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MULTISTAGE STOCHASTIC PROGRAMS: THE STATE–OF–THE–ART AND SELECTED BIBLIOGRAPHY[1]

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The paper gives a brief introduction into the problems of multistage stochastic programming with emphasis on the modeling issues (Section 2) and on the contemporary numerical advances (Section 3). Extensive classified bibliography is contained in the last Section.

1. INTRODUCTION

Mathematical modeling of economic, ecological and other complex systems with the goal to analyze them and to find optimal decisions has been studied for many years. The challenging problems connected with running market economies, of realistic approaches to environmental protection, etc., are that the decisions are to be made under uncertainty. Therefore the traditional deterministic optimization models are limited in practical applications because the models parameters (future demands, interest rates, water inflows, resources, etc.) are not completely known when some decision is needed. A typical approach of substituting expected values for all random parameters can lead to inferior solutions that discredit both the model designer and the use of optimization methods.

Moreover, in controlling or analyzing complex systems, various levels of uncertainties have to be taken into account: besides of requirements for proper treatment of nonhomogeneity of raw input materials, volatility of prices, demands or of water inflows one is asked to cope with future development of factors essential for running the system such as interest rates or innovations of technological progresses and to hedge against legislative changes and complete or partial changes of economic and other

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policies. In principle, these uncertainties can be modeled by various ways and one of them is stochastic programming.

Stochastic programming gives a probabilistic interpretation to the above mentioned uncertainties. It deals with optimization problems in which random parameters are explicitly spelled out and allows for incorporation of risk into optimization. It originated in the late fifties, cf. seminal papers by Beale [R1], Dantzig [R3], Tintner [R13] and Charnes and Cooper [R2]; some one dimensional examples of stochastic programming can be traced even in earlier papers on inventory, maintenance, etc.

The general formulation of stochastic programming problems, cf. [155], reads:

$$\text{Minimize} \quad \mathsf{E} f_0(\boldsymbol{x}, \omega) \tag{1}$$

$$\begin{aligned} \text{subject to} \quad & \mathsf{E}\{f_i(\boldsymbol{x}, \omega)\} \le 0, \quad i = 1, \dots, k \\ & \mathsf{E}\{f_i(\boldsymbol{x}, \omega)\} = 0, \quad i = k+1, \dots, k+r \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned} \tag{2}$$

where

ω is a random parameter with support Ω, probability distribution P and the corresponding expectation E,

$\mathcal{X}$ is a given nonempty closed set,

$f_0 : R^n \times \Omega \to R^1 \cup \{+\infty\}$, $f_i : R^n \times \Omega \to R^1$ are given functions. This formulation looks like an ordinary static deterministic program whose objective and constraints are of the form of expectations. Still it covers a whole spectrum of stochastic programming models – static, two-stage and multistage ones, models with probabilistic constraints and those of penalty or recourse type – provided that the important requirement of *nonanticipativity*, i. e., the requirement that decisions must occur before observations, has been properly treated. In the simplest case, $\mathcal{X} \subset R^n$ and the decision variable $\boldsymbol{x}$ in (1), (2) corresponds mostly to the main decision that has to be selected before the realizations of the random parameters ω can be observed. In more complicated cases, $\boldsymbol{x}$ consists of several subvectors, say $\boldsymbol{x}^1, \dots, \boldsymbol{x}^T$ that correspond to decisions to be taken at the stages $1, \dots, T$ of the decision process and, as we shall see in Section 2, it may be even convenient to model the decisions as random functions. The requirement of nonanticipativity can be incorporated into the definition of functions f_i or it can be formulated as an explicit additional constraint, see Wets [151] and Rockafellar and Wets [132] for the first developments of this idea.

A typical assumption is that the probability distribution P is known (decision making under risk) and independent of decision $\boldsymbol{x}$, on the other hand, various approaches have been designed that deal with stochastic programs under incomplete knowledge about the underlying distribution P (decision making under uncertainty); for a survey of stochastic programming applications under incomplete knowledge about P see [R6].

The existence of expectations in the constraints (2) for all feasible decisions x is guaranteed by special assumptions from case to case. Proper treatment of expectations of extended real functions that enter the objective (1) is not straightforward, see [155], on the other hand, using this formulation helps to concentrate on influence of specific constraints, e. g. of the nonanticipativity constraints, or to study problems in which the choice of solutions is limited by implicit, induced constraints on the solvability of the system. These constraints are not included explicitly in the system of constraints of (2), they can be sometimes detected at least partly by suitable preprocessing techniques, cf. [R16], or treated in course of numerical computations. Ability to choose decisions that perform well regardless these hidden constraints is one of strongpoints of stochastic programming.

The prevailing theoretical issue in models with probabilistic constraints turned to be the convexity property of the resulting deterministic program of type (1), (2) with indicator functions at the place of some of f_i; cf. Prékopa [R8] for the important breakthrough.

The theoretical results have been collected in various works, e. g., monographs [R7], [165] and collection [D]; for more recent results see survey [155] or the new textbook [85]. The progress in designing efficient algorithms (cf. Part II of [P-W], [E-W] and Part II of [B-W]) has resulted into special software packages suitable for solving large stochastic programs that arise in a variety of applications such as power generation planning, financial modeling or location analysis. The main stumbling block for algorithms is necessity to compute repeatedly values of multidimensional integrals (expectations of recourse functions or probabilistic constraints) that enter the nonlinear program (1), (2). To overcome this problem various approximation schemes, both stochastic and deterministic ones, were designed; see, e. g., [R18] and the references therein. In connection with evaluation of their properties and with the need for proper treatment of uncertainty about the probability distribution of random parameters, various error bounds have been derived and miscellaneous results on stability and postoptimality have been achieved; see, e. g., Part I of [P-W], Part I of [B-W] or survey papers [R4], [R5] and the references therein. A new area of interest is integer stochastic programming with many open theoretical problems and various interesting applications; one of the first papers is [126].

The present stage of knowledge and of computer technologies gives a chance to turn attention to the dynamic multistage stochastic programming problems. This area was mentioned already in the seminal paper of Dantzig [R3] and in his monograph [27] and the first theoretical results on multistage stochastic programs with recourse were obtained as generalization of those valid for two-stage stochastic programs, e. g., [113], [118] – [121], [149], [150]. Deep theoretical results closely connected with the crucial problem of modeling the multistage nature of the decision

process can be found, e. g., in [32] – [34], [128] – [135]; in these papers, multistage stochastic programs are treated as optimization problems in infinite dimensional spaces. For expositions concerning multistage stochastic programs with probabilistic constraints see, e. g., [47], [124], [162] – [166].

Besides of finance, the most popular areas for applications of multistage stochastic programs seem to be for the present production planning and management including electric power generation and transmission, transportation, optimal exploitation of exhaustible resources and water resources management; see Section 4 for references.

Except for bibliography (Section 4), we shall limit ourselves to multistage stochastic programs with recourse. After discussing briefly the modeling issues (Section 2), we shall turn our attention to scenario based multistage stochastic programs with recourse and in Section 3, we shall report on the relevant numerical methods.

2. MULTISTAGE MODELS

Consider first the *two-stage stochastic programming problems*: A decision $\boldsymbol{x} \in \mathcal{X}$ should be selected *before* realizations of random parameters can be observed or their values revealed. After this information becomes available the decision process continues by the second stage, i. e., by the choice of an auxiliary decision that depends on the first-stage decision and exploits the already obtained information. The second-stage decision is interpreted as an updating activity (portfolio revision, adjustment of the production plan, etc.) that brings about additional, *recourse* costs. The requirement that the first-stage decision $\boldsymbol{x}$ depends only on the past information and that it cannot depend on future observations of random parameters corresponds to the more general *nonanticipativity* property of the multistage decision processes.

It is important to realize that the stages do not necessarily correspond to time periods. The first-stage decisions consist of all decisions that have to be selected before the information is revealed whereas the second-stage decisions are allowed to adapt to this information; for a detailed explanation and examples see [49].

The model formulation that reflects the above verbally described decision scheme can be written in the following way:

Let $\mathcal{X}_1, \mathcal{X}_2$ be nonempty closed sets in R^{n_1}, R^{n_2}, respectively, and let (Ω, Σ, P) be a probability space, $f_{1i}, \forall i$ given functions on R^{n_1}, $f_{2i}, \forall i$ given functions on $\Omega \times R^{n_1} \times R^{n_2}$ that are P measurable for each $\boldsymbol{x}^1 \in R^{n_1}$, $\boldsymbol{x}^2 \in R^{n_2}$. The problem is to

$$\text{minimize} \qquad f_{10}(\boldsymbol{x}^1) + \int_\Omega f_{20}(\boldsymbol{x}^1, \boldsymbol{x}^2(\omega), \omega) P(\mathrm{d}\omega) \tag{3}$$

over all $\boldsymbol{x}^1 \in R^{n_1}$ that satisfy

$$\boldsymbol{x}^1 \in \mathcal{X}_1 \quad \text{and} \quad f_{1i}(\boldsymbol{x}^1) \le 0, \quad i = 1, \dots, m_1 \tag{4}$$

and $\boldsymbol{x}^2$ measurable such that P almost surely (a. s.)

$$\boldsymbol{x}^2(\omega) \in \mathcal{X}_2 \quad \text{and} \quad f_{2i}(\boldsymbol{x}^1, \boldsymbol{x}^2(\omega), \omega) \le 0, \quad i = 1, \dots, m_2, \tag{5}$$

an *optimization problem in a suitably chosen infinite dimensional space of measurable functions.*

An alternative formulation of the type (1), (2) reads

$$\text{minimize} \quad \mathsf{E}\{f_0(\boldsymbol{x}^1, \omega)\} \text{ subject to the constraints } (4) \tag{6}$$

with $f_0(\boldsymbol{x}^1, \omega)$ defined as follows:

$$f_0(\boldsymbol{x}^1, \omega) := f_{10}(\boldsymbol{x}^1) + \inf_{\boldsymbol{x}^2} \left[f_{20}(\boldsymbol{x}^1, \boldsymbol{x}^2, \omega) | \; f_{2i}(\boldsymbol{x}^1, \boldsymbol{x}^2, \omega) \le 0 \; \forall i, \; \boldsymbol{x}^2 \in \mathcal{X}_2 \right]. \tag{7}$$

The function $f_{20}(\boldsymbol{x}^1, \boldsymbol{x}^2, \omega)$ that appears in both formulations gives the cost of the recourse connected with the (not necessarily optimal) second-stage decision $\boldsymbol{x}^2$ in case that $\boldsymbol{x}^1$ is the accepted first-stage decision and

ω is the subsequently observed realization of the random parameters. The function $f_{10}(\boldsymbol{x}^1)$ corresponds to the costs that are independent of the second-stage decisions and it can be also defined as an expectation.

For the convex case and for $\boldsymbol{x}^2$ in (3), (5) restricted to the class of essentially bounded measurable functions, Rockafellar and Wets [R11] gave relatively weak conditions under which the introduced formulations are equivalent. The result applies, for instance, to $\mathcal{X}_1$ and $\mathcal{X}_2$ bounded. The first formulation is suitable for theoretical analysis such as optimality conditions or duality properties for problem (3) – (5). The results depend, inter alia, on the considered space of the measurable functions $\boldsymbol{x}^2(\omega)$. We refer to the series of papers [R9] – [R12] or to [32].

Special attention has been paid to the class of two-stage *linear* stochastic programs, known under the name *stochastic linear programs with recourse.* Their generic form that corresponds to the formulation (6), (7) reads

$$\text{minimize} \quad \mathsf{E}\left\{\boldsymbol{c}(\omega)^\top \boldsymbol{x} + Q(\boldsymbol{x},\omega)\right\} \quad \text{on the set} \quad \mathcal{K}_1 = \left\{\boldsymbol{x} \in R_+^{n_1} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\right\} \tag{8}$$

with the recourse costs $Q(\boldsymbol{x},\omega)$ defined for a given $\boldsymbol{x}$ and ω as the optimal value of the auxiliary *second-stage program*

$$\text{minimize} \quad \boldsymbol{q}(\omega)^\top \boldsymbol{y} \tag{9}$$

$$\text{subject to} \quad \boldsymbol{y} \in R_+^{n_2} \quad \text{that satisfy} \quad \boldsymbol{W}(\omega)\boldsymbol{y} + \boldsymbol{T}(\omega)\boldsymbol{x} = \boldsymbol{h}(\omega).$$

Notice that only the expectations of the random coefficients $\boldsymbol{c}(\omega)$ enter the above formulation (8) so that fixed costs $\boldsymbol{c}$ can be used without any loss of generality.

According to properties of the recourse matrix $\boldsymbol{W}(\omega)$ this stochastic linear program (SLP) is classified as

◦ SLP with *fixed recourse* if $\boldsymbol{W}(\omega) \equiv \boldsymbol{W}$, a fixed matrix,

◦ SLP with *fixed complete recourse* if $\boldsymbol{W}(\omega) \equiv \boldsymbol{W}$ and if for an arbitrary right hand side $\boldsymbol{w}$ the system $\boldsymbol{W}\boldsymbol{y} = \boldsymbol{w}$ has a nonnegative solution,

◦ SLP with *relatively complete recourse* if the second-stage problem (9) is a. s. feasible for an arbitrary $\boldsymbol{x} \in \mathcal{K}_1$ and $\omega \in \Omega$, etc.

Induced constraints concern the case when the second-stage program may happen to be infeasible for a first-stage decision $\boldsymbol{x} \in \mathcal{K}_1$ and a realization of the random parameter ω. From the point of view of the modeled problem, accepting such decision may lead to disaster (interruption of the production process, bankrupcy, environmental catastroph, etc.) and infinite costs $Q(\boldsymbol{x},\omega)$ can be used to reflect this fact; compare with possibly infinite values of $f_0(\boldsymbol{x},\omega)$ in (1). Another idea is to complete the constraints of the deterministic program (8) to avoid accepting first-stage decisions that may lead to infeasible second-stage program (9); we shall denote by $\mathcal{K}_2$ the set described by induced constraints. As to the structure of the resulting deterministic equivalent program

$$\text{minimize} \quad \mathsf{E}\left\{\boldsymbol{c}(\omega)^\top \boldsymbol{x} + Q(\boldsymbol{x},\omega)\right\} \quad \text{on the set} \quad \mathcal{K}_1 \cap \mathcal{K}_2 \tag{10}$$

it is possible to prove that (10) is a convex program provided that $\boldsymbol{W}$ is a fixed matrix. Additional conditions are needed to guarantee that the objective function is well defined; for instance, a sufficient condition is the existence of all second-order moments of the vector of all random parameters. For SLP with fixed recourse the set $\mathcal{K}_2$ of induced constraints can be written as

$$\mathcal{K}_2 = \{\boldsymbol{x} : \exists\, \boldsymbol{y} \geq 0 \quad \text{such that} \;\; \boldsymbol{W}\boldsymbol{y} = \boldsymbol{h}(\omega) - \boldsymbol{T}(\omega)\boldsymbol{x} \;\; \text{a. s.}\} \tag{11}$$

so that (10) is equivalent, compare (3) –(5), with

$$\text{minimize} \quad \mathsf{E}\left\{\boldsymbol{c}(\omega)^\top \boldsymbol{x} + \boldsymbol{q}(\omega)^\top \boldsymbol{y}(\omega)\right\} \tag{12}$$

$$\text{subject to} \quad \boldsymbol{x} \in \mathcal{K}_1 \quad \text{and} \quad \boldsymbol{y}(\omega) \geq 0 \quad \text{such that}$$

$$\boldsymbol{W}\boldsymbol{y}(\omega) + \boldsymbol{T}(\omega)\boldsymbol{x} = \boldsymbol{h}(\omega) \quad \text{a. s.}$$

Moreover, under additional conditions on the probability distribution P, such as $\boldsymbol{T}(\omega) \equiv \boldsymbol{T}$ or for a *discrete* distribution P, the set $\mathcal{K}_2$ is convex *polyhedral* so that it can be generated by a finite number of suitable cuts step by step. For a detailed survey see [R17]. Characterization of SLP with *random* recourse is more demanding. The cases where the above results remain valid are, e. g., SLP with random complete recourse and problems with discrete distribution P; cf. [R16]. In the latter case, for instance, one can index by $s = 1, \ldots, S$ the vectors of all possible realizations of the random coefficients in $\boldsymbol{q}$, $\boldsymbol{W}$, $\boldsymbol{T}$ and $\boldsymbol{h}$ and those of the corresponding second-stage variables $\boldsymbol{y}$, assign probabilities p_s to these realizations and arrives thus at the following linear program of a special dual block angular structure

$$\text{minimize} \qquad \boldsymbol{c}^\top \boldsymbol{x} + \sum_{s=1}^{S} p_s \boldsymbol{q}_s^\top \boldsymbol{y}_s \tag{13}$$

$$\text{subject to} \quad \begin{array}{lllllll} \boldsymbol{A}\boldsymbol{x} & & & & & & = \boldsymbol{b} \\ \boldsymbol{T}_1\boldsymbol{x} & + & \boldsymbol{W}_1\boldsymbol{y}_1 & & & & = \boldsymbol{h}_1 \\ \boldsymbol{T}_2\boldsymbol{x} & + & & \boldsymbol{W}_2\boldsymbol{y}_2 & & & = \boldsymbol{h}_2 \\ \vdots & & & & \ddots & & \vdots \\ \boldsymbol{T}_S\boldsymbol{x} & + & \ldots & & + & \boldsymbol{W}_S\boldsymbol{y}_S & = \boldsymbol{h}_S \end{array} \tag{14}$$

$$\boldsymbol{x} \geq 0,\ \boldsymbol{y}_s \geq 0,\ s = 1, \ldots, S.$$

The size of the program (13), (14) can be very large; for instance, consider just random right hand sides $\boldsymbol{h}$ consisting of m_2 independent random components with probability distributions approximated by alternative ones: it gives $m_1 + 2^{m_2}$ constraints in (14). Usefulness of special numerical techniques is obvious; see [E-W]. An alternative equivalent formulation of (13), (14)

$$\text{minimize} \qquad \boldsymbol{c}^\top \boldsymbol{x} + \sum_{s=1}^{S} p_s \boldsymbol{q}_s^\top \boldsymbol{y}_s \tag{13}$$

$$\begin{array}{llllll} \text{subject to} & \boldsymbol{A}\boldsymbol{x} & & & = \boldsymbol{b} & \\ & & \boldsymbol{T}_s\boldsymbol{x}_s + & \boldsymbol{W}_s\boldsymbol{y}_s & = \boldsymbol{h}_s & \qquad (14') \\ & \boldsymbol{x} - & \boldsymbol{x}_s & & = 0 & \end{array}$$

$$\boldsymbol{x} \geq 0, \quad \boldsymbol{x}_s \geq 0, \quad \boldsymbol{y}_s \geq 0, \quad s = 1, \ldots, S$$

helps to build procedures based on relaxation of the *nonanticipativity constraints*

$$\boldsymbol{x} = \boldsymbol{x}_s, s = 1, \ldots, S. \tag{15}$$

In the general *T-stage stochastic program* we think of a stochastic data process

$$\omega = \{\omega^1, \ldots, \omega^T\}$$

whose realizations are data trajectories in (Ω, Σ, P) and of a vector decision process

$$\boldsymbol{x} = \{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T\},$$

a measurable function of ω. The whole sequence of decisions and observations can be, e. g.,

$$\begin{aligned} &\boldsymbol{x}^1, \omega^1, \boldsymbol{x}^2(\boldsymbol{x}^1, \omega^1), \omega^2, \ldots, \omega^{T-1}, \\ &\boldsymbol{x}^T(\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^{T-1}, \omega^1, \ldots, \omega^{T-1}) = \boldsymbol{x}^T(\boldsymbol{x}^1, \omega^1, \ldots, \omega^{T-1}) \end{aligned} \tag{16}$$

and ω^T that contributes to the overall observed costs. The decision process is *nonanticipative* in that sense that decisions taken in any stage of the process do not depend on future *realizations* of random parameters or on future decisions. On the other hand, in the course of the decision process, the past information is exploited. The dependence of the decisions solely on the history can be expressed as follows: Denote $\Sigma_{t-1} \subseteq \Sigma$ the σ-field generated by the observations $\{\omega^1, \ldots, \omega^{t-1}\}$ of the part of the stochastic data process that precedes the stage t. The dependence of the tth stage decision $\boldsymbol{x}^t$ only on these past observations means that $\boldsymbol{x}^t$ is Σ_{t-1}-adaptable or, in other words, that $\boldsymbol{x}^t$ is measurable with respect to Σ_{t-1}. In each of stages, the choice of a decision is limited by constraints that may depend on the previous decisions and observations. Assumption of nonanticipativity of these constraints in the sense that no additional constraint can enter later as a consequence of future observations corresponds to the assumption of relatively complete recourse for two stage

models and assuming nonanticipativity of constraints means that no induced, hidden constraints can appear.

Once more, two formulations can be used:

Let $\mathcal{X}_t$ be given nonempty sets in R^{n_t}, $t = 1, \ldots, T$ and denote

$$\mathcal{X}_t(\omega) = \left\{ \boldsymbol{x}^{t\bullet} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^t) : f_{ti}(\boldsymbol{x}^{t\bullet}, \omega) \leq 0,\ i = 1, \ldots, m_t,\ \boldsymbol{x}^{t\bullet} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_t \right\} \tag{17}$$

the set of the tth stage constraints, $t = 2, \ldots, T$ and by $f_0(\boldsymbol{x}, \omega)$ the overall cost connected with the decision process (16).

The *T-stage stochastic program* **is to find**

$$\begin{aligned} & \boldsymbol{x}^1 \in \mathcal{X}_1 \quad \textbf{such that} \quad f_{1i}(\boldsymbol{x}^1) \leq 0, \quad i = 1, \ldots, m_1 \\ & \boldsymbol{x}^t \quad \Sigma_{t-1}\ \textbf{-measurable, } t = 2, \ldots, T \\ & \boldsymbol{x}^{t\bullet} \in \mathcal{X}_t(\omega), t = 2, \ldots, T \qquad \textbf{a. s.} \\ \textbf{that minimizes} \quad & \mathsf{E}\left\{ f_0(\boldsymbol{x}^1, \boldsymbol{x}^2(\omega), \ldots, \boldsymbol{x}^T(\omega), \omega) \right\}. \end{aligned} \tag{18}$$

Allowing for extended real objective function f_0 opens for instance the possibility to incorporate the constraints $\boldsymbol{x}^{t\bullet} \in \mathcal{X}_t(\omega)$, $t = 2, \ldots, T$ into the objective function and to study the influence of the nonanticipativity constraints, cf. [58], [132], or to include, at least theoretically, the prospective induced constraints.

The second formulation is based on a recursive evaluation of the overall objective function, compare with (7), that spells out the nonanticipativity in an explicit way:

$$\textbf{minimize} \quad \mathsf{E}\{f_0(\boldsymbol{x}, \omega)\} := f_{10}(\boldsymbol{x}^1) + \mathsf{E}_{\omega^1}\left\{ \varphi_{10}(\boldsymbol{x}^1, \omega^1) \right\} \tag{19}$$

$$\textbf{subject to} \quad \boldsymbol{x}^1 \in \mathcal{X}_1 \quad \textbf{and} \quad f_{1i}(\boldsymbol{x}^1) \leq 0, \quad i = 1, \ldots, m_1,$$

where for $t = 2, \ldots, T, \varphi_{t-1,0}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{t-1}, \omega^1, \ldots, \omega^{t-1})$ is the optimal value of the stochastic program

$$\textbf{minimize} \quad f_{t0}(\boldsymbol{x}^t) + \mathsf{E}_{\omega^t}\left\{ \varphi_{t0}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^t, \omega^1, \ldots, \omega^{t-1}, \omega^t) \right\} \tag{20}$$

with respect to $\boldsymbol{x}^t \in \mathcal{X}_t$ that fulfils

$$f_{ti}(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^{t-1}, \boldsymbol{x}^t, \omega^1, \ldots, \omega^{t-1}) \leq 0, \quad i = 1, \ldots, m_t$$

and $\varphi_{T,0} \equiv 0$.

The latter formulation resembles the backward recursion common in *stochastic dynamic programming problems*. **In these models, the goal is to provide a sequence of decision rules that can be used in particular stages of the decision process and in any state of the system, that allow the decision maker to pass from observations to decisions in an optimal way, e. g., for minimal total expected costs. To get the decision rules, one applies**

the principle of optimality that in turn requires evaluation of the optimal values in all stages in dependence on multidimensional parameters, e. g., on $x^1, \dots, x^{t-1}, \omega^1, \dots, \omega^{t-1}$ in (20). Dimensionality puts serious limitations even on *discrete* stochastic dynamic programming problems or stochastic control problems with discrete time and finite number of states. Moreover, for application of the backward induction, special structure of the problem (e. g., separability) is essential.

On the contrary, the main interest in the multistage stochastic programming problems lies in the first-stage decisions. Even if it is possible to characterize the decision rules [63], it is not necessary to design a full backward recursion as in dynamic programming and, due to large dimensionality of the stochastic programming problems, such procedure will be hardly tractable. For a qualified discussion of the relationship between the stochastic control problems with discrete time and the multistage stochastic programs see, e. g., [34], [89], [130], [135], [146], [156]. An important issue for the both classes of stochastic dynamic decision models is the proper choice of the horizon. In the context of stochastic programming, there are attempts to decrease the contribution of costs for recourse activities in distant stages by discounting (see, e. g., [7]), to decrease the number of stages by aggregation or by exploitation of a special structure of the solved problem and there are suggestions how to treat the effects of replacement of an infinite horizon problem by a finite horizon one [60], [72].

For results concerning the equivalence of the two formulations of multistage stochastic programs with recourse for the decision space L^∞ see, e. g., [132] – [134]. Optimality conditions and duality results can be in principle obtained from the standard results for nonlinear programs of the type (1), (2) that corresponds to the formulation (19). Naturally, optimality conditions and dual problems obtained in this way involve only the expectation functionals. To get pointwise conditions and dual problems expressed with respect to all $\omega \in \Omega$ one can choose a suitable decision space of measurable functions in the formulation (18) and exploit methods of abstract optimization. Convex problems allow for application of conjugate duality results; special care has been devoted to the impact of nonanticipativity constraints and to the corresponding dual variables or Lagrange multipliers that can be interpreted as prices for nonanticipativity (see [34], [R9], [45], [52], [54]) and to the "singular" multipliers attached to the induced constraints [133]. For an extension of results on optimality conditions to the nonconvex problems of the type (18) treated in L^p space see the recent paper [59].

A crucial problem for multistage stochastic programs is modeling the information structure; see [155] for a variant of model (1), (2) that corresponds to situations when the observation process reveals only a partial information.

A popular form of the multistage stochastic *linear* program (MSLP) with recourse reads

$$\text{minimize} \qquad c_1^\top x^1 + \mathsf{E}_{\omega^1}\left\{\varphi_1(x^1, \omega^1)\right\} \tag{21}$$

subject to constraints

$$\text{subject to constraints} \qquad A_1 x^1 = b_1$$

$$l_1 \le \; x^1 \le \; u_1,$$

where the functions $\varphi_{t-1}, t = 2, \ldots, T$, are defined recursively as

$$\varphi_{t-1}(x^{t-1}, \omega^{t-1}) = \inf_{x^t}\left[c_t(\omega^{t-1})^\top x^t + \mathsf{E}_{\omega^t}\left\{\varphi_t(x^t, \omega^t)\right\}\right] \tag{22}$$

$$\text{subject to} \qquad B_t(\omega^{t-1}) \quad x^{t-1} + A_t(\omega^{t-1}) \quad x^t = b_t(\omega^{t-1})$$

$$l_t \le x^t \le u_t,$$

and $\varphi_T \equiv \; 0$.

For the sake of simplicity, we denote here by ω^{t-1} the random vector that generates the coefficients b_t, c_t and matrices A_t, B_t in the decision problem of the tth stage, $t = 2, \ldots, T$ (compare with the scheme (16) or with (20)), we assume a Markovian structure of the constraints and of the objective and we suppose that the corresponding expectations E are well defined. The bounds l_t, u_t $\forall\, t$ are nonrandom and for the first stage, known values of all elements of b_1, c_1, A_1 are assumed. The assumption of *fixed recourse* means that A_t are known nonstochastic matrices for all t. The main decision variable is x^1 that corresponds to the first stage. If the random parameters are *stage independent*, characteristic properties such as convexity of the resulting deterministic program can be obtained from the results for two-stage problems with fixed recourse, see, e. g., [113], [149], [150]; Olsen [118] – [121] allows for dependence of the random right hand sides. It is clear how to formulate the MSLP of the form (18) and when to expect (at least intuitively) the equivalence between the two different formulations that are used to model the same decision problem; once more, general statements on equivalence between these two formulations are not trivial, e. g. [128], [129], [134]. The results on equivalence hold true, for instance, for MSLP with *finite discrete distribution* P of all coefficients A_t, B_t, h_t, c_t, $\forall\, t$ in which case (18) reduces into a large linear program of a special structure similar to (13), (14) that is convenient for decomposition purposes. This observation opens possibilities of an algorithmic solution.

3. NUMERICAL TECHNIQUES

Various numerical techniques have been developed for multistage stochastic programs with recourse and with a given *discrete* multidimensional probability distribution of the random parameters whose atoms, say ω_s, $s =$

$1, \ldots, S$, are called *scenarios*. The origin of this discrete distribution can be very diverse; it can be obtained as an approximation of a true continuous distribution, based on a sample information, or related to scenarios provided by an expert. Similarly as for two-stage problems, the specific assumption of discrete distribution allows for transformation of multistage stochastic *linear* programs into large scale linear programs that can be, in principle, solved by general purpose algorithms adjusted to the special structure of the solved problem; see, e. g., [70]. However, the size of the resulting linear program can be prohibitively large; it grows exponentially with the number of scenarios taken into account and with the number of stages so that the direct approaches based on standard linear programming software are of limited use. There are various ideas how to reach a manageable size of the problem: to wave the stochastic character of the data and to replace the random parameters by some fixed "base" values, e. g., by expectations [16]; to wave the possibility of adapting the decisions according to the past information [81]; to use an appropriate labeling for to avoid ambiguity in definition of data and of variables, cf. [11], [64], [92]; to aggregate some of periods into one stage, e. g., [4], [50], [91], [108], [167]; to aggregate scenarios [137], [138], [154] or both scenarios and periods [7]; to select "important" scenarios using statistical techniques [30] or expert's opinion; to decompose the problem into manageable ones and to use parallel procedures. Regardless of the chosen solution technique the desired optimal solution depends on the used scenarios and on their probabilities and its performance under other "out of sample" scenarios should be a subject of postoptimality, stability or simulation studies.

In *primal decomposition methods*, the subproblems are related to individual stages. The original computer code by Birge [9] extends the L-shaped method [R14] to three stage problems with random right-hand sides. Its generalization by Gassmann [64], known under name MSLiP, concentrates to increase of efficiency of the performance and allows for random elements in the matrix of the system of constraints and for more than three stages. To save labor in implementation development, performance evaluation and comparisons of existing and future software for multistage stochastic programs, a standard input format [11] has been suggested.

The standard form of the T-stage stochastic linear program with fixed recourse and with a finite number of scenarios (compare (13), (14) or (21), (22)) is

$$\text{minimize} \quad \boldsymbol{c}_1^\top \boldsymbol{x}_0 + \sum_{k_2=1}^{K_2} p_{k_2} \boldsymbol{c}_2^\top \boldsymbol{x}_{k_2} + \sum_{k_3=K_2+1}^{K_3} p_{k_3} \boldsymbol{c}_3^\top \boldsymbol{x}_{k_3} + \cdots + \sum_{k_T=K_{T-1}+1}^{K_T} p_{k_T} \boldsymbol{c}_T^\top \boldsymbol{x}_{k_T} \tag{23}$$

subject to constraints

$$\begin{array}{lllll}
\boldsymbol{A}_1\boldsymbol{x}_0 & & & = \boldsymbol{b}_0 & \\
\boldsymbol{B}_{k_2}\boldsymbol{x}_0 + \boldsymbol{A}_2\boldsymbol{x}_{k_2} & & & = \boldsymbol{b}_{k_2}, & k_2 = 1,\ldots,K_2 \\
& \boldsymbol{B}_{k_3}\boldsymbol{x}_{a(k_3)} + \boldsymbol{A}_3\boldsymbol{x}_{k_3} & & = \boldsymbol{b}_{k_3}, & k_3 = K_2+1,\ldots,K_3 \\
& \ddots \quad \ddots & & \vdots & \\
& & \boldsymbol{B}_{k_T}\boldsymbol{x}_{a(k_T)} + \boldsymbol{A}_T\boldsymbol{x}_{k_T} & = \boldsymbol{b}_{k_T}, & k_T = K_{T-1}+1,\ldots,K_T
\end{array} \tag{24}$$

$$\boldsymbol{l}_t \le \boldsymbol{x}_{k_t} \le \boldsymbol{u}_t, \qquad k_t = K_{t-1}+1,\ldots,K_t, \quad t = 1,\ldots,T$$

and it necessarily depends on the used scenarios, i. e., on finitely many sequences of possible realizations $(\boldsymbol{B}_{k_t}, \boldsymbol{b}_{k_t}),\ t{=}2,\ldots,T$ of the random right hand sides and random transition matrices in the constraints for the stage t and on the path probabilities p_{k_t} of subsequences of these realizations $p_{k_t} > 0\ \forall\, k_t$, $\sum_{k_t=K_{t-1}+1}^{K_t} p_{k_t} = 1,\ t{=}2,\ldots,T$ that identify the discrete distribution P. The probabilities p_s of individual scenarios ω_s are equal to the *path probabilities* $p_{k_T}, \quad k_T = K_{T-1}+1,\ldots,K_T$ and they can be obtained by multiplication of the (conditional) arc probabilities of the corresponding realizations. In program (23), (24), $\boldsymbol{x}_{k_1} \equiv \boldsymbol{x}_0$ denotes the first-stage decision variable and $a(k_t)$ denotes the immediate ancestor of k_t, so that $a(k_2) = k_1,\ k_2 = 1,\ldots,K_2$. Induced constraints are not included in in the system (24) but they can be treated in course of numerical procedure.

The input information that leads to (23), (24) can be represented in the form of so called *scenario tree* or *event tree*. Each path through the tree from its root to one of its leaves corresponds to one scenario, i. e., to a particular sequence of realizations. The nodes where the branching occurs correspond to stages and they are indexed by k_t. In our notation, the root is indexed by $k_1 = 0$, there are $K_t - K_{t-1}$ nodes indexed by $k_t = K_{t-1}+1,\ldots,K_t$ for the stage t (with $K_1 = 0$); particularly, the $K_T - K_{T-1}$ leaves indexed by k_T correspond to scenarios. For each node of the scenario tree, an entire set of decision variables is introduced in program (23), (24); for instance, the vector of the first-stage decision variables $\boldsymbol{x}_0$ corresponds to the root and subvectors $\boldsymbol{x}_{k_t}$ of the tth stage decision variables are assigned to the nodes k_t, respectively.

A natural idea is to decompose the large linear program to many relatively small problems corresponding to individual scenarios. For instance, the problem corresponding to scenario $\omega_* = \{(\boldsymbol{B}_{K_j}, \boldsymbol{b}_{K_j}),\ j = 2,\ldots,T\}$ in (24), if such scenario exists, is the following linear program:

$$\text{Minimize} \quad \boldsymbol{c}_1^\top\boldsymbol{x}_0 + \boldsymbol{c}_2^\top\boldsymbol{x}_{K_2} + \boldsymbol{c}_3^\top\boldsymbol{x}_{K_3} + \cdots + \boldsymbol{c}_T^\top\boldsymbol{x}_{K_T} \tag{25}$$

subject to constraints

$$\begin{array}{llll}
\boldsymbol{A}_1\boldsymbol{x}_0 & & & = \boldsymbol{b}_0 \\
\boldsymbol{B}_{K_2}\boldsymbol{x}_0 + \boldsymbol{A}_2\boldsymbol{x}_{K_2} & & & = \boldsymbol{b}_{K_2} \\
& \boldsymbol{B}_{K_3}\boldsymbol{x}_{K_2} + \boldsymbol{A}_3\boldsymbol{x}_{K_3} & & = \boldsymbol{b}_{K_3} \\
& \ddots \quad \ddots & & \vdots \\
& & \boldsymbol{B}_{K_T}\boldsymbol{x}_{K_T} + \boldsymbol{A}_T\boldsymbol{x}_{K_T} & = \boldsymbol{b}_{K_T}
\end{array} \tag{26}$$

$$\boldsymbol{l}_t \leq \boldsymbol{x}_{K_t} \leq \boldsymbol{u}_t, \quad t = 1, \ldots, T.$$

The x_0-part of the optimal solution of (25), (26) gives the first-stage solution that is optimal for the used scenario ω_*. However/ this solution need not be either optimal or feasible for other scenarios. Decisions made conditional on knowledge of a scenario, i. e., based on an assumed realization of ω, do not evidently fulfil requirements of the considered decision model such as to get *scenario independent optimal first-stage solution*. Therefore, in designing algorithms based on decomposition with respect to individual scenarios one must take care for nonanticipativity constraints: At any stage t of the decision process, the final decision $\boldsymbol{x}^t$ can be based only on the previous decisions and on the already observed segment of data represented in (24) by one subsequence $\{(\boldsymbol{B}_{k_j}, \boldsymbol{b}_{k_j}),\ j = 1, \ldots, t-1\}$. In other words, decisions accepted at the stage t that are based on the same history must be equal. In the introduced form (23), (24) requirement of nonanticipativity has been included directly into the constraints.

For *decomposition methods with respect to scenarios* nonanticipativity constraints can be included in the form of a large system of simple binding linear equations that spell out explicitly the corresponding requirements (compare (15))

$$\boldsymbol{x}^t(\omega_s) = \boldsymbol{x}^t(\omega_j) \tag{27}$$

for those s, j that have the same data up to t.

For a given scenario ω_s we denote by $\boldsymbol{x}(\omega_s) = \boldsymbol{x}^{T\bullet}(\omega_s)$ the vector of all decision variables $\boldsymbol{x}^t(\omega_s),\ t = 1, \ldots, T$ in the individual scenario problem such as (25), (26), p_s the path probability of ω_s, $\boldsymbol{c}(\omega_s)$ the vector of coefficients in the objective function and $\mathcal{X}(\omega_s)$ the corresponding set of feasible decisions described by linear constraints that are fixed by the assumed values of scenario ω_s. The multistage stochastic linear program based on scenarios $\omega_s, s = 1, \ldots, S$, can be now written as

$$\text{minimize} \quad \sum_{s=1}^{S} p_s \boldsymbol{c}(\omega_s)^\top \boldsymbol{x}(\omega_s) \tag{28}$$

$$\text{subject to} \quad \boldsymbol{x}(\omega_s) \in \mathcal{X}(\omega_s), s = 1, \ldots, S \tag{29}$$

$$\text{and} \quad \boldsymbol{x} = \boldsymbol{U}\boldsymbol{x} \tag{30}$$

where $\boldsymbol{x}$ contains carefully grouped decision vectors $\boldsymbol{x}(\omega_s)\ \forall\, s$ and $\boldsymbol{U}$ is the $0-1$ matrix of coefficients of the nonanticipativity constraints of the form (27). For instance, nonanticipativity of the first-stage decisions, i. e., the condition $\boldsymbol{x}^1(\omega_1) = \boldsymbol{x}^1(\omega_2) = \cdots = \boldsymbol{x}^1(\omega_S)$ can be expressed in the form (30) with $\boldsymbol{U}$ equal to the following block permutation matrix

$$\begin{pmatrix} \mathbf{0} & \boldsymbol{I} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix}.$$

In standard formulations of decomposition algorithms, the nonanticipativity constraints (30) correspond to the Master problem. Similarly as for two-stage problems, Lagrangian dual decomposition methods improve efficiency of scenario decomposition methods, cf. [34]. For instance, *augmented Lagrangian method* proposed in [106] relaxes the nonanticipativity constraints by adding a quadratic penalty term $r\|\boldsymbol{x} - \boldsymbol{U}\boldsymbol{x}\|^2$ to the common Lagrangian function and uses suitable diagonal approximation of the penalty term to recover separability of the resulting objective function. These ideas can be generalized to multistage stochastic programs with linear constraints and convex quadratic or convex separable objective function; cf. [5], [94], [116], [140].

The *progressive hedging algorithm* designed by Rockafellar and Wets [137], [154] is another decomposition type procedure based on augmented Lagrangian method, this time with a projection matrix at the place of $\boldsymbol{U}$. It aims at progressive construction of compromising *aggregate* solutions that sufficiently approximate the optimal solution of the original stochastic program. It is not limited to linear problems; for its applications and extensions see, e. g., [80], [109] – [111], [127] or [147].

Further recent algorithmic advances concentrate to exploitation of the structure of the problem such as separability of the recourse matrices [95] or a network form of the constraints [108], [112], [114], to parallel computations, e. g., [12], [28], [29], [83], [141], [153], [167] and to possible reduction of the number of scenarios; cf. the EVPI model reduction suggested in [33], [34] and included, e. g., into the system MIDAS [35] – [38], or importance sampling techniques [30], [84].

All numerical techniques for solving multistage stochastic programs are demanding as to the input data (including scenarios) and to the programming efforts. Their final performance depends on the structure of the solved problem and on the mentioned input. In this context, *generation of scenarios* (discrete approximation of the underlying probability distribution) is a very important task. In real-life applications, it has been solved from case to case and both parametric and nonparametric statistical methods have been applied to historical data. An approach to generation of one-dimensional scenarios was described in detail in [167] and other subsequent papers. Generation of vector valued scenarios is more complicated even if the dimension can be reduced by means of principal components method, factor analysis and regression analysis or by sampling, see, e. g., [20], [21] – [23], [30], [35], [36], [87], [104], [105], [108]. If a multistage stochastic program is to be applied repeatedly as a part of a decision support system, additional routines for data processing, scenario generation, simulation and evaluation of results and of their robustness should be developed along the whole path from theory to implementation and complemented by estimates of error bounds and by suitable postoptimization techniques. For results concerning error bounds for multistage stochastic programs see, e. g., [46], [148], postoptimization is treated in

[43], stability results are presented in [56]. One of very important open problems is a proper design of the information structure, namely, of the interstage dependence that cannot be overlooked and that complicates scenario generation and using sampling methods in general.

At this point, interdisciplinary nature of research in multistage stochastic programming becomes evident. It is characterized not only by the trade-off between advanced modeling, available data and algorithmic procedures but also by a strong interplay between optimization, statistics, numerical methods and computer science.

4. BIBLIOGRAPHY ON MULTISTAGE STOCHASTIC PROGRAMMING

(For full annotation of collections of papers see References)

Classified as

ALG	–	computation, software, algorithms
APPL	–	non-financial application
B	–	approximation and bounds
CON	–	connection with stochastic control
EX	–	example
FIN	–	application in finance
GEN	–	general reference
P	–	for probabilistic constraints
S	–	for scenario generation
STAB	–	stability, postoptimization
TH	–	theoretical paper

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