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CONTROL TECHNIQUES FOR CHAOTIC DYNAMICAL SYSTEMS

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The recent interest in the area of the control of chaos is remarked and the various approaches to this problem presented in the literature are briefly summarized. The Harmonic Balance (HB) technique for the approximate analysis of systems with complex behaviour is outlined and the general idea of using this technique for control problems is proposed. Two possible approaches are indicated to show the flexibility of the method, the clearness of the required conditions and the simplicity of the solutions to be expected.

1. INTRODUCTION

The study of nonlinear systems which exhibit complex dynamic phenomena as bifurcations and chaos has produced in the last years a large number of contributions from all the areas of the nonlinear science. Most of the developed research has been essentially related to aspects of the analysis of such complex systems but recently a significant attention has been directed towards problems which are viewed and denoted as concerning *the control of the chaos*. In this framework the proposed approaches again come from various scientific fields and, apart from the specific techniques which are employed, they refer to different statements, goals and constraints. The usual paradigm is that complex and chaotic behaviours have to be avoided by leading the controlled system to more regular regimes, even if some ideas have been formulated on the opportunity of synthesizing chaotic systems as expression of healty dynamics, in biological as well as in engineering processes [13, 43]. A general view of reasons and applications of these theories is reported in [25].

The contributions generally concerning *the suppression of the chaos* may be divided into two main groups: in the first one specific chaotic systems are considered, while in the second one classes of systems, depending on parameters and presenting bifurcations and possibly chaotic regions, are the object of the study. The former approach states a quantitative goal in the state space – as an equilibrium point or a limit cycle or some given trajectory – and it tends to entrain the system to this dynamics by means of suitable controls. Of course, several solutions have been proposed having in mind different possible constraints which must hold. In particular, there is a number of suggested methods which use low energy controls, without changing some main characteristics outside one or more restricted regions of interest.

This also agrees with the speculative idea offered by the chaotic systems of obtaining large results with small controls. Most of these methods are related to the algorithm proposed by Ott, Grebogi and Yorke [33, 38, 36], with some possible modifications [12], where a feedback variable structure controller is employed, switching in a suitable state space region around the required solution and stabilizing it. Applications are presented in [11, 34, 40, 21, 37]. Other limited energy techniques use weak periodic forcing signals as open loop controls and they tame the chaos on the basis of a parametric resonance mechanism [27, 4]. In the methods of the first group, another open loop approach, actually a high energy method, is due to Hübler and Jackson (see, for instance, [19, 23, 24] and [20]) which offers the possibility of transferring complex systems to and between a wide variety of dynamics. The first group can be completed by other high energy methods including more typical control theory solutions based on feedback structures and related to linear, optimal [45], stochastic [14], adaptive [41, 32] and nonlinear control [6, 7]. A wide survey concerning these approaches is presented in [7].

A smaller number of contributions can be considered as belonging to the second way of controlling chaos. In this case, the object under study is a class of dynamical systems, which depends on parameters and it is globally viewed in its state-parameter space, while the above considerations have been essentially directed towards state space behaviours. Now, the goal to reach is essentially qualitative, typically bifurcations appearing in the original system are delayed in order to increase the range of parameter values for which the system exhibits regular motions. In particular, Abed et al. ([26, 46, 1]) lead to this result by a linear dynamic feedback controller (washout filter), without at the same time changing the set of the equilibrium points (see also [22] for a similar approach). Moreover, such papers use an additive nonlinear feedback controller in order to suitably stabilize the above bifurcation whenever it occurs.

This presentation of previous works on chaos control (see also [2, 9, 42] for more theoretical aspects) is necessarily brief and does not enter in the details presented in the original papers, where the various features of the methods with respect to transient behaviour, robustness, effect of noise, etc., are considered as well as the modifications involved in considering systems described by difference or differential equations. On the other hand, the above approaches are illustrated by a number of applications to a wide variety of systems especially in physics and engineering (see, for example, [35, 10]), with numerical simulations and experimental results obtained on real processes. In particular, there is a growing interest on chaos synchronization (see, for example, [5]), while a large number of applications of different control techniques to the well-known Chua's circuit [8] is in [28, 29].

In such a framework the purpose of this paper is to present a new approach to the chaos control which can be used along several ways presented in the first part of this section. The approach follows from an analysis technique which has been recently proposed in [15, 16, 17], based on control system ideas and employing *the frequency Harmonic Balance* (HB) to study the system behaviour. This technique leads to formulate simple structural (non-numerical) conditions among the system parameters which approximately express the occurrence of complex dynamics phenomena (homoclinic orbits, onset of period doubling). The method is developed for a quite general class of systems, represented by a canonical input-output structure to which almost all the classical and more recent chaotic equations can be reduced.

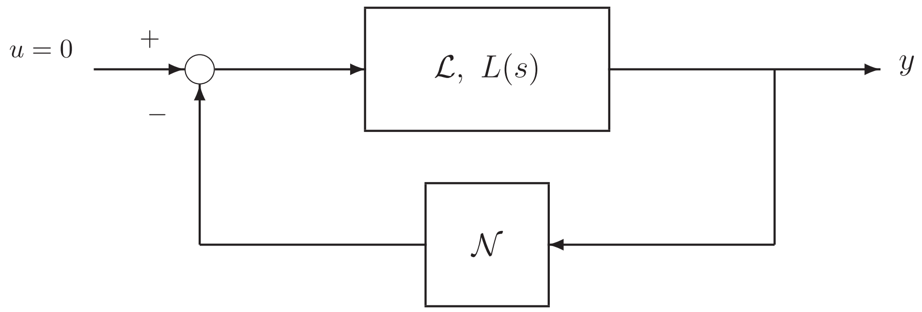

Fig. 1. Canonical feedback structure.

The global view derived for the system dynamics allows one to study as avoiding or delaying bifurcations and chaos by means of suitable modifications, possibly slight or simple to do, on the various part of the system. This can concern specific systems where parameters are given, as well as classes of systems depending on parameters. Therefore, the two main situations above evidenced can be taken into account, at least in principle.

2. THE HB TECHNIQUE FOR ANALYSING COMPLEX SYSTEMS

Assume that the dynamical system under study can be described in the form

$$\dot{x} = Ax - bn(x, z) \quad (1)$$
$$\dot{z} = Mx + m(x) \quad (2)$$

where $x \in R^p$ and $z \in R^q$ are state variables, $n \in R$ and $m \in R^q$ are nonlinear functions, and A, b, M are constant matrices of appropriate dimensions. The system of eqs. (1) and (2) can be simply decomposed as in Figure 1 where a new scalar variable y, namely the output, has been introduced and two SISO subsystems, the former linear with state x denoted by $\mathcal{L}$ and the latter nonlinear with state z denoted by $\mathcal{N}$, are connected in feedback configuration [16].

The linear block $\mathcal{L}$ will be modeled by its transfer function $L(s)$, where s denotes the complex variable. For $q = 0$ the block $\mathcal{N}$ often reduces to the nonlinear output function $n(\cdot)$: the structure of Figure 1 will be called *Lur'e System* (LS) referring to the classical problem of absolute stability [44, 31]. Otherwise, the whole system will be denoted as an *Extended Lur'e System* (ELS): the output of the block $\mathcal{N}$ retains an explicit expression in terms of y. In the scheme of Figure 1 the presence of constant and/or sinusoidal forcing terms, entering to the summing point or affecting the nonlinear subsystem $\mathcal{N}$, can also be included. The possibility of representing by these systems almost all the most famous and studied continuous chaotic equations (Duffing, Van der Pol, Lorenz, Rossler, Chua, etc.) has been recently shown [16, 17].

Now, some concepts are recalled and definitions are given concerning constant and periodic solutions of eqs. (1) and (2) and in particular of system of Figure 1. The tool for approximately investigating some of these elements is the well-known describing function method based on the HB of signals along this loop (see, for example, [39, 3, 30]). Once assumed for y the form

$$y_o(t) = A + B\cos\omega t, \quad B, \omega > 0 \tag{3}$$

the nonlinear subsystem $\mathcal{N}$ is characterized in the corresponding steady state time output $n_o(t)$ by the bias gain

$$N_o(A, B, \omega) \doteq \frac{1}{2\pi A}\int_{-\pi}^{\pi} n_o(t)\, d\omega t, \tag{4}$$

and by the complex first harmonic gain

$$N_1(A, B, \omega) \doteq \frac{1}{\pi B}\int_{-\pi}^{\pi} n_o(t)\, e^{j\omega t} d\omega t\ . \tag{5}$$

The following elements are put in evidence:

- *Equilibrium points* (EPs): the constant output values y_e corresponding to the equilibrium solutions of eqs. (1) and (2). They satisfy

$$y + \tilde{n}(y)L(0) = 0 \tag{6}$$

  where $\tilde{n}(y)$ simbolically indicates the steady state output of the subsystem $\mathcal{N}$ when its input is the constant y. The stability features of the EPs can be studied by suitably linearising $\tilde{n}$. For LSs $\tilde{n}$ reduces to the nonlinear function n.

- *Predicted Limit Cycles* (PLCs): approximate periodic solutions of eqs. (1) and (2) derived by the describing function method. According to (3), (4), and (5) the conditions are [39, 3, 30]

$$A[1 + N_o(A, B, \omega)\, L(0)] = 0 \tag{7}$$

$$1 + N_1(A, B, \omega)L(j\omega) = 0. \tag{8}$$

  These eqs. follow by imposing the HB to the system of Figure 1, where the transfer function L has been evaluated at its steady state gains, and must be solved with respect to the parameters A, B and ω (see (3)). The stability features of such solutions can also be evaluated [39, 3, 30]. They form limit cycles in the state space which are called *predicted* since they derive from a heuristic analysis and their shape and even existence are uncertain. The reliability of the prediction depends on the distortion along the loop (see below).

- *Distortion*: concerning a PLC of frequency ω it is the amount of the neglected higher harmonics. It can be expressed by

$$\Delta \doteq \frac{\|\tilde{y}_o(t) - y_o(t)\|_2}{\|y_o(t)\|_2} \tag{9}$$

where $\tilde{y}_o$ represents the output of the open loop path of Figure 1 as shown in Figure 2. Small values of Δ indicate that the open loop system is an efficient *low-pass* filter and that the corresponding PLC solution is reliable [39, 3, 30].

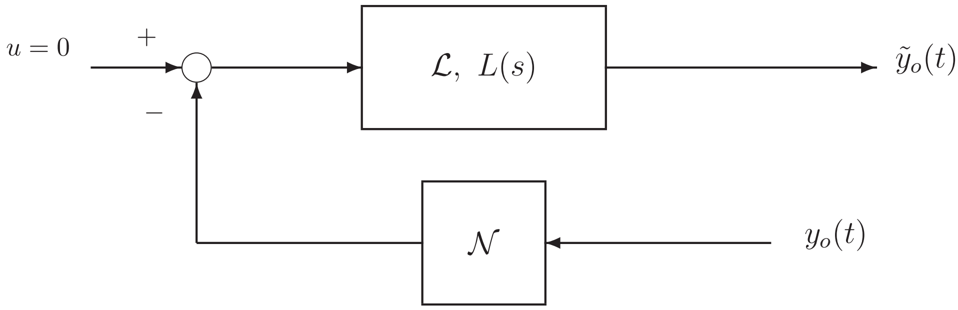

Fig. 2. Open loop path for distortion definition.

The above concepts are used in [15, 16, 17] to recognize by heuristic but structural (non-numerical) and simple conditions when eqs. (1) and (2) can give rise to complex behaviours. Essentially, *two independent* (and alternative for LSs) *mechanisms* have been put in evidence and verified:

i) the period doubling. It happens when a stable PLC exists satisfying (7) and (8) and in addition the condition

$$1 + N_{1/2} L(j\omega/2) = 0 \tag{10}$$

holds. Here, $N_{1/2}$ is the incremental complex gain of the nonlinear subsystem $\mathcal{N}$ around the PLC and it is defined by perturbing this solution by a small $\omega/2$ frequency term and using the describing function approach [16]. $N_{1/2}$ depends on the derivative of N_o and N_1 with respect to their arguments [3]. In presence of a low value of Δ a true period doubling (flip) bifurcation is found out, while a medium value of Δ may indicate the proximity of chaotic behaviours.

ii) for LSs, the interaction of a stable PLC and an unstable EP. It is defined by the condition

$$y_o(t) = A + B\cos\omega t = y_e, \quad \text{for some } t \tag{11}$$

where y_e is an EP different from that to which $y_o(t)$ reduces when B tends to zero. This mechanism approximately reveals a homoclinic orbit [15, 16]. For low values of Δ a true limit cycle and an EP really exist, while medium values of Δ indicate chaos.

3. THE HB TECHNIQUE FOR THE CONTROL OF COMPLEX SYSTEMS

Given the system of Figure 1 assume that some complex dynamics has been recognized via the above conditions on the basis of the HB techniques. Apart from the transfer function $L(s)$ such conditions essentially involve N_o, N_1 in eqs. (7), (8), (11), and $N_{1/2}$ in eq. (10), computed at A, B, ω of the PLC $y_o(t)$, plus $\tilde{n}$ at the EPs y_e in eqs. (6) and (11). There are also inequalities to be satisfied for suitable derivatives of N_o, N_1 and $\tilde{n}$ in the points of interest to predict the prescribed stability features. Moreover, in order to guarantee that the distortion Δ belongs to a certain range of values some other constraints can be written, involving in general the frequency response of the subsystem $\mathcal{N}$.

Now, assume that a feedback controller $\mathcal{N}_c$, generally nonlinear, is placed in parallel to the original system $\mathcal{N}$ as shown in Figure 3.

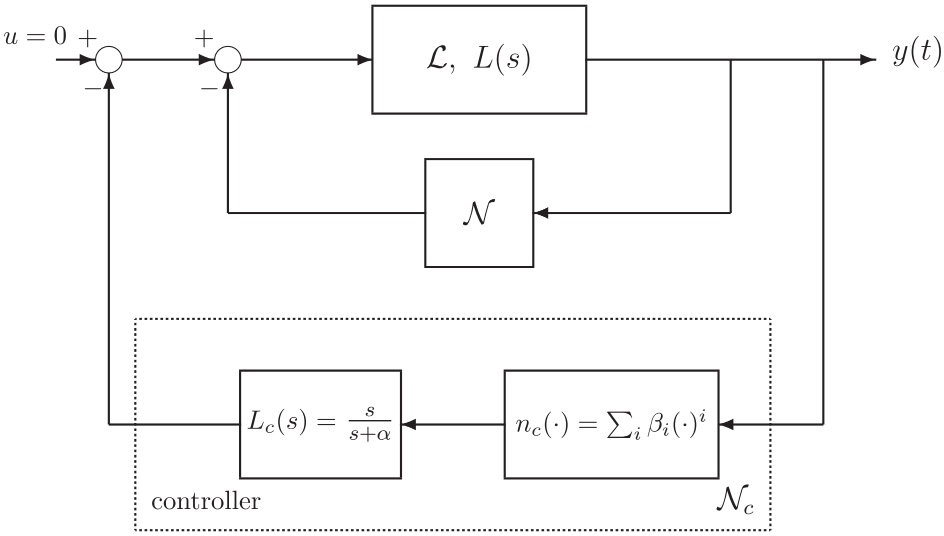

Fig.3. The structure of the controlled system.

The above conditions for complex dynamics can again be considered where the involved terms are those corresponding to the nonlinear subsystem $\mathcal{N} + \mathcal{N}_c$.

The main idea to control chaos, in the general sense of Section 1, is to select $\mathcal{N}_c$ in such a way that the describing functions $N_o + N_{oc}$, $N_1 + N_{1c}$, $N_{1/2} + N_{1/2c}$, the function $\tilde{n} + \tilde{n}_c$ (the new symbols clearly refer to the controller) and the controlled system distortion Δ_c no longer satisfy the conditions of Section 2 for the existence of complex phenomena.

To illustrate this approach and to show its flexibility in taking into account different objectives and constraints, consider for example (see Figure 3) a controller formed by a polynomial nonlinearity of a suitable order, linear in the coefficients, as

$$n_c(y) = \sum_i \beta_i y^i \tag{12}$$

and by a linear filter of transfer function

$$L_c(s) = \frac{s}{s+\alpha} \ . \qquad (13)$$

This last element, called a washout filter in [26, 46], has been introduced to preserve the EPs of the original system, as its main characteristics which must be usually untouched, and to have an almost constant gain in the interested bandwith at higher frequencies. The coefficients β_i and α are unknown and must be determined to define the controller.

Two possible controls of the chaos are briefly outlined:

1. Assume that for a LS a chaotic behaviour following *the interaction mechanism* ii) of Section 2 has been discovered. The stabilization of the fundamental limit cycle of parameters A, B, ω can be provided by the main conditions

$$N_{oc}(A, B, \omega) = 0, \quad N_{1c}(A, B, \omega) = 0 \qquad (14)$$

and

$$\Delta_c < k < \Delta. \qquad (15)$$

In fact, eqs. (14) preserve the original PLC, while eq. (15) reduces under a sufficiently small positive k the amount of higher harmonics on the loop of Figure 1, by canceling those of the original system by the corresponding ones of the controller. So, the prediction of such a stable PLC is made reliable.

2. Assume that for an ELS a *period doubling* has been recognized. This phenomenon (point i) of Section 2) can be avoided for the controlled system by imposing the inequality

$$|N_{1/2c} L(j\omega/2)| > h \qquad (16)$$

where h is a suitable positive number. Again eqs. (14), and possibly some bounds on Δ_c similar to (15), will ensure to maintain the same periodic solution of the original system. Eq. (16) is used to disagree the period doubling condition (10): by an appropriate selection of the subsystem $\mathcal{N}_c$ it can be expected of controlling the occurrence of this condition.

Observe that problems 1. and 2. are two different questions which can be posed in controlling a chaotic system as seen in Section 1. In any case the proposed solutions determine low energy controls, due to Eqs. (14) and to the presence of the washout filter in the controller $\mathcal{N}_c$. These and other situations can be faced by the HB technique since such methods approximately give separate structural conditions on the various complex behaviours. Finally, notice that for the controller structure specifically proposed, eqs. (14) result to be linear in the unknown parameters β_i, while (15) and (16) are quadratic. Therefore, this approach can allows one to synthesize in a quite simple form a control which satisfies the required specifications. An application is in [18].

4. CONCLUSION

The paper has presented a general view of several problems that are considered when complex or chaotic behaviours have to be removed in a dynamical system. An approach to these control problems can be derived by the use of the Harmonic Balance technique, a well-known approximate tool already applied to the analysis of chaotic systems. The basic concepts and results of this method are briefly recalled and the main idea for controlling chaos via a feedback compensator is presented. As an indicative application a suitable nonlinear structure is proposed and the conditions to come to its synthesis are outlined.

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REFERENCES

[1] E. H. Abed, H. O. Wang and R. C. Chen: Stabilization of period doubling bifurcations and implications for control of chaos. In: Proc. 31st IEEE Conference on Decision and Control, Tucson 1992, pp. 2119–2124.
[2] F. Albertini and E. D. Sontag: Some connections between chaotic dynamical systems and control systems. In: Proc. 1st European Control Conference, Grenoble 1991, pp. 158–163.
[3] D. P. Atherton: Nonlinear Control Engineering. Van Nostrand Reinhold, London 1975.
[4] Y. Braiman and I. Goldhirsch: Taming chaotic dynamics with weak periodic perturbations. Phys. Rev. Lett. *66* (1991), 2545–2548.
[5] T. L. Carrol and L. M. Pecora: A circuit for studying the synchronization of a chaotic system. Internat. J. Bifur. Chaos Appl. Sci. Engng. *2* (1992), 659–667.
[6] G. Chen and X. Dong: On feedback control of chaotic nonlinear dynamic systems. Internat. J. Bifur. Chaos Appl. Sci. Engng. *2* (1992), 407–412.
[7] G. Chen and X. Dong: From chaos to order – Prespectives and methodologies in controlling chaotic nonlinear dynamical systems. Systems Control and Computing, Tech. Rep. 92–07, University of Houston, Houston 1992.
[8] L. O. Chua: The genesis of Chua's circuit. Archiv für Elektronik und Übertragungstechnik *47* (1992), 250–257.
[9] F. Colonius and W. Klienman: On control sets and feedback for nonlinear systems. In: Preprints 2nd IFAC Symposium NOLCOS, Bordeaux 1992, pp. 29–36.
[10] A. Dabrowski, Z. Galias and M. J. Ogorzalek: A study of identification and control in a real implementation of Chua's circuit. In: Preprints 2nd IFAC Workshop on System Structure and Control, Prague 1992, pp. 278–281.
[11] W. L. Ditto, S. N. Rauseo and M. L. Spano: Experimental control of Chaos. Phys. Rev. Lett. *26* (1990), 3211–3214.
[12] U. Dressler and G. Nitsche: Controlling Chaos using time delay coordinates. Phys. Rev. Lett. *68* (1992), 1–4.
[13] W. H. Fleming (ed.): Report of the panel on future directions in control theory: a mathematical perspective. Society for Industrial and Applied Mathematics, Philadelfia 1988.
[14] T. B. Fowler: Application of stochastic control techniques to chaotic nonlinear systems. IEEE Trans. Automat. Control *AC-34* (1989), 201–205.
[15] R. Genesio and A. Tesi: Chaos prediction in nonlinear feedback systems. IEE Proc. D Control Theory and Applications *138* (1991), 313–320.
[16] R. Genesio and A. Tesi: Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems. Automatica *28* (1992), 531–548.

[17] R. Genesio and A. Tesi: A harmonic balance approach for chaos prediction: the Chua's circuit. Internat. J. Bifur. Chaos Appl. Sci. Engng. *2* (1992), 61–79.

[18] R. Genesio and A. Tesi: Distortion Control of chaotic systems: the Chua's circuit'. J. Circuits, Systems and Computers *3* (1993), 151–171.

[19] A. Hübler: Adaptive control of chaotic systems. Helv. Phys. Acta *62* (1989), 343–346.

[20] A. Hübler: Modeling and control of complex systems: paradigms and applications. In: Modeling Complex Phenomena (L. Lam and V. Naroditsky, eds.), Springer–Verlag, New York 1992, pp. 5–65.

[21] E. R. Hunt: Stabilizing high-period orbits in a chaotic system: the diode resonator. Phys. Rev. Lett. *67* (1992), 1953–1955.

[22] H. Hyötyniemi: Postponing Chaos using a robust stabilizer. In: Proc. 1st IFAC Symposium on Design Methods of Control Systems, Zurich 1991, pp. 568–572.

[23] E. A. Jackson: The entrainment and migration controls of multiple-attractor systems. Phys. Lett. A *151* (1990), 478–484.

[24] E. A. Jackson: On the control of complex dynamic systems. Phys. Lett. D *50* (1991), 341–366.

[25] J. H. Kim and J. Stringer (eds.): Applied Chaos. John Wiley, New York 1992.

[26] H. C. Lee. and E. H. Abed: Washout filters in the bifurcation control of high alpha flight dynamics. In: Proc. 1991 IEEE Amer. Contr. Conf., Boston 1991.

[27] R. Lima and M. Pettini: Suppression of Chaos by resonant parametric perturbations. Phys. Rev. *41* (1991), 1726–1733.

[28] R. Madan (ed.): Special Issue on Chua's circuit: A paradigm for chaos. Part I. J. Circuits, Systems and Computers *3* (1993), 1.

[29] R. Madan (ed.): Special Issue on Chua's circuit: A paradigm for chaos. Part II. J. Circuits, Systems and Computers *3* (1993), 2.

[30] A. I. Mees: Dynamics of Feedback Systems. John Wiley, New York 1981.

[31] M. Mohler: Nonlinear systems: Dynamics and Control. Vol. I. Prentice Hall, Englewood Cliffs 1991.

[32] F. Mossayebi, H. K. Qammar and T. T. Hartley: Adaptive estimation and synchronization of chaotic systems. Phys. Lett. A *161* (1991), 255–262.

[33] E. C. Ott, C. Grebogi and J. A. Yorke: Controlling Chaos. Phys. Rev. Lett. *64* (1990), 1196–1199.

[34] B. Peng, V. Petrov, and K. Showalter: Controlling Chemical Chaos. J. Phys. Chem. *95* (1991), 4957–4959.

[35] R. Rajasekar and M. Lakshmanan: Controlling of chaos in Bonhoeffer-van der Pol oscillator. Internat. J. Bifur. Chaos Appl. Sci. Engng. *2* (1992), 201–204.

[36] F. J. Romeiras, E. Ott, C. Grebogi and W. P. Dayawansa: Controlling chaotic dynamical systems. In: Proc. 1991 IEEE Amer. Control Conference, Boston 1991.

[37] R. Roy, T. W. Murphy Jr., T. D. Maier, Z. Gills, and E. R. Hunt: Dynamical control of a chaotic laser: experimental stabilization of a globally coupled system. Phys. Rev. Lett. *68* (1992), 1259–1262.

[38] T. Shinbrot, E. Ott, C. Grebogi and J. A. Yorke: Using Chaos to direct trajectories to targets. Phys. Rev. Lett. *65* (1990), 3215–3218.

[39] D. D. Šiljak: Nonlinear Systems: the Parameter Analysis and Design. John Wiley, New York 1969.

[40] J. Singer, Y. Z. Wang and H. H. Bau: Controlling a chaotic system. Phys. Rev. Lett. *66* (1991), 1123–1125.

[41] S. Sinha, R. Ramaswamy and J. S. Rao: Adaptive control in nonlinear dynamics. Phys. Lett. D *43* (1990), 118–128.

[42] T. Taylor: Chaos and its applications in control. In: Proc. 31st IEEE Conference on Decision and Control, Tucson 1992, pp. 2102–2106.

[43] A. Vaněček: Strongly nonlinear and other control systems. Problems Control Inform. Theory *20* (1991), 3–12.
[44] M. Vidyasagar: Nonlinear System Analysis. Prentice–Hall, Englewood Cliffs 1978.
[45] T. L. Vincent and J. Yu: Control of a chaotic system. Dynamics and Control *1* (1991), 35–52.
[46] H. O. Wang and E. H. Abed: Bifurcation control of chaotic dynamical systems. In: Preprints 2nd IFAC Symposium NOLCOS, Bordeaux, France, 1992.

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