $\chi_{B'_1} \le \chi_{B_2}$ $(\Leftrightarrow B'_1 \subset B_2)$. $\square$

Corollary 2.7. Every fuzzy topological space with a fuzzy pair-base of rank 1 is paracompact fuzzy (and S-paracompact, and S^*-paracompact).

Proof. If (X, T) has a fuzzy pair-base of rank 1, by the above proposition and [6, Theorem 2.4] we have that $(X, [T])$ is proto-metrizable. Then $(X, [T])$ is paracompact, and (X, T) verifies all the good extensions of paracompactness. $\square$

Open problem. Is it possible to give a characterization of proto-metrizable fuzzy topological spaces using the notion of fuzzy pair-base of rank 1?, i. e., is the Gruenhage and Zenor's theorem extendable to fuzzy topological spaces?

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