

The existence of closed 3-forms of \tilde{G}_2 -type on 7-manifolds*

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Abstract

In this note we construct a first example of a closed 3-form of \tilde{G}_2 -type on $S^3 \times S^4$. We prove that $S^3 \times S^4$ does not admit a homogeneous 3-form of \tilde{G}_2 -type. Thus our example is a first example of a closed 3-form of \tilde{G}_2 -type on a compact 7-manifold which is not stably homogeneous.

Key words: closed 3-forms, \tilde{G}_2 -forms.

1 Introduction.

Let $\Lambda^k(V^n)^*$ be the space of k -linear anti-symmetric forms on a given linear space V^n . For each $\omega \in \Lambda^k(V^n)^*$ we denote by I_ω the linear map

$$I_\omega : V^n \rightarrow \Lambda^{k-1}(V^n)^*, x \mapsto (x] \omega) := \omega(x, \dots).$$

A k -form ω is called **multi-symplectic**, if I_ω is a monomorphism. Two k -forms are equivalent, if they are in the same orbit of the action of $Gl(V^n)$ on $\Lambda^k(V^n)^*$.

The classification of multi-symplectic 3-forms on \mathbb{R}^7 has been done by Bures and Vanzura [1]. There are together 8 types of these forms, among them there are two stable forms of G_2 -type and \tilde{G}_2 -type. More precisely the orbits of these stable 3-forms under the action of $Gl(V^7)$ are open sets in $\Lambda^3(V^7)^*$ and their corresponding isotropy groups are the compact group G_2 and its dual non-compact group \tilde{G}_2 .

There are many known results on 7-manifolds admitting a (closed) 3-form of G_2 -type, see e.g. [2],[3], [6],[7], [9], [10], [11]. Manifolds which admit a (closed) 3-form of \tilde{G}_2 -type are less known. In particular, known examples of closed 3-forms of \tilde{G}_2 -type on compact 7-manifolds up to now are homogeneous examples or obtaining from those by adding a small closed 3-form. We call such a closed 3-form of \tilde{G}_2 -type stably homogeneous.

In this note we construct a first example of a closed 3-form ω^3 of \tilde{G}_2 -type on a manifold $X^7 = S^3 \times S^4$ by identifying X^7 with a submanifold of the group $SU(3)$ provided with the Cartan 3-form (Theorem 2.1.) In section 3 we prove that the codimension of the action of the full automorphism group of (X^7, ω^3) on X^7 is 1 (Theorem 3.1), thus (X^7, ω^3) is not homogeneous. Moreover we prove that

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X^7 admits no homogeneous \tilde{G}_2 -structure (Proposition 3.2). So our example is a \tilde{G}_2 -structure on $S^3 \times S^4$ with maximal symmetry.

This note also contains an appendix which gives a necessary and sufficient condition for a closed 7-manifold to admit a \tilde{G}_2 -structure. As a corollary we obtain many examples of open 7-manifolds admitting a closed 3-form of \tilde{G}_2 -type.

2 New example of a closed 3-form of \tilde{G}_2 -type on $S^3 \times S^4$.

On each semi-simple Lie group G there exists a natural bi-invariant 3-form ϕ^3 which is defined at the Lie algebra $\mathfrak{g} = T_e G$ as follows

$$\phi^3(X, Y, Z) = \langle X, [Y, Z] \rangle,$$

where \langle, \rangle denotes the Killing form on \mathfrak{g} . This 3-form ϕ^3 is also called the Cartan 3-form.

We claim that the Cartan 3-form ω^3 is multi-symplectic. To show the injectivity of the linear map I_{ϕ^3} we notice that if $X \in \ker I_{\phi^3}$, then

$$\langle X, [Y, Z] \rangle = 0 \text{ for all } Y, Z \in \mathfrak{g}.$$

But this condition is incompatible with the semi-simplicity of \mathfrak{g} .

Let us consider the group $G = SU(3)$. For each $1 \leq i \leq j \leq 3$ let $g_{ij}(g)$ be the complex function on $SU(3)$ induced from the standard unitary representation ρ of $SU(3)$ on \mathbb{C}^3 : $g_{ij}(g) := \langle \rho(g) \circ e_i, \bar{e}_j \rangle$. Here $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ is a unitary basis of \mathbb{C}^3 . Let us denote by X^7 the co-dimension 1 subset in $SU(3)$ which is defined by the equation $Im(g_{11}(g)) = 0$.

2.1. Theorem. *The subset X^7 is diffeomorphic to the manifold $S^3 \times S^4$. Moreover X^7 is provided with a closed 3-form of \tilde{G}_2 -type which is the restriction of ϕ^3 to X^7 .*

Proof. Let $SU(2)$ be the subgroup in $SU(3)$ consisting of all $g \in SU(3)$ such that $\rho(g) \circ e_1 = e_1$. We denote by π the natural projection

$$\pi : SU(3) \rightarrow SU(3)/SU(2).$$

We identify $SU(3)/SU(2)$ with the sphere $S^5 \subset \mathbb{C}^3$ using the standard representation ρ of $SU(3)$ on \mathbb{C}^3 . Namely we set

$$\tilde{\rho}(g \cdot SU(2)) := \rho(g) \circ e_1.$$

We denote by Π the composition $\tilde{\rho} \circ \pi : SU(3) \rightarrow SU(3)/SU(2) \rightarrow S^5$. Let $S^4 \subset S^5$ be the geodesic sphere which consists of points $v \in S^5$ such that $Im e^1(v) = 0$. Here $\{e^i, i = 1, 2, 3\}$ are the complex 1-forms on \mathbb{C}^3 which are dual to $\{e_i\}$. The pre-image $\Pi^{-1}(S^4)$ consists of all $g \in SU(3)$ such that

$$Im e^1(\rho(g) \circ e_1) = 0.$$

$$\iff Im(g_{11}) = 0.$$

So X^7 is $SU(2)$ -fibration over S^4 . But this fibration is the restriction of the $SU(2)$ -fibration $\Pi^{-1}(D^5)$ over the half-sphere D^5 to the boundary $\partial D^5 = S^4$. Hence X^7 is a trivial $SU(2)$ -fibration. This proves the first statement of Theorem 2.2.

Let us denote by $SO(2)^1$ the orthogonal group of the real subspace $\mathbb{R}^2 \subset \mathbb{C}^3$ such that \mathbb{R}^2 is the span of e_1 and e_2 over \mathbb{R} . Clearly $SO(2)^1$ is a subgroup of $SU(3)$.

We denote by $m_L(g)$ (resp. $m_R(g)$) the left multiplication (resp. the right multiplication) by an element $g \in SU(3)$.

2.2. Lemma. X^7 is invariant under the action of $m_L(SU(2)) \cdot m_R(SU(2))$. For each $v \in S^4$ there exist an element $\alpha \in SO(2)^1$ and an element $g \in SU(2)$ such that $\Pi(g \cdot \alpha) = v$. Consequently for any point $x \in X^7$ there are $g_1, g_2 \in SU(2)$ and $\alpha \in SO(2)^1$ such that

$$(2.2.1) \quad x = g_1 \cdot \alpha \cdot g_2,$$

Proof. Using the identification that $X^7 = \Pi^{-1}(S^4)$ we note that the orbits of $m_R(SU(2))$ -action on X^7 are the fibers $\Pi^{-1}(v)$. Now the first statement follows by straightforward calculations. Let $v = (\cos \alpha, z_2, z_3) \in S^4$, where $z_i \in \mathbb{C}$. We choose $\alpha \in SO(2)^1$ so that

$$\rho(\alpha) \circ e_1 = (\cos \alpha, \sin \alpha) \in \mathbb{R}^2.$$

Clearly α is defined by v uniquely up to sign \pm . We set

$$w := (\sin \alpha, 0) \in \mathbb{C}^2 = \langle e_2, e_3 \rangle_{\otimes \mathbb{C}}.$$

We notice that

$$|z_2|^2 + |z_3|^2 = \sin^2 \alpha.$$

Since $SU(2)$ acts transitively on the sphere S^3 of radius $|\sin \alpha|$ in $\mathbb{C}^2 = \langle e_2, e_3 \rangle_{\otimes \mathbb{C}}$, there exists an element $g \in SU(2)$ such that $\rho(g) \circ w = (z_2, z_3)$. Clearly

$$\Pi(g \cdot \alpha) = v.$$

This proves the second statement. The last statement of Lemma 2.2 follows from the second statement and the fact that $X^7 = \Pi^{-1}(S^4)$ \square

Continuation of the proof of the second part of Theorem 2.1. Let us write here a canonical expression of a 3-form ω^3 of \tilde{G}_2 -type (see e.g. [2], [1])

$$(2.3) \quad \omega^3 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_3.$$

Here α_i are 2-forms on V^7 which can be written as

$$\alpha_1 = y_1 \wedge y_2 + y_3 \wedge y_4, \quad \alpha_2 = y_1 \wedge y_3 - y_2 \wedge y_4, \quad \alpha_3 = y_1 \wedge y_4 + y_2 \wedge y_3$$

and $(\theta_1, \theta_2, \theta_3, y_1, y_2, y_3, y_4)$ is an oriented basis of $(V^7)^*$.

Using Lemma 2.2 we reduce the proof of the second statement of Theorem 2.1 to verifying that the value of $\phi^3_{|X^7}$ at any point $\alpha \in SO(2)^1 \subset X^7$ is a 3-form of \tilde{G}_2 -type. First we shall compute that

value at point $e \in SO(2)^1 \subset X^7$ and then we shall compute that value at any point $\alpha \in SO(2)^1$.

Step 1. We shall use the Killing metric to identify the Lie algebra $su(3)$ with its co-algebra g . In what follows we shall not distinguish co-vectors and vectors, poly-vector and exterior forms on $su(3)$. Clearly we have

$$T_e X^7 = \{v \in su(3) : \text{Im } g_{11}(v) = 0\}.$$

Now we identify $gl(\mathbb{C}^3)$ with $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ and we denote by e_{ij} the element of $gl(\mathbb{C}^3)$ of the form $e_i \otimes (e_j)^*$. Let δ_i be the 1-forms in $T_e X^7$ which are defined as follows:

$$\delta_1 = \frac{i}{\sqrt{2}}(e_{22} - e_{33}), \delta_2 = \frac{1}{\sqrt{2}}(e_{23} - e_{32}), \delta_3 = \frac{i}{\sqrt{2}}(e_{23} + e_{32}).$$

Furthermore, ω_i are 2-forms on $T_e X^7$ which have the following expressions:

$$\begin{aligned} 2\omega_1 &= -(e_{12} - e_{21}) \wedge i(e_{12} + e_{21}) + (e_{13} - e_{31}) \wedge i(e_{13} + e_{31}), \\ 2\omega_2 &= -(e_{12} - e_{21}) \wedge (e_{13} - e_{31}) - i(e_{12} + e_{21}) \wedge i(e_{13} + e_{31}), \\ 2\omega_3 &= -(e_{12} - e_{21}) \wedge i(e_{13} + e_{31}) + i(e_{12} + e_{21}) \wedge (e_{13} - e_{31}). \end{aligned}$$

2.4. Lemma. *The restriction of the Cartan 3-form to X^7 at the point e is*

$$(2.4.1) \quad \phi_{|T_e X^7}^3 = \sqrt{2}\delta_1 \wedge \delta_2 \wedge \delta_3 + \frac{1}{\sqrt{2}}\omega_1 \wedge \delta_1 + \frac{1}{\sqrt{2}}\omega_2 \wedge \delta_2 + \frac{1}{\sqrt{2}}\omega_3 \wedge \delta_3,$$

Proof. We set

$$\begin{aligned} f_1 &:= \frac{e_{12} - e_{21}}{\sqrt{2}}, \quad f_2 := \frac{i(e_{12} + e_{21})}{\sqrt{2}} \\ f_3 &:= \frac{e_{13} - e_{31}}{\sqrt{2}}, \quad f_4 := \frac{i(e_{13} + e_{31})}{\sqrt{2}} \end{aligned}$$

Then $(\delta_1, \delta_2, \delta_3, f_1, f_2, f_3, f_4)$ form an ortho-normal basis of $T_e X^7$ w.r.t. to the restriction of the Killing metric to X^7 . Next we observe that $(\delta_1, \delta_2, \delta_3)$ form a $su(2)$ -algebra. Furthermore we have

$$(2.5.1) \quad [\delta_2, \delta_3] = \sqrt{2}\delta_1, [\delta_1, \delta_2] = \sqrt{2}\delta_3, [\delta_1, \delta_3] = -\sqrt{2}\delta_2.$$

$$(2.5.2) \quad [f_1, f_2] = i(e_{11} - e_{22}), [f_1, f_3] = -\frac{1}{\sqrt{2}}\delta_2, [f_1, f_4] = -\frac{1}{\sqrt{2}\delta_3}.$$

$$(2.5.3) \quad [f_2, f_3] = \frac{1}{\sqrt{2}}\delta_3, [f_2, f_4] = -\frac{1}{\sqrt{2}}\delta_2.$$

$$(2.5.4) \quad [f_3, f_4] = i(e_{11} - e_{33}).$$

Using (2.5.1) - (2.5.4) we compute all the values $\phi^3(X_1, X_2, X_3)$ easily, where either (X_1, X_2) are (δ_i, δ_j) or (X_1, X_2) are (f_i, f_j) for some (i, j) and X_3 is one of the basic vectors (δ_i, f_i) . In this way we get the equality (2.4.1). \square

Now compare (2.4.1) with (2.3) we observe that these two 3-forms are $Gl(R^7)$ equivalent (e.g. by rescaling δ_i with factor $(1/2)$). This proves that $\phi^3_{|T_e X^7}$ is a 3-form of \tilde{G}_2 -type. This completes the first step.

Step 2. Using step 1 it suffices to show that

$$(2.6) \quad D m_L(\alpha^{-1})(T_\alpha X^7) = T_e X^7$$

for any $\alpha \in SO(2)^1 \subset X^7$, $\alpha \neq e$.

Since $X^7 \supset \alpha \cdot SU(2)$, we have

$$(2.7) \quad su(2) \subset D m_L(\alpha^{-1})(T_\alpha X^7).$$

Denote by $SO(3)$ the standard orthogonal group of $\mathbb{R}^3 \subset \mathbb{C}^3$. Since $\alpha \in SO(3) \subset X^7$, we have $D m_L(\alpha^{-1})(T_\alpha SO(3)) \subset D m_L(\alpha^{-1})(T_\alpha X^7)$. In particular we have

$$(2.8) \quad \langle (e_{12} - e_{21}), (e_{13} - e_{31}) \rangle_{\otimes \mathbb{R}} \subset D m_L(\alpha^{-1})(T_\alpha X^7).$$

Since $SU(2) \cdot \alpha \subset X^7$, we have

$$(2.9) \quad Ad(\alpha^{-1})su(2) \subset D m_L(\alpha^{-1})(T_\alpha X^7).$$

Using the formula

$$Ad(\alpha^{-1}) = \exp(-ad(t \cdot \frac{e_{12} - e_{21}}{\sqrt{2}})), t \neq 0$$

we get immediately from (2.7), (2.8), (2.9) the following inclusion

$$\langle i(e_{12} + e_{21}), i(e_{13} + e_{31}) \rangle_{\otimes \mathbb{R}} \subset D m_L(\alpha^{-1})(T_\alpha X^7)$$

which together with (2.7), (2.8) imply the desired equality (2.6).

This completes the proof of Theorem 2.1. \square

3 The orbits of the action of the automorphism group of (X^7, ω^3) on X^7 .

Denote by $Aut(X^7, \omega^3)$ the full automorphism group of the manifold X^7 equipped by the 3-form ω^3 constructed in Theorem 2.1. In this section we shall prove the following

3.1. Theorem. *The co-dimension of the $Aut(X^7, \omega^3)$ -action on X^7 is equal to 1.*

Lemma 2.2 implies that the group $SU(2) \times SU(2)$ is a subgroup of the automorphism group $Aut(X^7, \omega^3)$. Further taking into account (2.2.1) we note that the dimension of a generic orbit of the $Aut(X^7, \omega^3)$ -action on X^7 is at least 6. Hence Theorem 3.1 is a corollary of the following Proposition, which implies that our manifold (X^7, ω^3) is not stably homogeneous.

3.2. Proposition. *Let G be a Lie group which acts transitively on X^7 . Then there does not exist a G -invariant 3-form of \tilde{G}_2 -type on X^7 .*

Proof of Proposition 3.2. Since $S^3 \times S^4$ is connected, we can assume that G is connected, because the identity component of G acts also transitively on $S^3 \times S^4$. Denote by \tilde{G} the simply connected covering of G . Then \tilde{G} acts transitively and almost effectively on X^7 . We shall prove the following

3.3. Proposition. *Let X^n be a compact connected space with $\pi_1(X^n) = 0 = \pi_2(X^n)$. Suppose that \tilde{G} is a connected and simply connected group which acts transitively on X^n . Let $\tilde{G}_{s,u}$ be a maximal compact group of \tilde{G} . Then $\tilde{G}_{s,u}$ acts transitively on X^n .*

Proof. We denote by \tilde{H} the isotropy of the action of \tilde{G} on X^n . Since X^n is simply connected, the subgroup \tilde{H} is connected. From the homotopy exact sequence

$$\pi_2(X^n) = 0 \rightarrow \pi_1(\tilde{H}) \rightarrow \pi_1(\tilde{G}) = 0$$

we obtain that \tilde{H} is also simply connected. Using the Levi-Maltsev decomposition theorem we write

$$\tilde{G} = \tilde{G}_s \times V^k, \quad \tilde{H} = \tilde{H}_s \times V^r,$$

where \tilde{G}_s is a semisimple Lie subgroup of \tilde{G} , V^k is a solvable normal subgroup of \tilde{G} and \tilde{H}_s is semisimple Lie subgroup of \tilde{H} and V^r is a solvable normal subgroup of \tilde{H} . Moreover we can assume that \tilde{H}_s is a subgroup of \tilde{G}_s after applying an inner automorphism group of \tilde{G} . Using the fibration

$$V^r \rightarrow \tilde{G}/\tilde{H} \rightarrow \tilde{G}/\tilde{H}_s$$

we conclude that the quotient space \tilde{G}/\tilde{H}_s is homotopic to \tilde{G}/\tilde{H} . Further using the fibration $V^k \rightarrow \tilde{G}/\tilde{H}_s \rightarrow \tilde{G}_s/\tilde{H}_s$ we conclude that \tilde{G}_s/\tilde{H}_s has the same homotopy type of X^7 .

Denote by h_s the Lie algebra of \tilde{H}_s and by $h_{s,u}$ the maximal compact Lie sub-algebra of the semi-simple Lie algebra h_s . The Lie subalgebra $h_{s,u}$ is the Lie algebra of a maximal compact Lie group $\tilde{H}_{s,u}$ of \tilde{H}_s . We can also assume that $h_{s,u}$ is a subalgebra of a maximal compact Lie subalgebra $g_{s,u}$ of g_s . Denote by $\tilde{G}_{s,u}$ the maximal compact subgroup of \tilde{G}_s whose Lie algebra is $g_{s,u}$. Using the Iwasawa decomposition we conclude that the quotient space $\tilde{G}_{s,u}/\tilde{H}_{s,u}$ has the same homotopy type of X^n . Since $\tilde{G}_{s,u}$ and $\tilde{H}_{s,u}$ are compact and using

$$H_{i+1}(X^n, \mathbb{Z}) = 0, \text{ if } i \geq n, \quad H_n(X^n, \mathbb{Z}) = \mathbb{Z},$$

we obtain that $\dim \tilde{G}_{s,u} = \dim \tilde{H}_{s,u} + n$. Thus to show that the action of $\tilde{G}_{s,u}$ on X^n is transitively, it suffices to prove

$$(3.4) \quad g_{s,u} \cap h = h_{s,u},$$

where \mathfrak{h} is the Lie algebra of \bar{H} . We have the Iwasawa and Levi-Maltsev decomposition

$$\mathfrak{h} = \mathfrak{h}_{s,u} + \mathfrak{h}_f + \mathfrak{r},$$

where \mathfrak{r} is the radical of \mathfrak{h} which is the Lie algebra of V^r . Suppose that $v \in ((g_{s,u} \cap \mathfrak{h}) \setminus \mathfrak{h}_{s,u})$. Since $\mathfrak{h}_{s,u} \in g_{s,u}$ we can assume that $v \in (\mathfrak{h}_f + \mathfrak{r})$ after adding some vector w in $\mathfrak{h}_{s,u}$, if necessary. Let $v = v_f + v_r$ be the decomposition of v into the components in \mathfrak{h}_f and \mathfrak{r} . Assume that $v_f \neq 0$. Then the the closure of the 1-parameter subgroup $\exp tv$ in \bar{H} is noncompact, since its projection to the quotient group $(\bar{H}/V^r) = \bar{H}_s$ is non-compact. Since \bar{H} is a closed subgroup of \bar{G} , we conclude that the closure of $\exp tv$ in \bar{G} is non-compact. On the other hand, the subgroup $\exp tv$ lies in the compact subgroup $\bar{G}_{s,u}$. Thus we arrive at a contradiction. Hence $v_f = 0$, so $v = v_r$. If $v_r \neq 0$, then the closure of the subgroup $\exp tv$ in \bar{H} lies in V^r and it is also non-compact. Hence v_r also vanishes. This proves (3.4) and completes the proof of Proposition 3.3. \square

Continuation of the proof of Proposition 3.2. Using Proposition 3.2 it suffices to consider only the transitive action of the compact, connected and simply connected group $\bar{G}_{s,u}$ on $X^7 = S^3 \times S^4$. Next we can assume that this action is almost effective, since otherwise the quotient $\bar{G}_{s,u}/N$ acts effectively and transitively on X^7 , where N is the kernel of the action. The group $\bar{G}_{s,u}/N$ is compact, connected, and semi-simple. Its compact universal covering acts almost effectively on X^7 .

Next we observe that the isotropy group of the action of $\bar{G}_{s,u}$ on X^7 is $\bar{H}_{s,u}$, since the isotropy group of this action must be connected. From now on we shall assume that $\bar{G}_{s,u} = \bar{H}_{s,u}$ admits a $\bar{G}_{s,u}$ -invariant 3-form ω^3 of \tilde{G}_2 -type on X^7 .

Let $e = e \cdot \{H\}$ be a reference point on X^7 . Denote by ρ the isotropy representation of $\mathfrak{h}_{s,u}$ on $T_e(\bar{G}/\bar{H}) = \mathbb{R}^7$. Since $\mathfrak{h}_{s,u}$ is semi-simple, the representation ρ is faithful, i.e. the kernel of ρ is empty. Clearly $\rho(\mathfrak{h}_{s,u})$ must be a sub-algebra of the Lie algebra $\tilde{\mathfrak{g}}_2$ of \tilde{G}_2 , in particular the rank of $\mathfrak{h}_{s,u}$ is at most 2.

3.5. Lemma. *We have $\mathfrak{h}_{s,u} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.*

Proof. Since ρ is faithful, we identify $\mathfrak{h}_{s,u}$ with its image in $\tilde{\mathfrak{g}}_2$. Clearly the algebra $\mathfrak{h}_{s,u}$ is also a subalgebra of a maximal compact Lie subalgebra in $\tilde{\mathfrak{g}}_2$, so $\mathfrak{h}_{s,u}$ is either $\mathfrak{su}(2)$ or $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

Suppose that $\mathfrak{h}_{s,u} = \mathfrak{su}(2)$. Then $\dim \bar{G}_{s,u} = 10$ and therefore $\bar{G}_{s,u}$ must be a product of classical Lie groups. In particular we know that $\pi_4(\bar{G}_{s,u})$ is a finite group (see e.g. [8, §25.4].)

We also have $\pi_3(\bar{H}_{s,u}) = \mathbb{Z}$. Further we observe that the inclusion $\pi_3(\bar{H}_{s,u}) \rightarrow \pi_3(\bar{G}_{s,u})$ is injective, since the subgroup $\bar{H}_{s,u}$ realizes a non-trivial element in $H_3(\bar{G}_{s,u}, \mathbb{R})$. Now let us consider the homotopy exact sequence

$$(3.6) \quad \pi_4(\bar{G}) \rightarrow \pi_4(X^7) = \mathbb{Z}_2 \oplus \mathbb{Z} \rightarrow 0 = \ker(\pi_3(\bar{H}) \rightarrow \pi_3(\bar{G})).$$

The exact sequence (3.6) implies that $\pi_4(\bar{G})$ contains a subgroup \bar{Z} which is impossible. Hence $\mathfrak{h}_{s,u}$ cannot be $\mathfrak{su}(2)$. \square

3.7. Lemma. *We have $\mathfrak{g}_{s,u} = \mathfrak{su}(2) \oplus \mathfrak{so}(5)$.*

Proof. Since $\dim g_{s,u} = 6 + 7 = 13$, it is easy to see that $g_{s,u}$ must be the product of classical compact Lie groups. Furthermore we notice that, since the dimension of $\bar{G}_{s,u}/\bar{H}_{s,u}$ is odd, the rank of $\bar{G}_{s,u}$ must be strictly greater than the rank of $\bar{H}_{s,u}$. Let $g_{s,u}^1$ be a simple component of $g_{s,u}$. Then the rank of $g_{s,u}^1$ is less than or equal to 2, otherwise the dimension of $g_{s,u}$ is greater than or equal to 15. Thus $g_{s,u}$ must be a sum of simple components of rank 1 or 2. Next, by dimension reason, $g_{s,u}$ cannot be a sum of only components of rank 1 and it cannot contain more than one simple component of rank 2. Looking at the table of simple Lie algebras we arrive at Lemma 3.7. \square

3.8. Lemma. *The subalgebra $h_{s,u}$ lies in the component $so(5)$ of $g_{s,u}$.*

Proof. We notice that $\bar{G}_{s,u}$ has the same homotopy type as \bar{G} and hence it is simply connected. Thus $\bar{G}_{s,u}$ is the product $SU(2) \times Spin(5)$. Analogously $\bar{H}_{s,u}$ must be $SU(2) \times SU(2)$. Denote by e_1 the composition of the embedding $e : h_{s,u} \rightarrow g_{s,u}$ with the projection $p : g_{s,u} \rightarrow su(2)$. To prove Lemma 3.8 it suffices to show that the kernel of e_1 is equal to $h_{s,u}$.

Suppose that the kernel of e_1 is not equal to $h_{s,u}$. Then this kernel must be one of the components $su(2)$ of $h_{s,u}$ since e_1 can not be injective. Denote this kernel by $su(2)^2$ and let $su(2)^1$ be the other component of $h_{s,u}$. The isomorphism $e_1 : su(2)^1 \rightarrow su(2)$ lifts to an isomorphism denoted by \tilde{e}_1 from the corresponding component $SU(2)^1$ of $\bar{H}_{s,u}$ to the component $SU(2)$ of $\bar{G}_{s,u}$. Denote by \tilde{e}_2 the homomorphism from $SU(2)^1 \times SU(2)^2$ to $Spin(5)$. Since $\tilde{e}_1(SU(2)^2) = Id$, the restriction of \tilde{e}_2 to $SU(2)^2$ is an embedding. We shall construct a map

$$I : \bar{G}_{s,u}/\bar{H}_{s,u} \rightarrow Spin(5)/\tilde{e}_2(SU(2)^2)$$

and show that I is a homeomorphism. For each point $(a \cdot b) \cdot \{\bar{H}_{s,u}\} \in \bar{G}_{s,u}/\bar{H}_{s,u}$, where $a \in SU(2)$ and $b \in Spin(5)$ we set

$$(3.9) \quad I((a, b) \cdot \{\bar{H}_{s,u}\}) = (b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \in Spin(5)/\tilde{e}_2(SU(2)^2).$$

The map (3.9) is well-defined, since for any $(\theta_1, \theta_2) \in SU(2)^1 \times SU(2)^2$ we have

$$\begin{aligned} I(a \cdot \tilde{e}_1(\theta_1), b \cdot \tilde{e}_2(\theta_1, \theta_2)) \cdot \{\bar{H}_{s,u}\} &= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a \cdot \tilde{e}_1(\theta_1))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\ &= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a) \cdot \theta_1)]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\ &= (b \cdot \tilde{e}_2(\theta_1, \theta_2) \cdot [\tilde{e}_2(\theta_1)]^{-1} \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\ &= (b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1}) \cdot \{\tilde{e}_2(SU(2)^2)\} \\ &= (b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot \tilde{e}_2(\theta_2)) \cdot \{\tilde{e}_2(SU(2)^2)\} = I((a, b) \cdot \{\bar{H}_{s,u}\}). \end{aligned}$$

Substituting $a = 1$ into (3.9) we obtain that I is surjective. Now suppose that $I((a; b) \cdot \{\bar{H}_{s,u}\}) = I((a', b') \cdot \{\bar{H}_{s,u}\})$. Then

$$b' \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a'))]^{-1} = b \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot \tilde{e}_2(\theta_2) = b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1},$$

for some $\theta_2 \in SU(2)^2$. Hence

$$(3.10) \quad b' = b \cdot \tilde{e}_2(\theta_2) \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a))]^{-1} \cdot [\tilde{e}_2(\tilde{e}_1^{-1}(a'))] = b \cdot \tilde{e}_2(\theta_2) \cdot \tilde{e}_2[(\tilde{e}_1^{-1}(a^{-1})) \cdot \tilde{e}_1^{-1}(a')].$$

Set $\theta_1 = \tilde{e}^{-1}(a^{-1} \cdot a')$. Then $a' = a \cdot \tilde{e}_1(\theta_1)$, and we get from (3.10)

$$b' = b \cdot \tilde{e}_2(\theta_2) \cdot \tilde{e}_2(\theta_1) = b \cdot \tilde{e}_2(\theta_1) \cdot \tilde{e}_2(\theta_2).$$

Hence I must be injective. Thus we have proved that I is a homeomorphism. Hence $\pi_4(\text{Spin}(5)/\tilde{e}_2(SU(2)^2)) = \mathbb{Z}$. Now let us consider the homotopy exact sequence

$$\pi_4(\text{Spin}(5)) \xrightarrow{j} \pi_4(\text{Spin}(5)/\tilde{e}_2(SU(2)^2)) \rightarrow \pi_3(\tilde{e}_2(SU(2)^2)) \xrightarrow{i} \pi_3(\text{Spin}(5)).$$

Clearly i is injective. Hence j must be surjective. But $\pi_4(\text{Spin}(5))$ is a finite group. Therefore $\pi_4(\text{Spin}(5)/\tilde{e}_2(SU(2)^2))$ is also a finite group. Thus we arrive at a contradiction, since $\pi_4(X^7)$ contains an infinite subgroup. This proves Lemma 3.8. \square

Completion of the proof of Proposition 3.2. From Lemma 3.8 it follows that the restriction of the isotropy action of $h_{s,u}$ on each component $su(2)^i$ contains the sum of three trivial representations. Now look at the table of the irreducible 7-dimensional representation π_1 of g_2 (see e.g. [13, table 1]) we know that the weights of this representation are $\pm\varepsilon_i, 0$. Here $i = \overline{1, 3}$ and $\varepsilon_i - \varepsilon_j \pm \varepsilon_i$ are the roots of g_2 . The complexification of the maximal compact algebra $so(4)$ in g_2 is the direct sum $so(4)_{\otimes \mathbb{C}} = \langle h_{\varepsilon_1}, e_{\varepsilon_1}, e_{-\varepsilon_1} \rangle_{\otimes \mathbb{C}} \oplus \langle h_{\varepsilon_2 - \varepsilon_3}, e_{\varepsilon_2 - \varepsilon_3}, e_{\varepsilon_3 - \varepsilon_2} \rangle_{\otimes \mathbb{C}}$. Hence the restriction of the π_1 to $sl(2) = \langle h_{\varepsilon_1}, e_{\varepsilon_1}, e_{-\varepsilon_1} \rangle_{\otimes \mathbb{C}}$ is the sum of two irreducible components of dimension 2 and the adjoint representation of dimension 3. Thus it has no trivial component. We arrive at a contradiction. This proves Proposition 3.2. \square

4 Appendix. A necessary and sufficient condition for a closed 7-manifold to admit a 3-form of \tilde{G}_2 -type.

A.1. Theorem. *Suppose that M^7 is a compact 7-manifold. Then M^7 admits a 3-form of \tilde{G}_2 -type, if and only if M^7 is orientable and spinnable. Equivalently the first and the second Stiefel-Whitney classes of M^7 vanish.*

Proof. If M^7 admits a \tilde{G}_2 -structure, it admits also a G_2 -structure, since a maximal compact group of \tilde{G}_2 is also a compact subgroup of the group G_2 . Applying the Gray criterion for the existence of a G_2 -structure [9] we obtain the ‘‘only if’’ statement. Now let us prove the ‘‘if’’ part. By a result of Dupont [4] any compact orientable 7-manifold admits three linearly independent vector fields. Hence M^7 admits an $SO(4)$ -structure. In particular M^7 admits an $SO(3) \times SO(4)$ -structure. Denote by $Spin(3, 4)$ the Lie subgroup in $Spin(7, \mathbb{C})$ whose Lie algebra is $so(3, 4)$. It is known (see e.g. [2 Theorem 5]) that $\pi_1(Spin(3, 4)) = \mathbb{Z}_2$. Denote by $(SO(3) \times SO(4))^*$ the connected subgroup in $Spin(3, 4)$ whose Lie algebra is $so(3) \times so(4)$. Clearly $(SO(3) \times SO(4))^*$ is a maximal compact Lie subgroup of $Spin(3, 4)$. Taking into account the isomorphism $Spin(7)/(SO(3) \times SO(4))^* = SO(7)/(SO(3) \times SO(4))$ we conclude that M^7 admits an $(SO(3) \times SO(4))^*$ -structure and hence a $Spin(3, 4)$ -structure. Now we shall prove that the $Spin(3, 4)$ -structure on M^7 is reduced to a \tilde{G}_2 -structure. It is easy to see that the quotient $Spin(3, 4)/\tilde{G}_2$ is the pseudo-sphere $S^7(4, 4)$ in the space $\mathbb{R}e_0 \times \mathbb{R}^7$ of the spinor representation of $Spin(3, 4)$. This pseudo-sphere bundle over M^7 admits a section e_0 . Hence M^7 admits a \tilde{G}_2 -structure. \square

Using Theorem A.1 and the Gromov h-principle (see e.g. [5]) for open 3-forms of \tilde{G}_2 -type we can get a lot examples of open 7-manifolds admitting a closed 3-form of \tilde{G}_2 -type.

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