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OPTIMAL CONTROL OF NONLINEAR DELAY SYSTEMS WITH IMPLICIT DERIVATIVE AND QUADRATIC PERFORMANCE

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The existence of optimal control for nonlinear delay systems having an implicit derivative with quadratic performance criteria is proved. The results are established by an iterative technique and using the Darbo fixed point theorem.

1. INTRODUCTION

The problem of optimal control of nonlinear systems with quadratic performance criteria has been studied by many authors [2, 3, 4, 5, 6, 8, 11] by means of fixed point principles. Malek–Zavarei [9] has established an iterative approach for obtaining the suboptimal control for linear systems with multiple state and control delays and with quadratic cost while Balachandran and Ramaswamy [5] have extended the iterative technique to nonlinear multiple-delay systems.

Dacka [7] has introduced a new method of analysis to study the controllability of nonlinear systems with an implicit derivative, based on the measure of non-compactness of a set and the Darbo fixed point theorem. This method has been extended to a larger class of nonlinear dynamical systems by Balachandran [2, 3]. In [3] he has proved the existence theorems for the optimal control of nonlinear multiple-delay systems by suitably adopting the techniques of Dacka [7] and Malek–Zavarei [9]. In this paper we shall extend the procedure of [3] to prove the existence of optimal control for nonlinear delay systems having an implicit derivative with quadratic performance criteria.

2. MATHEMATICAL PRELIMINARIES

Let $(X, \|\cdot\|)$ be a Banach space and E a bounded subset of X. In this work, the following definition of the measure of the non-compactness of a set E is used (Sadovskii [10]).

$$\mu(E) = \inf\{r > 0 : E \text{ can be covered by a finite number of balls whose radii are smaller than } r\}$$

The following version of the Darbo fixed point theorem being a generalization of the Schauder fixed point theorem shows the usefulness of the measure of non-compactness.

Theorem 1. If S is a non-empty bounded closed convex subset of X and $P : S \to S$ is a continuous mapping, such that for any set $E \subset S$ we have

$$\mu(PE) \leq b\mu(E),$$

where b is a constant, $0 \leq b < 1$, then P has a fixed point.

For the space of continuous functions $C(I; R^n)$ with norm

$$\|x\| = \{|x_i(t)| : i = 1, 2, \ldots, n : t \in [t_0, T] = I\}$$

the measure of non-compactness of a set E is given by

$$\mu(E) = \tfrac{1}{2}\, w_0(E) = \frac{1}{2} \lim_{h \to 0^+} w(E, h),$$

where $w(E, h)$ is the common modulus of continuity of the functions which belong to the set E, that is

$$w(E, h) = \sup_{x \in E} [\sup |x(t) - x(s)| : |t - s| \leq h]$$

and for the space of continuously differentiable functions $C^1(I; R^n)$ with norm

$$\|x\|_{C(I;R^n)} = \|x\|_{C(I;R^n)} + \|Dx\|_{C(I;R^n)}$$

we have

$$\mu(E) = \tfrac{1}{2}\, w_0(DE),$$

where

$$DE = \{\dot{x} : x \in E\}\,.$$

3. STATEMENT OF PROBLEM

Consider the nonliner delay system of the form

$$\dot{x}(t) = A(t)\, x(t) + \sum_{i=1}^{M} f_i(x(t - \delta_i)) + C(t)\, u(t)$$
$$+ \sum_{j=1}^{N} g_j(u(t - \tau_j)) + \sigma(x(t),\, \dot{x}(t),\, t), \quad t \geq t_0 \tag{1a}$$

$$x(t) = \theta(t), \quad t_0 - \Delta \leq t \leq t_0 \tag{1b}$$

$$u(t) = \alpha(t), \quad t_0 - \Gamma \le t \le t_0, \tag{1c}$$

where $x(t)$ and $u(t)$ are respectively, state and control vectors. Here $A(t)$ and $C(t)$ are real continuous matrices of appropriate dimensions defined on the appropriate interval; $f_i,\ i = 1, 2, 3, \dots, M,\ g_j,\ j = 1, 2, \dots, N$ are continuous functions defined on appropriate intervals; σ is a continuous function; t_0 is the initial process time; $\theta(t)$ and $\alpha(t)$ are specified initial functions; $\delta_i,\ i = 1, 2, \dots, M$, and $\tau_j,\ j = 1, 2, \dots, N$ are given positive scalars, and

$$\Delta = \max_i \delta_i \quad \text{and} \quad \Gamma = \max_j \tau_j .$$

Assume that the matrices $A(t)$ and $C(t)$ and the functions f_i and g_i are bounded on I and $N^* = \sup \|C(t)\|$. Moreover, the continuous function σ is bounded and for each $z,\ \overline{z} \in R^n$, and $t \in I$ we have

$$|\sigma(t, x, z) - \sigma(t, x, \overline{z})| \le b_1 |x - \overline{z}| \tag{2}$$

where b_1 is a non-negative constant such that $0 \le b_1 < \frac{1}{2}$.

The cost functional to be minimized is

$$J = \tfrac{1}{2}\, x'(T)\, F\, x(T) + \frac{1}{2} \int_{t_0}^{T} [x'(t)\, Q(t)\, x(t) + u'(t)\, R(t)\, u(t)]\, \mathrm{d}t, \tag{3}$$

where the prime denotes transposition; the matrix F is symmetric positive semi-definite; the matrix $Q(t)$ is symmetric positive semi-definite and continuous; and the matrix $R(t)$ is symmetric, positive definite and continuous. The problem is to find a control $u(t),\ t_0 \le t \le T$, which for fixed final time T and free final state $x(T)$ minimizes the cost functional J in equation (3).

4. EXISTENCE THEOREMS

The following theorem is important in obtaining an optimal control scheme for the problem under consideration.

Theorem 2. Consider the sequence of nonlinear state equations

$$\dot{x}_k(t) = A(t)\, x_k(t) + \sum_{i=1}^{M} f_i\left(x_{k-1}(t - \delta_i)\right) + C(t)\, u_k(t)$$
$$+ \sum_{j=1}^{N} g_j\left(u_{k-1}(t - \tau_j)\right) + \sigma(x_k(t),\, \dot{x}_k(t),\, t), \quad k = 1, 2, 3, \dots \tag{4a}$$

with

$$x_0(t) = \phi(t, t_0)\, \theta(t_0), \quad t \ge t_0 \tag{4b}$$

$$u_0(t) = \beta(t), \qquad t \geq t_0 \tag{4c}$$

$$x_k(t) = \theta(t), \quad t_0 - \Delta \leq t \leq t_0, \quad k = 0, 1, 2, 3, \ldots \tag{4d}$$

$$u_k(t) = \alpha(t), \quad t_0 - \Gamma \leq t \leq t_0, \quad k = 0, 1, 2, 3, \ldots \tag{4e}$$

and the sequence of associated cost functionals

$$J_k = \tfrac{1}{2}\, x_k'(T)\, F\, x_k(T) + \frac{1}{2}\int_{t_0}^{T} [x_k'(t)\, Q(t)\, x_k(t) + u_k'(t)\, R(t)\, u_k(t)]\, \mathrm{d}t, \quad k = 0, 1, 2, \ldots, \tag{5}$$

where $\beta(t)$ is an arbitrary continuous function and $\phi(t, s)$ is the state transition matrix corresponding to the matrix

$$A(t) - S(t)\, K(t), \quad \text{where} \quad S(t) = C(t)\, R^{-1}(t)\, C'(t) \tag{6}$$

and $K(t)$ is a symmetric positive definite solution of the matrix Riccati equation

$$\dot{K}(t) + K(t)\, A(t) + A'(t)\, K(t) - K(t)\, S(t)\, K(t) + Q(t) = 0 \tag{7a}$$

$$\text{with the terminal condition} \qquad K(T) = F. \tag{7b}$$

Suppose that for the kth optimization problem the optimal state trajectory is $x_k^*(t)$ and the optimal control is $u_k^*(t)$. If the sequences $\{x_k^*(t)\}$ and $\{u_k^*(t)\}$ converge uniformly to $x^*(t)$ and $u^*(t)$ respectively, then these are the optimal state and control for the optimal control problem given by equations (1) and (3).

Since the system is nonlinear we can not obtain the results directly from (4). Hence for each fixed k, $\{z_k\} \subset C^1(I; R^n)$, we shall consider the following fixed sequence of linear delay systems

$$\dot{x}_k(t) = A(t)\, x_k(t) + \sum_{i=1}^{M} f_i(x_{k-1}(t - \delta_i)) + C(t)\, u_k(t) + \sum_{j=1}^{N} g_j(u_{k-1}(t - \tau_j)) + \sigma(z_k(t),\, \dot{z}_k(t),\, t). \tag{8}$$

For the linear optimal control problem (8) and (5) we have from [11]

$$\begin{aligned} u_k(z_k, t) &= -K^*(t)\, x_k(t) - q_k^*(z_k,\, \dot{z}_k,\, t) \\ &= -R^{-1}(t)\, C'(t)\, K(t)\, x_k(t) - R^{-1}(t)\, C'(t)\, q_k(z_k,\, \dot{z}_k,\, t) \end{aligned} \tag{9a}$$

where

$$\dot{K}(t) + K(t)\, A'(t) + A(t)\, K(t) - K(t)\, S(t)\, K(t) + Q(t) = 0 \tag{9b}$$

with the terminal condition

$$K(t) = F \tag{9c}$$

$$\dot{q}_k(z_k,\, \dot{z}_k,\, t) = -[A(t) - S(t)\, K(t)]'\, q_k(z_k,\, \dot{z}_k,\, t) - K(t)\, h_{k-1}(z_k, \dot{z}_k, t) \tag{9d}$$

$$q_k(z_k,\, \dot{z}_k,\, T) = 0 \tag{9e}$$

here

$$h_{k-1}(z_k,\, \dot{z}_k,\, t) = \sum_{i=1}^{M} f_i(x_{k-1}(t-\delta_i)) + \sum_{j=1}^{N} g_j(u_{k-1}(t-\tau_j)) + \sigma(z_k,\, \dot{z}_k, t). \tag{9f}$$

For this linear optimal regulator problem, if there exists a solution $x_k(t)$ which agrees with a predetermined function $z_k(t)$, then this function is also recognized as a solution for the problem in Theorem 2. From this point of view, the controllability problem for nonlinear systems has been studied by several authors (see survey article by Balachandran and Dauer [4]).

Next we shall prove the following main theorem. For this we fix k.

Theorem 3. If the nonlinear delay systems (4a) with quadratic performance (5) satisfied the condition (2), then the optimal control exists and is given by

$$\begin{aligned} u_k(x_k, t) &= -K^*(t)\, x_k(t) - q_k^*(x_k,\, \dot{x}_k,\, t) \\ &= -R^{-1}(t)\, C'(t)\, K(t)\, x_k(t) - R^{-1}(t)\, C'(t)\, q_k(x_k,\, \dot{x}_k,\, t) \end{aligned} \tag{10a}$$

where $K(t)$ satisfies (9b), (9c) and

$$\dot{q}_k(x_k,\, \dot{x}_k,\, t) = -[A'(t) - S(t)\, K(t)]' q_k(x_k, \dot{x}_k,\, t) - K(t)\, h_{k-1}(x_k,\, \dot{x}_k,\, t) \tag{10b}$$

$$q_k(x_k,\, \dot{x}_k,\, T) = 0. \tag{10c}$$

P r o o f. The solution of (8) with condition (4d) is given by

$$x_k(t) = \Phi(t, t_0)\, \theta(t_0) + \int_{t_0}^{t} \Phi(t, s)\, C(s)\, u_k(s)\, \mathrm{d}s + \int_{t_0}^{t} \Phi(t, s)\, h_{k-1}(z_k,\, \dot{z}_k, s)\, \mathrm{d}s, \tag{11}$$

where $\Phi(t, s)$ is the fundamental matrix solution for the homogeneous linear equation of (8). If we substitute (9a) into (11), we get

$$\begin{aligned} x_k(t) \quad &= \quad \Phi(t, t_0)\, \theta(t_0) - \int_{t_0}^{t} \Phi(t, s)\, C(s)\, K^*(s)\, x_k(s)\, \mathrm{d}s \\ &\quad - \int_{t_0}^{t} \Phi(t, s)\, C(s)\, q_k^*(z_k,\, \dot{z}_k,\, s)\, \mathrm{d}s + \int_{t_0}^{t} \Phi(t, s)\, h_{k-1}(z_k,\, \dot{z}_k,\, s)\, \mathrm{d}s. \end{aligned} \tag{12}$$

As (12) represents a nonlinear relation between $z_k(s)$ and $x_k(s)$ on I, it is sufficient for the existence of optimal control (10a) that at least one fixed point exists for the nonlinear map. Hence (12) is equivalent to (13) for the existence of fixed points

$$x_k(t) = \Phi(t,t_0)\,\theta(t_0) - \int_{t_0}^{t} \Phi(t,s)\,C(s)\,K^*(s)\,z_k(s)\,\mathrm{d}s - \int_{t_0}^{t} \Phi(t,s)\,C(s)\,q_k^*(z_k,\,\dot z_k,\,s)\,\mathrm{d}s + \int_{t_0}^{t} \Phi(t,s)\,h_{k-1}(z_k,\,\dot z_k,\,s)\,\mathrm{d}s. \tag{13}$$

If the nonlinear function $\sigma(z_k,\,\dot z_k,\,t)$ satisfies the condition (2) then from (9d), $q_k(z_k,\,\dot z_k,\,t)$ also satisfies the same condition (Balachandran and Somasundaram [6]) and there exists some positive constant b_2 such that

$$|q_k^*(y_k,\,z_k,\,t) - q_k^*(\overline{y}_k,\,z_k,\,t)| \le (b_2/N^*)\,|y_k - \overline{y}_k|, \tag{14}$$

where the positive constant N^* is already defined and $0 \le b_2 < \frac{1}{2}$. Now the equation (13) can be written in the operator form

$$x_k(t) = P(z_k)\,(t), \tag{15}$$

where P is a nonlinear operator on $C^1(I;R^n)$. This operator is continuous, since all the functions involved in the operator are continuous. Consider the closed convex subset

$$H = \left\{z_k \in C^1(I;R^n) : \|z_k\| \le N_1,\ \|Dz_k\| \le N_2\right\},$$

where N_1 and N_2 are certain positive constants depending on the bounds of A, f_i, C, g_j, σ, K^* and q^*. The operator P maps H into itself. As can easily be seen, all the functions $P(z_k)\,(t)$ with $z_k \in H$ are equicontinuous, since they have uniformly bounded derivatives. We shall now find an estimate for the modulus of continuity of the functions $DP(z_k)\,(t)$ for $t,\,s \in I$. Observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(P(z_k)\,(t)) &= A(t)\,\Phi(t,t_0)\,\theta(t_0) - \int_{t_0}^{t} A(t)\,\Phi(t,s)\,C(s)\,K^*(s)\,z_k(s)\,\mathrm{d}s \\
&\quad - C(t)\,K^*(t)\,z_k(t) - \int_{t_0}^{t} \Phi(t,s)\,C(s)\,q_k^*(z_k,\,\dot z_k,\,s)\,\mathrm{d}s \\
&\quad - C(t)\,q_k^*(z_k(t),\,\dot z_{k(t)},t) + \int_{t_0}^{t} \Phi(t,s)\,h_{k-1}(z_k,\,\dot z_k,\,s)\,\mathrm{d}s + h_{k-1}(z_k,\,\dot z_k,\,t) \\
&= A(t)\,P(z_k)\,(t) - C(t)\,K^*(t)\,z_k(t) - C(t)\,q_k^*(z_k(t),\,\dot z_k(t),\,t) + h_{k-1}(z_k(t),\,\dot z_k(t),\,t).
\end{aligned}$$

Now,

$$\begin{aligned}
|DP(z_k)\,(t) - DP(z_k)(s)| &\le |A(t)\,P(z_k)\,(t) - A(s)\,P(z_k)\,(s)| \\
&\quad + |C(t)\,K^*(t)\,z_k(t) - C(s)\,K^*(s)\,z_k(s)| \\
&\quad + |C(t)\,q_k^*(z_k(t),\,\dot z_k(t),\,t) - C(s)\,q_k^*(z_k(s),\,\dot z_k(s),\,s)| \\
&\quad + |h_{k-1}(z_k(t),\,\dot z_k(t),\,t) - h_{k-1}(z_k(s),\,\dot z_k(s),\,s)|\,.
\end{aligned} \tag{16}$$

Following Balachandran [2], for the first two terms of the right-hand side of (16) we may give the upper estimate as $\beta_0(|t-s|)$, where β_0 is a non-negative function such that $\lim_{h\to 0^+}\beta_0(h)=0$. Similarly, for the last two terms we have the upper estimate as

$$b_2|\dot z_k(t)-\dot z_k(s)|+\beta_1|(t-s)| \quad \text{and} \quad b_1|\dot z_k(t)-\dot z_k(s)|+\beta_2(|t-s|)$$

respectively. Letting $\beta=\beta_0+\beta_1+\beta_2$ and $b=b_1+b_2$, then

$$|DP(z_k)\,(t)-DP(z_k)\,(s)|\le b\,|\dot z_k(t)-\dot z_k(s)|+\beta(|t-s|)$$

and we infer that

$$w(DP(z_k),\,h)\le bw(Dz_k,\,h)+\beta(h).$$

Hence we conclude that for any set $E\subset H$

$$\mu(PE)\le b\mu(E).$$

Thus, by the Darbo fixed point theorem the operator P has at least one fixed point: therefore there exists a function $z_k^*\in C^1(I;R^n)$ such that

$$x_k^*(t)=z_k^*(t)=P(z_k^*(t)). \tag{17}$$

This $x_k^*(t)$ satisfies the condition given in (10). □

P r o o f o f T h e o r e m 2 . Thus from (10) for the kth optimization problem, the optimal control is

$$u_k^*(x_k^*,t)=-R^{-1}(t)\,C'(t)\,K(t)\,x_k^*(t)-R^{-1}(t)\,C'(t)\,q_k(x_k^*,\dot x_k^*,t) \tag{18a}$$

$$\dot q_k(x_k^*,\dot x_k^*,t)=-\left[A'(t)-S(t)\,K(t)\right]'\,q_k(x_k^*,\dot x_k^*,t)-K(t)\,h_{k-1}(x_k^*,\dot x_k^*,t). \tag{18b}$$

The optimal state trajectory $x_k^*(t)$ is the solution to

$$\dot x_k^*(t)=[A(t)-S(t)\,K(t)]\,x_k^*(t)-S(t)\,q_k(x_k^*,\dot x_k^*,t)+h_{k-1}(x_k^*,\dot x_k^*,t), \qquad t_0\le t\le T. \tag{18c}$$

From (18b) we observe that $q_k(x_k^*,\dot x_k^*,t)$ depends on a known function and $h_{k-1}(x_k^*,\dot x_k^*,t)$. Also observe that the homogeneous parts of equations (18b) and (18c) are adjoint. The solution to equation (18c) with boundary condition (4d) is

$$x_k^*(t)=\phi(t,t_0)\,\theta(t_0)=\int_{t_0}^{t}\phi(t,t_0)\,[-S(s)\,q_k(x_k^*,\dot x_k^*,s)+h_{k-1}(x_k^*,\dot x_k^*,s)]\,\mathrm{d}s, \qquad t_0\le t\le T,\ k=1,2,3,\dots, \tag{18d}$$

where $\phi(t,s)$ is the state transition matrix corresponding to the matrix $A(t)-$

$S(t)\,K(t)$. By observing the equation (4b) and (18d) with the hypothesis of Theorem 2, we find the sequences $\{u_k^*(t)\}$ and $\{q_k(x_k^*, \dot{x}_k^*, t)\}$ converge, because these sequences are related to $\{x_k^*(t)\}$ and $\{\dot{x}_k^*(t)\}$ by continuous transformations. Hence the limit of the sequence $\{x_k^*(t)\}$ is the solution to

$$\dot{x}^*(t) = [A(t) - S(t)\,K(t)]\,x^*(t) - S(t)\,q(x^*(t), \dot{x}^*(t),\, t) + \sum_{i=1}^{m} f_i(x^*(t-\delta_i))$$
$$+ \sum_{j=1}^{N} g_j(u^*(t-\tau_j)) + \sigma(x^*(t), \dot{x}^*(t),\, t), \quad t \geq t_0 \tag{19a}$$

$$x^*(t) = \theta(t), \quad t_0 - \Delta \leq t \leq t_0 \tag{19b}$$

$$u^*(t) = \alpha(t), \quad t_0 - \Gamma \leq t \leq t_0, \tag{19c}$$

where $x^*(t)$, $u^*(t)$ and $q(x^*, \dot{x}^*, t)$ are respectively the limits of the sequence $\{x_k^*(t)\}$, $\{u_k^*(t)\}$ and $\{q_k(x_k^*, \dot{x}_k^*, t)\}$. From (18a),

$$u^*(x^*, t) = -R^{-1}(t)\,C'(t)\,K(t)\,x^*(t) - R^{-1}(t)\,C'(t)\,q(x^*, \dot{x}^*, t), \quad t \geq t_0. \tag{20}$$

Substituting $u^*(x^*, t)$ in (20), for $u(t)$ in equation (1a) and comparing the result with equation (19a) shows that $x^*(t)$ and $u^*(x^*, t)$ are the optimal state trajectory and the optimal control respectively for the optimization problem given by equations (1) and (3). This completes the proof of Theorem 2. □

5. COMPUTATIONAL PROCEDURE

The Riccati differential equation (7) must first be solved. Using the terminal condition (7b), equation (7a) may be solved backward in time to obtain $K(t)$. Choosing an arbitrary function $\beta(t)$ in equation (4c), $h_0(t)$ can then be determined from equation (9f). As equation (10b) for q_1 is a differential equation with a final time end condition, a future value of the state x_1 is required for the numerical value of q_1. This can be obtained from equation (13). Now by using the boundary condition (4d), $x_1^*(t)$ can be calculated. The optimal control u_1^* can be determined from (18a). This procedure is repeated for consecutive integral values of k. An m_1th order optimal state trajectory $x_{m_1}^*(t)$ and optimal control $u_{m_1}^*(x_{m_1}^*; t)$ can be obtained if the procedure is continued up to $k = m_1$.

Remark. The optimal control scheme obtained in the above theorems is not easy to implement and hence further research is required in this direction.

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