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STABILITY IN STOCHASTIC PROGRAMMING — THE CASE OF UNKNOWN LOCATION PARAMETER

Vlasta Kaňková

The assumption of a complete knowledge of the distribution in stochastic optimization problem is only seldom justified in real-life situations. Consequently, statistical estimates of the unknown probability measure, if they exist, can be only utilized to obtain some estimates of the optimal value and the optimal solution.

The empirical distribution function is usually used everywhere when the theoretical distribution function is fully unknown [1], [5], [17]. This substitution leads to the "good" statistical estimates [2], [9], [10], [14], [16]. However, unfortunately, it is also well-known that the corresponding approximative problem need not be a concave problem even in the case when the theoretical original one possesses this property. In particular, this happens rather often in the case of the chance constrained stochastic programming problems.

If we can assume that the theoretical distribution function belongs to a parametric family, then we can employ estimates of the unknown parameter to get some estimates of the optimal value and the optimal solution [3], [16]. In this paper, we shall consider the case when the unknown parameter can be introduced as a location parameter. We obtain the estimates of the optimal value and the optimal solution with statistical properties fully determined by the properties of the original parameter estimates. Moreover, the approximative problems belong to the same type of the optimization problems as the original one. However, to obtain these results we have to study the stability problem with respect to the location parameter, first.

At the end of the paper we shall try to apply some obtained results to stochastic optimization problem considered with respect to the discrete time interval $1 \div N$. Namely surely, the main importance of the former results will be found just in such dynamic models.

1. INTRODUCTION

Let (Ω, S, P) be probability space, $\xi = \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)]$ be an s-dimensional random vector defined on (Ω, S, P), $g_i(x, z)$, $i = 0, 1, 2, \ldots, \ell$, be real-valued, continuous functions defined on $E_n \times E_s$, $X \subset E_n$ be a nonempty set (E_n, $n \geq 1$ denotes an n-dimensional Euclidean space).

The general optimization problem with random elements can be introduced as the problem to find

$$\max\{g_0(x, \xi(\omega)) \,|\, x \in X : \; g_i(x, \xi(\omega)) \leq 0, \; i = 1, 2, \ldots, \ell\}. \tag{1}$$

If the solution x has to be found without knowing realization of the random vector $\xi(\omega)$, then it is necessary, first, to determine the decision rule. This means to assign to the original stochastic optimization problem (1) some deterministic one, called the deterministic equivalent. Two well-known types of deterministic equivalents can be introduced as the following problems (cf. [4]):

I. Find

$$\max\{\mathsf{E}\,\overline{g}(x,\xi(\omega))|x \in X\}.$$

This type includes, among others, the problems with penalty function and two-stage stochastic programming problems.

II. Find

$$\max\{\mathsf{E}\,g(x,\xi(\omega))\,|\,x \in X(\alpha)\},$$

$$\text{such that} \qquad X(\alpha) = \{x \in X : P\{\omega : g_i(x,\xi(\omega)) \le 0,\ \ i = 1,2,\ldots,\ell\} \ge \alpha\}\,.$$

This deterministic equivalent is called the chance constrained stochastic programming problem in the literature.

In what follows, $\alpha \in \langle 0,1\rangle$ is a parameter, $\overline{g}(x,z), g(x,z)$ are some real-valued functions defined on $E_n \times E_s$, E denotes the operator of mathematical expectation.

Remark. In detail, the introduced definitions of deterministic equivalents are given in [4] for linear case only.

We shall restrict our investigation to the special form of the function $g_i(x,z)$, $i = 1,2,\ldots,\ell$, in the case of the deterministic equivalent II. In detail, we shall assume in this case that

$$\ell = s, \quad g_i(x,z) = f_i(x) - z_i, \quad i = 1,2,\ldots,\ell, \quad z = (z_1,\ldots,z_\ell), \tag{2}$$

where $f_i(x),\ i = 1,2,\ldots,\ell$, are real-valued, continuous functions defined on E_n.

If (generally) $A \subset E_s$ is a nonempty parametric set,
$F_0(z)$ is an s-dimensional distribution function,
$\mathcal{P}_a$, $a \in A$, denotes a parametric family of distribution functions such that

$$F_a \in \mathcal{P}_a,\ a \in A \iff F_a(z) = F_0(z-a), \tag{3}$$

then we can denote the set $X(\alpha)$ by $X_a(\alpha)$, that is

$$X(\alpha) = X_a(\alpha) = \{x \in X : P_a\{\omega : f_i(x) \le \xi_i(\omega),\ i = 1,2,\ldots\ell\} \ge \alpha\}, \tag{4}$$

where P_a is the probability measure corresponding to the distribution function F_a.

Remark. It is evident that there exists an inaccuracy in relation (4). The exact form should be

$$X(\alpha) = X_a(\alpha) = \{x \in X : P_a\{\omega : f_i(x) \le \xi_i^a(\omega),\ i = 1,2,\ldots,\ell\} \ge \alpha\},$$

where $\xi^a(\omega) = (\xi_1^a(\omega), \ldots, \xi_\ell^a(\omega))$ is some random vector with the distribution function $F_a(z)$.

If in addition $\hat{a}(N) = \hat{a}(N, \omega)$, $N = 1, 2, \ldots$, denote some statistical estimates of the parameter $a \in A$, then it is easy to see that $\max\limits_{x \in X} \mathsf{E}_{\hat{a}(N)}\, \overline{g}(x, \xi(\omega))$ estimates the value $\max\limits_{x \in X} \mathsf{E}_a\, \overline{g}(x, \xi(\omega))$ in the case of the deterministic equivalent I. In the case of the deterministic equivalent II the theoretical value $\max\limits_{X_a(\alpha)} \mathsf{E}_a\, g(x, \xi(\omega))$ can be estimated by the value $\max\limits_{X_{\hat{a}(N)}(\alpha)} \mathsf{E}_{\hat{a}(N)}\, g(x, \xi(\omega))$ (E_a denotes mathematical expectation considered with respect to the distribution function P_a).

The aim of this paper is to study the just introduced estimates, first. (Of course, it will be done under the assumptions that the theoretical distribution function of the random vector $\xi(\omega)$ belongs to the parametric family of the distributions given by (3).) Further, we shall apply these results to time dependent sequences of stochastic optimization problems.

Remarks.

1. The choice of the functions $\overline{g}(\cdot, \cdot)$ and $g(\cdot, \cdot)$ depends on the character of the original stochastic problem.

2. It can generally happen that some symbols mentioned above are not reasonable. However, this situation cannot appear under the assumptions considered in this paper.

2. SOME AUXILIARY ASSERTIONS AND DEFINITIONS

Lemma 1. Let $X \subset E_n$, $A \subset E_s$ be nonempty sets. If

1. $\overline{g}(x, z)$ is a continuous function on $X \times E_s$,

2. for every $x \in X$, $\overline{g}(x, z)$ is a Lipschitz function of $z \in E_s$ with Lipschitz constant $\bar{L}$ independent of $x \in E_n$,

3. for every $x \in X$ there exists a finite $\mathsf{E}_0\, \overline{g}(x, \xi(\omega))$,

then

$$\left| \mathsf{E}_{a(1)}\, \overline{g}(x, \xi(\omega)) - \mathsf{E}_{a(2)}\, \overline{g}(x, \xi(\omega)) \right| \le \overline{L} \cdot \|a(1) - a(2)\|$$

for every $x \in X$, $a(1)$, $a(2) \in A$ ($\|\cdot\|$ denotes the Euclidean norm in E_s).

Proof. First, it follows from the assumptions 2, 3 of Lemma 1 that for every $x \in X, a \in A$ there exists a finite $\mathsf{E}_a\, \overline{g}(x, \xi(\omega))$. Furthermore, we get immediately from the definition of mathematical expectation that in virtue of (3)

$$\begin{aligned} &\left| \mathsf{E}_{a(1)}\, \overline{g}(x, \xi(\omega)) - \mathsf{E}_{a(2)}\, \overline{g}(x, \xi(\omega)) \right| = \\ = \quad &\left| \int g(x, z)\, \mathrm{d}F_{a(2)}(z + a(2) - a(1)) - \int g(x, z)\, \mathrm{d}F_{a(2)}(z) \right|, \end{aligned}$$

and hence we obtain the assertion of Lemma 1 on the bases of the assumption 2. □

Lemma 2. If $\alpha \in (0,1)$, $X = E_n^+, \underline{a}, \overline{a} \in A$ are arbitrary such that $\underline{a} \le \overline{a}$ componentwise, then

$$X_{\underline{a}}(\alpha) \subset X_{\overline{a}}(\alpha). \tag{5}$$

($E_n^+ = \{x \in E_n : x = (x_1, \dots, x_n), x_i \ge 0,\ i = 1, 2, \dots, n\}$.)

P r o o f. Let $\underline{a} = (\underline{a}_1, \dots, \underline{a}_\ell)$, $\overline{a} = (\overline{a}_1, \dots, \overline{a}_\ell)$, α be arbitrary fulfilling the assumptions of Lemma 2.
If $X_{\underline{a}}(\alpha) = \emptyset$ relation (5) is trivially fulfilled, so in the rest of the proof we assume that $X_{\underline{a}}(\alpha) \ne \emptyset$. To verify the assumption (5) (in this case) it is sufficient to prove the validity of the implication

$$x \in X_{\underline{a}}(\alpha) \implies x \in X_{\overline{a}}(\alpha). \tag{6}$$

However, since for every $x \in X_{\underline{a}}(\alpha)$ it holds

$$\begin{aligned} \alpha &\le P_{\underline{a}}\{\omega : f_i(x) \le \xi_i(\omega),\ i = 1, 2, \dots, \ell\} = \\ &= P_{\overline{a}}\{\omega : f_i(x) + \overline{a}_i - \underline{a}_i \le \xi_i(\omega),\ i = 1, 2, \dots, \ell\} \end{aligned} \tag{7}$$

and since $\overline{a} - \underline{a} \ge 0$ componentwise, we obtain the validity of the implication (6) immediately. □

Lemma 3. Let $\alpha \in (0,1)$, $\underline{a}$, $\overline{a} \in A$, $\underline{a} = (\underline{a}_1, \dots, \underline{a}_\ell)$, $\overline{a} = (\overline{a}_1, \dots, \overline{a}_\ell)$, $X = E_n^+$. If

1. there exists $a \in E_1^+$, $a > 0$ such that $\underline{a}_i + a = \overline{a}_i$, $i = 1, 2, \dots, \ell$,
2. there exists real-valued constant $\gamma_1 > 0$, such that $f_i(x') - f_i(x) \ge \gamma_1 \sum_{j=1}^{n}(x'_j - x_j)$, $i = 1, 2, \dots, \ell$, for every $x = (x_1 \dots, x_n)$, $x' = (x'_1, x'_2, \dots, x'_n) \in E_n$, $x \le x'$ componentwise,
3. the probability measure corresponding to the distribution function $F_0(\cdot)$ is absolutely continuous with respect to the Lebesgue measure in E_ℓ,
4. $X_{\underline{a}}(\alpha) \ne \emptyset$,

then $X_{\overline{a}}(\alpha) \ne \emptyset$, and

$$\triangle[X_{\overline{a}}(\alpha),\, X_{\underline{a}}(\alpha)] \le \frac{a}{\gamma_1}\sqrt{n}.$$

($\triangle[\cdot, \cdot]$ denotes the Hausdorff distance of sets, see e. g. [10].)

P r o o f. It follows from the definition of the Hausdorff distance and from the assertion of Lemma 2 that to prove the assertion of Lemma 3 it is enough to prove the following inequality

$$\sup_{x \in X_{\overline{a}}(\alpha)} \inf_{x' \in X_{\underline{a}}(\alpha)} \rho(x, x') \le \frac{a}{\gamma_1}\sqrt{n}, \tag{8}$$

where $\rho(\cdot, \cdot)$ denotes the Euclidean metric in E_n.

So let $x \in X_{\overline{a}}(\alpha)$ be arbitrary. It is easy to see that to prove relation (8) it is sufficient to find $x' = x'(x)$, $x' \in X_{\underline{a}}(\alpha)$ such that

$$\rho(x, x') \le \frac{a}{\gamma_1}\sqrt{n}.$$

If $x \in X_{\underline{a}}(\alpha)$, $x = (x_1, \dots, x_n)$, then we can set $x' = x$, evidently. It remains to consider the case $x \notin X_{\underline{a}}(\alpha)$. If we define in this case the point $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ by $x_i^* = x_i - \frac{a}{\gamma_1}$, $i = 1, 2, \dots, n$, we get $\|x - x^*\| \leq \sqrt{n}\frac{a}{\gamma_1}$. Two different cases can happen

a) there exists an $r \in \{1, 2, \dots, n\}$ such that $x_r^* \geq 0$,

b) $x_j^* < 0$ for every $j \in \{1, 2, \dots, n\}$.

Let us, first, consider the case a). In this case we can define the point $x' = (x_1', x_2', \dots, x_n')$ by

$$x_r' = x_r^*,\ x_j' = x_j \ \text{ for } \ j \neq r.$$

It follows from the assumptions that $f_i(x') < f_i(x)$, $i = 1, 2, \dots, \ell$, and moreover

$$f_i(x) - f_i(x') \geq a, \ \ i = 1, 2, \dots, \ell.$$

However, it means

$$f_i(x') \leq f_i(x) - a, \ \ i = 1, 2, \dots, \ell.$$

Furthermore, since $x \in X_{\overline{a}}(\alpha)$ we obtain

$$\begin{aligned} \alpha &\leq P_{\overline{a}}\{\omega : f_i(x) \leq \xi_i(\omega),\, i = 1, 2, \dots, \ell\} \leq \\ &\leq P_{\overline{a}}\{\omega : f_i(x') + a \leq \xi_i(\omega),\, i = 1, 2, \dots, \ell\} = \\ &= P_{\underline{a}}\{\omega : f_i(x') \leq \xi_i(\omega),\, i = 1, 2, \dots, \ell\} \end{aligned}$$

and so also

$$x' \in X_{\underline{a}}(\alpha).$$

Since $\rho(x, x') = \frac{a}{\gamma_1}$ we have finished the proof of the assertion in the case a).

Now we shall consider the case b). However, since then $\|x\| \leq \sqrt{n}\frac{a}{\gamma_1}$, the assertion of Lemma 3 follows from the assumptions 3, 4 and the properties of the probability measure. □

Lemma 4. Let $\alpha \in (0, 1)$, $a(1), a(2) \in A$ be arbitrary, $X_{a(1)}(\alpha) \neq 0$, $X_{a(2)}(\alpha) \neq \emptyset$, $X = E_n^+$. Let, further, the assumptions 2, 3 of Lemma 3 be fulfilled. If there exist vectors $\overline{a}$, $\underline{a} \in A$, $\overline{a} = (\overline{a}_1, \dots, \overline{a}_\ell)$, $\underline{a} = (\underline{a}_1, \dots, a_\ell)$ such that $\overline{a}_i - \underline{a}_i = \overline{a}_1 - \underline{a}_1$, $i = 1, 2, \dots, \ell$, $X_{\underline{a}}(\alpha) \neq \emptyset$, $X_{\overline{a}(\alpha)} \neq \emptyset$ and simultaneously $\underline{a} \leq a(1) \leq \overline{a}$, $\underline{a} \leq a(2) \leq \overline{a}$ componentwise, then

$$\triangle[X_{a(1)}(\alpha),\ X_{a(2)}(\alpha)] \leq \sqrt{n}\frac{a}{\gamma_1} \ \text{ where } \ a = \overline{a}_1 - \underline{a}_1.$$

Proof. First, it follows from Lemma 2 that $X_{\underline{a}}(\alpha) \subset X_{a(1)}(\alpha) \subset X_{\overline{a}}(\alpha)$ and simultanously $X_{\underline{a}}(\alpha) \subset X_{a(2)}(\alpha) \subset X_{\overline{a}}(\alpha)$.

Moreover, it follows from the above facts and from the definition of the Hausdorff distance that

$$\triangle[X_{a(1)}(\alpha), X_{a(2)}(\alpha)] \leq \triangle[X_{\underline{a}}(\alpha), X_{\overline{a}}(\alpha)],$$

and hence the assertion of Lemma 4 follows immediately from the assertion of Lemma 3. □

Lemma 5. Let $X = E_n^+$ and $A \subset E_\ell$ be a nonempty set. Let further $\alpha \in (0, 1)$, $a \in A$ be arbitrary such that $X_a(\alpha) \neq \emptyset$. If the assumptions 2, 3 of Lemma 3 are fulfilled, then $X_a(\alpha)$ is a compact set.

Proof. Let a, α fulfil the assumptions of Lemma 5. Since it follows from Lemma 4 of [9] that $X_a(\alpha)$ is a bounded set, the assertion of Lemma 5 will be proved if we verify the validity of the implication

$$x_N \in X_a(\alpha),\ N = 1, 2, \ldots, \lim_{N\to\infty} x_N = x \implies x \in X_a(\alpha). \tag{9}$$

It follows immediately from the assumptions that for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that

$$\begin{aligned} &\alpha \le P_a\{\omega : f_i(x_N) \le \xi_i(\omega),\ i = 1, 2, \ldots, \ell\} \le \\ \le\ & P_a\{\omega : f_i(x) \le \xi_i(\omega),\ i = 1, 2, \ldots, \ell\} + \\ +\ & \sum_{i=1}^{\ell} P_a\{\omega : \xi_i(a) \in [f_i(x) - \varepsilon,\ f_i(x) + \varepsilon],\ \xi_j(\omega) > f_j(x) - \varepsilon,\ j \neq i,\ j = 1, 2, \ldots, \ell\} \end{aligned}$$

for $N > N_0(\varepsilon)$.
However, since according to the assumptions it follows from the former inequality that $\alpha \le P_a\{\omega : f_i(x) \le \xi_i,\ i = 1, 2, \ldots, \ell\}$ too, we see that the assertion of Lemma 5 holds. □

At the end of this part we shall present one result of convex analysis. However, first, we shall recall the definition of strongly concave functions [13], [15].

Definition 1. Let $h(x)$ be a real-valued function defined on a convex set $\mathcal{K} \subset E_n$. $h(x)$ is a strongly concave function with a parameter $\rho > 0$ if

$$h(\lambda x_1 + (1-\lambda) x_2) \ge \lambda h(x_1) + (1-\lambda) h(x_2) + \lambda(1-\lambda)\rho \,\|x_1 - x_2\|^2$$

for every $x_1, x_2 \in \mathcal{K},\ \lambda \in \langle 0, 1\rangle$.

Lemma 6. Let $\mathcal{K} \subset E_n$ be a non-empty, compact, convex set. Let further $h(x)$ be strongly concave with a parameter $\rho > 0$, continuous, real-valued function defined on $\mathcal{K}$. If $x_0 \in \mathcal{K}$ is defined by the relation

$$x_0 = \arg\max_{x\in\mathcal{K}} h(x) \tag{10}$$

then

$$\|x - x_0\|^2 \le \frac{2}{\rho}[h(x_0) - h(x)],$$

for every $x \in \mathcal{K}$.

Proof. Since it follows from the definition of strongly concave functions with a parameter $\rho > 0$ that

$$h(\lambda x_1 + (1-\lambda)x_2) \ge \lambda h(x_1) + (1-\lambda) h(x_2) + \lambda(1-\lambda)\rho \,\|x_1 - x_2\|^2$$

for every $x_1,\, x_2 \in \mathcal{K},\ \lambda \in \langle 0, 1\rangle$, we get

$$\lambda(1-\lambda)\rho\,\|x - x_0\|^2 \le \lambda(h(x_0) - h(x)) + h(\lambda x + (1-\lambda)\,x_0) - h(x_0)$$

for x_0 given by (10) and $x \in \mathcal{K}$ arbitrary. Since further

$$h(\lambda x + (1-\lambda)x_0) - h(x_0) \le 0 \ \text{ for every } \ \lambda \in (0,1)$$

we can see that the assertion of Lemma 6 holds. □

Remarks.

1. An assumptions under which a quadratic form is a strongly concave (respectively strongly convex) function are introduced for example in [13].
2. The assertion of Lemma 6 has been already presented for example in [15].

3. STABILITY RESULTS

Let $a(1),\, a(2) \in A,\ \alpha \in (0,1)$ be arbitrary. In this section we shall present an upper bound on the expression

$$\left|\max_{x\in X} \mathsf{E}_{a(1)}\, \overline{g}(x, \xi(\omega)) - \max_{x\in X} \mathsf{E}_{a(2)}\, \overline{g}(x, \xi(\omega))\right|$$

in the case of the deterministic equivalent I and further an upper bound on the expression

$$\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right|$$

in the case of the deterministic equivalent II. We shall see that similar upper bounds also exist for the optimal solution in some special cases.

First, we shall deal with the deterministic equivalent I. To this end, let us assume

i) $\overline{g}(x,z)$ is a continuous function on $X \times E_s$,

ii) for every $x \in X$, $\overline{g}(x,z)$ is a Lipschitz function of $z \in E_s$ with Lipschitz constant $\overline{L}$ independent of $x \in E_n^+$,

iii) a) X is a convex set,

b) for every $z \in E_s$, $\overline{g}(x,z)$ is a strongly concave function of $x \in E_n$ with a parameter $\rho > 0$.

We shall define the point $\bar{x}_a$ (if it exists) for $a \in A$ by

$$\overline{x}_a = \arg\max_{x\in X} \mathsf{E}_a\, \overline{g}(x, \xi(\omega)).$$

(It is easy be see that the point $\bar{x}_a$ for $a \in A$ is uniquely defined, under the assumption iii.)

We shall present the following theorem.

Theorem 1. Let $X \subset E_n$ be a nonempty, compact set, $A \subset E_s$ be a nonempty set and let the assumptions i), ii) be fulfilled. If there exists a finite $\mathsf{E}_a \, \overline{g}(x, \xi(\omega))$ for $a = a(1)$, $a = a(2)$, $a(1)$, $a(2) \in A$, $x \in X$, then

$$\left| \max_{x \in X} \mathsf{E}_{a(1)} \, \overline{g}(x, \xi(\omega)) - \max_{x \in X} \mathsf{E}_{a(2)} \, \overline{g}(x, \xi(\omega)) \right| \le \overline{L} \, \|a(1) - a(2)\|. \tag{11}$$

If, moreover, the assumption iii) is fulfilled, then

$$\|\overline{x}_{a(1)} - \overline{x}_{a(2)}\|^2 \le \frac{4}{\rho} \, \overline{L} \, \|a(1) - a(2)\|. \tag{12}$$

P r o o f. First, it follows from Lemma 1 that $|\mathsf{E}_{a(1)} \, \overline{g}(x, \xi(\omega)) - \mathsf{E}_{a(2)} \, \overline{g}(x, \xi(\omega))|$ is uniformly bounded by the constant $\overline{L} \, \|a(1) - a(2)\|$. Consequently, the assertion given by relation (11) is valid.

So it remains to prove the assertion given by (12). Since it follows from Lemma 1 and from (just proven) relation (11) that

$$|\mathsf{E}_{a(1)} \, \overline{g}(x, \xi(\omega)) - \mathsf{E}_{a(2)} \, \overline{g}(x, \xi(\omega))| \le \overline{L} \, \|a(1) - a(2)\|$$

for every $x \in X$, and simultaneously

$$|\mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(1)}, \xi(\omega)) - \mathsf{E}_{a(2)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega))| \le \overline{L} \, \|a(1) - a(2)\|,$$

we obtain, employing the triangular inequality successively,

$$\begin{aligned} & |\mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(1)}, \xi(\omega)) - \mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega))| \le \\ \le \; & |\mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(1)}, \xi(\omega)) - \mathsf{E}_{a(2)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega))| + \\ + \; & |\mathsf{E}_{a(2)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega)) - \mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega))| \\ \le \; & \overline{L} \, \|a(1) - a(2)\| + \overline{L} \, \|a(1) - a(2)\|. \end{aligned}$$

However, since further it follows from Lemma 6 that

$$\|\overline{x}_{a(1)} - \overline{x}_{a(2)}\|^2 \le \frac{2}{\rho} \left[\mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(1)}, \xi(\omega)) - \mathsf{E}_{a(1)} \, \overline{g}(\overline{x}_{a(2)}, \xi(\omega)) \right]$$

we can see that the relation (12) is valid, too. □

Theorem 1 presents stability results in the case of the deterministic equivalent I. Further, we shall try to present similar results for the deterministic equivalent II.

To get some results in the case of the deterministic equivalent II, we shall assume that

i') $g(x, z)$ is

a) a continuous function on $X \times E_\ell$,

b) for every $z \in E_\ell$ a Lipschitz function on E_n^+ with Lipschitz constant L' independent of $z \in E_\ell$,

c) for every $x \in X$ a Lipschitz function of $z \in E_\ell$ with Lipschitz constant L

independent of $x \in E_n$,

ii') $g_i(x, z)$, $i = 1, 2, \ldots, \ell$, fulfil relations (2) with continuous functions $f_i(x)$, $i = 1, 2, \ldots, \ell$, for which there exists a real-valued constant $\gamma_1 > 0$ such that

$f_i(x') - f_i(x) \geq \gamma_1 \sum\limits_{j=1}^{n} (x'_j - x_j), \ i = 1, 2, \ldots, n$

for every $x = (x_1, \ldots, x_n), \ x' = (x'_2, \ldots, x'_n) \in E_n, \ x \leq x'$ componentwise,

iii') the probability measure corresponding to the distribution function $F_0(\cdot)$ is absolutely continuous with respect to the Lebesque measure in E_ℓ,

iv') for $a \in A$, $\alpha \in (0, 1)$, $X_a(\alpha)$ is a convex set,

v') there exists a convex set X^* such that $X_a(\alpha) \subset X^*$ for $\alpha \in (0, 1)$, $a \in A$ and further for every $z \in E_\ell$ $g(x, z)$ is a strongly concave function of $x \in X^*$ with a parameter $\rho > 0$,

vi') there exists a convex set X^* such that $X_a(\alpha) \subset X^*$ for $\alpha \in (0, 1)$, $a \in A$ and, further, for every $z \in E_\ell$, $g(x, z)$ is a strictly concave function, i. e. $g(\lambda x_1 + (1-\lambda)x_2) > \lambda g(x_1) + (1 - \lambda) g(x_2)$ for every x_1, $x_2 \in X$, $\lambda \in (0, 1)$.

If the points $\underline{x}_a$, for $a \in A$, fulfil the relation

$$\underline{x}_a \in \arg \max_{X_a(\alpha)} \mathsf{E}_a \, g(x, \xi(\omega)), \tag{13}$$

then the following theorem takes place.

Theorem 2. Let $X = E_n^+$, $A \subset E_\ell$ be nonempty sets, $\alpha \in (0, 1)$. If the assumptions i'), ii'), iii') are fulfilled and if for $x \in X$, $a = a(1)$, $a = a(2)$, $a(1)$, $a(2) \in A$ a finite $\mathsf{E}_a \, g(x, \xi(\omega))$ exists and simultaneously $X_a(\alpha) \neq \emptyset$,
then

$$\left| \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)} \, g(x, \xi(\omega)) - \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)} \, g(x, \xi(\omega)) \right| \leq [L + \frac{L' \sqrt{n}}{\gamma_1}] \, \|a(1) - a(2)\|. \tag{14}$$

Furthermore, there exist points $x' \in X_{a(1)}(\alpha)$, $x'' \in X_{a(2)}(\alpha)$ such that

$$\begin{aligned} \|\underline{x}_{a(1)} - x''\| &\leq \sqrt{n} \, \tfrac{\|a(1) - a(2)\|}{\gamma_1}, \\ \|\underline{x}_{a(2)} - x'\| &\leq \sqrt{n} \, \tfrac{\|a(1) - a(2)\|}{\gamma_1} \end{aligned} \tag{15}$$

and simultaneously

$$\begin{aligned} \left| \mathsf{E}_{a(1)} \, g(\underline{x}_{a(1)}, \xi(\omega)) - \mathsf{E}_{a(2)} \, g(x'', \xi(\omega)) \right| &\leq \left[L + \tfrac{L' \sqrt{n}}{\gamma_1} \right] \|a(1) - a(2)\|, \\ \left| \mathsf{E}_{a(1)} \, g(x', \xi(\omega)) - \mathsf{E}_{a(2)} \, g(\underline{x}_{a(2)}, \xi(\omega)) \right| &\leq \left[L + \tfrac{L' \sqrt{n}}{\gamma_1} \right] \|a(1) - a(2)\|. \end{aligned}$$

If, moreover, the assumptions iv'), v') are fulfilled and $a(1) \le a(2)$ componentwise, then also

$$\|\underline{x}_{a(1)} - \underline{x}_{a(2)}\|^2 \le \frac{4}{\rho}\left[L + \frac{L'\sqrt{n}}{\gamma_1}\right] \|a(1) - a(2)\|.$$

P r o o f. First, we shall prove relation (14). To this end we employ the triangular inequality

$$\begin{aligned} &\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right| \le \\ \le \; &\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right| + \\ + \; &\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega)) - \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right|. \end{aligned} \tag{16}$$

Since it follows from Lemma 5 and Theorem 1 that

$$\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right| \le L\,\|a(1) - a(2)\|,$$

to prove (14) it is sufficient to prove that

$$\left|\max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega)) - \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega))\right| \le L'\sqrt{n}\,\frac{\|a(1) - a(2)\|}{\gamma_1}. \tag{17}$$

If we define vectors $\underline{a}$, $\overline{a} \in E_\ell$, $\overline{a} = (\overline{a}_1, \dots, \overline{a}_\ell)$, $\underline{a} = (\underline{a}_1, \dots, \underline{a}_\ell)$ by

$$\begin{aligned} \underline{a}_i &= a_i(1) - \|a(1) - a(2)\|, \\ \overline{a}_i &= a_i(1) + \|a(1) - a(2)\|, \quad i = 1, 2, \dots, \ell, \\ a(1) &= (a_1(1), \dots, a_\ell(1)), \\ a(2) &= (a_1(2), \dots, a_\ell(2)), \end{aligned}$$

we get $\underline{a} \le a(1) \le \overline{a}$, $\underline{a} \le a(2) \le \overline{a}$ componentwise.

Two cases can happen

a) $\underline{a}$, $\overline{a} \in A$, $X_{\underline{a}}(\alpha) \ne \emptyset$,

b) either $\underline{a}, \overline{a} \notin A$ for at least one element from the pair $(\underline{a}, \overline{a})$ or $X_{\underline{a}}(\alpha) = \emptyset$.

First we shall consider the case a).

Since it follows from the assumptions that $\mathsf{E}_{a(2)}\, g(x, \xi(\omega))$ is a Lipschitz function with Lipschitz constant L', we shall obtain relation (17) on applying Lemma 4. So we have finished the proof of the assertion given by (14) in the case a). It remains to consider the case b). However, it is easy to see that on the transformation bases we obtain the assertion in this case, too.

Now, we shall give the proof of relation (15). But this follows immediately from Lemma 1, Lemma 3, Lemma 4, Lemma 5 and the assumptions.

We have finished the proof of the first part of the assertion of Theorem 2. It remains to verify the validity of the second part. Since it follows from Lemma 2, Lemma 5 and Lemma 6 that

$$\|\underline{x}_{a(2)} - \underline{x}_{a(1)}\|^2 \leq \frac{2}{\rho} \left| \mathsf{E}_{a(2)}\, g(\underline{x}_{a(2)}, \xi(\omega)) - \mathsf{E}_{a(2)}\, g(\underline{x}_{a(1)}, \xi(\omega)) \right|,$$

we see that the assertion will be proved if we verify the validity of the inequality

$$\left| \mathsf{E}_{a(2)}\, g(\underline{x}_{a(2)}, \xi(\omega)) - \mathsf{E}_{a(2)}\, g(\underline{x}_{a(1)}, \xi(\omega)) \right| \leq 2 \left[L + \frac{L' \sqrt{n}}{\gamma_1} \right] \|a(1) - a(2)\|.$$

To this end we shall employ the triangular inequality

$$\begin{aligned} & \left| \mathsf{E}_{a(2)}\, g(\underline{x}_{a(2)}, \xi(\omega)) - \mathsf{E}_{a(2)}\, g(\underline{x}_{a(1)}, \xi(\omega)) \right| \leq \\ \leq \; & \left| \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega)) - \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) \right| + \\ + \; & \left| \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \mathsf{E}_{a(2)}\, g(\underline{x}_{a(1)}, \xi(\omega)) \right|. \end{aligned}$$

However, since it follows from the assertion of relation (14) that

$$\left| \max_{X_{a(2)}(\alpha)} \mathsf{E}_{a(2)}\, g(x, \xi(\omega)) - \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) \right| \leq \left[L + \frac{L' \sqrt{n}}{\gamma_1} \right] \|a(1) - a(2)\|$$

and since it follows from Lemma 1 and Lemma 5 that

$$\left| \max_{X_{a(1)}(\alpha)} \mathsf{E}_{a(1)}\, g(x, \xi(\omega)) - \mathsf{E}_{a(2)}\, g(\underline{x}_{a(1)}, \xi(\omega)) \right| \leq L\, \|a(1) - a(2)\|$$

we can see that we have verified also the last assertion of Theorem 2. □

Remark. Evidently, if we omit the assumption $a(1) \leq a(2)$, then it is possible to prove some similar assertion to the one presented in the second part of Theorem 2, too.

The results obtained in this section will be the foundation for convergence results of statistical estimates.

4. CONVERGENCE RESULTS

If we denote by $\hat{a}(N) = \hat{a}(N, \omega)$, $N = 1, 2, \ldots$, a sequence of statistical estimates of the parameter a, then we can already present the following theorem.

Theorem 3. Let $X \subset E_n$ be a nonempty, compact set, $A \subset E_s$ be a nonempty set, and let a finite $\mathsf{E}_a\, g(x, \xi(\omega))$ exist for $a \in A$, $x \in X$. Let, further, assumptions i), ii) be fulfilled. If $\hat{a}(N) = \hat{a}(N, \omega)$, $N = 1, 2, \dots$, is a sequence of statistical estimates of the parameter $a \in \mathrm{int} A$, then

$$\mathrm{p}\lim_{N\to\infty} \hat{a}(N,\omega) = a \quad \Longrightarrow \quad \mathrm{p}\lim_{N\to\infty} \max_X \mathsf{E}_{\hat{a}(N,\omega)}\, \overline{g}(x, \xi(\omega)) = \\ = \max_X \mathsf{E}_a\, \overline{g}(x, \xi(\omega))$$

and

$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \text{ a.s.} \ \Longrightarrow \\ \lim_{N\to\infty} \max_X \mathsf{E}_{\hat{a}(N,\omega)}\, \overline{g}(x, \xi(\omega)) = \max_X \mathsf{E}_a\, \overline{g}(x, \xi(\omega)) \ \text{ a.s.}$$

If moreover, the assumption iii) holds, then also

$$\mathrm{p}\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \Longrightarrow \ \mathrm{p}\lim_{N\to\infty} \|\underline{x}_{\hat{a}(N,\omega)} - \overline{x}_a\|^2 = 0,$$
$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \text{ a.s.} \ \Longrightarrow \ \lim_{N\to\infty} \|\underline{x}_{\hat{a}(N,\omega)} - \overline{x}_a\|^2 = 0 \ \text{ a.s.}$$

P r o o f. The assertion of Theorem 3 follows immediately from Theorem 1 and elementary properties of the probability measure. □

Theorem 3 deals with the deterministic equivalent I. There are presented the assumptions under which the convergence of parameter estimates to the theoretical parameter value in some sense vouches the convergence of the optimal value estimates and the optimal solution estimates in the same sense. Further, we shall try to introduce similar results for the deterministic equivalent II.

Theorem 4. Let $X = E_n^+$, $A \subset E_\ell$ be a nonempty set and $\alpha \in (0,1)$. Let, further, the assumptions i'), ii'), iii') be fulfilled and a finite $\mathsf{E}_a\, g(x, \xi(\omega))$ exists for $a \in A$, $x \in X$. If $\hat{a}(N) = \hat{a}(N,\omega)$, $N = 1, 2, \dots$, is a sequence of statistical estimates of the parameter $a \in \mathrm{int}\, A$ such that there exists neighbourhood $U(a) \subset A$ for which $X_{a'}(\alpha) \neq \emptyset$, $a' \in U(a)$, then

$$\mathrm{p}\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \Longrightarrow \ \mathrm{p}\lim_{N\to\infty} \max_{X_{\hat{a}(N,\omega)}(\alpha)} \mathsf{E}_{\hat{a}(N,\omega)}\, g(x, \xi(\omega)) = \\ = \max_{X_a(\alpha)} \mathsf{E}_a\, g(x, \xi(\omega))$$

and

$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \text{ a.s.} \ \Longrightarrow \ \lim_{N\to\infty} \max_{X_{\hat{a}(N,\omega)}(\alpha)} \mathsf{E}_{\hat{a}(N,\omega)}\, g(x, \xi(\omega)) \\ = \max_{X_a(\alpha)} \mathsf{E}_a\, g(x, \xi(\omega)) \ \text{ a.s.} \tag{18}$$

Moreover, if the assumptions iv'), vi') are fulfilled, then also

$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \ \text{ a.s.} \ \Longrightarrow \ \lim_{N\to\infty} \|\underline{x}_{\hat{a}(N,\omega)} - \underline{x}_a\| = 0 \ \text{ a.s.} \tag{19}$$

P r o o f . The assertion given by relation (18) follows immediately from Theorem 2 and elementary properties of the probability measure. So, it remains to prove the assertion given by relation (19).

First, it follows from the assumptions iv'), vi') that the $\underline{x}_{\hat{a}(N,\omega)}$, $\underline{x}_a$, $N = 1, 2, \ldots,$ $\omega \in \Omega$, $a \in A$ are uniquely defined for enough large N. So, if we denote

$$\Omega' = \{\omega \in \Omega : \lim_{N\to\infty} \hat{a}(N,\omega) = a \quad \text{and simultaneously}$$
$$\lim_{N\to\infty} \max_{X_{\hat{a}(N,\omega)}(\alpha)} \mathsf{E}_{\hat{a}(N,\omega)}\, g(x, \xi(\omega)) = \max_{X_a(\alpha)} \mathsf{E}_a\, g(x, \xi(\omega))\, \},$$

then, according to Lemma 1 of [16] and relation (18), it is easy to see that relation (19) will be proved if we verify the implication

$$\omega \in \Omega' \Longrightarrow \lim_{N\to\infty} \|\underline{x}_{\hat{a}(N,\omega)} - \underline{x}_a\| = 0.$$

We shall prove this implication by contradiction. We shall assume that there exists $\omega' \in \Omega'$ such that

$$\lim_{N\to\infty} \|\underline{x}_{\hat{a}(N,\omega')} - \underline{x}_a\| \neq 0.$$

It follows from Lemma 4 and Lemma 5 that there exists a compact set $\overline{X} \subset X$ and a natural number $N_0 = N_0(\omega')$ such that

$$X_{\hat{a}(N,\omega')}(\alpha) \subset \overline{X},\ X_a(\alpha) \subset \overline{X} \ \text{ for } \quad N > N_0.$$

Since $\overline{X}$ is a compact set we can see that there exists a subsequence $\{\hat{a}(N_k, \omega')\}_{N_k=1}^{+\infty}$ of the sequence $\{\hat{a}(N, \omega')\}_{N=1}^{+\infty}$ and a point $x' \in \overline{X}$, $x' \neq x_a$ such that

$$\lim_{N\to\infty} \|\underline{x}_{\hat{a}(N_k,\omega')} - x'\| = 0.$$

According to Lemma 4 and Lemma 5 it must hold that

$$x' \in X_a(\alpha)$$

and further, since $\mathsf{E}_a\, g(x, \xi(\omega))$ is a strictly concave function, it also holds that

$$\mathsf{E}_a\, g(x', \xi(\omega)) \neq \mathsf{E}_a\, g(\underline{x}_a, \xi(\omega)). \tag{20}$$

Employing the triangular inequality and Lemma 1, we obtain simultaneously

$$\begin{aligned}
&\left|\mathsf{E}_{\hat{a}(N_k,\omega')}\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) - \mathsf{E}_a\, g(x', \xi(\omega))\right| \leq \\
&\left|\mathsf{E}_{\hat{a}(N_k,\omega')}\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) - \mathsf{E}_a\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega))\right| \\
&+ \left|\mathsf{E}_a\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) - \mathsf{E}_a\, g(x', \xi(\omega))\right| \leq \\
&L\, \|\hat{a}(N_k, \omega') - a\| + \left|\mathsf{E}_a\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) - \mathsf{E}_a\, g(x', \xi(\omega))\right|.
\end{aligned} \tag{21}$$

Since, further, we can easily see that $\mathsf{E}_a\, g(x, \xi(\omega))$ is a continuous function we get

$$\lim_{N\to\infty} \mathsf{E}_a\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) = \mathsf{E}_a\, g(x', \xi(\omega))$$

and employing (21) also that

$$\lim_{N\to\infty} \left| \mathsf{E}_{\hat{a}(N_k,\omega')_k}\, g(\underline{x}_{\hat{a}(N_k,\omega')}, \xi(\omega)) - \mathsf{E}_a\, g(x', \xi(\omega)) \right| = 0.$$

However, according to (20) this contradicts with $\omega' \in \Omega'$. □

Further, we shall study the convergence rate. It is easy to see that the convergence rate of the $\hat{a}(N, \omega)$ fully determines the convergence rate of the optimal value estimates. Moreover, a similar assertion also holds for optimal solution estimate in the case of the deterministic equivalent I.

Theorem 5. Let $X \subset E_n$ be a nonempty, compact set, $A \subset E_s$ be a nonempty set, and a finite $\mathsf{E}_a\, g(x, \xi(\omega))$ exists for $a \in A$, $x \in X$. If assumptions i), ii) are fulfilled and if $\hat{a}(N) = \hat{a}(N, \omega)$, $N = 1, 2, \ldots$, is a sequence of statistical estimates of the parameter $a \in \text{int}\, A$ such that there exists a real-valued sequence ν_N, $N = 1, 2, \ldots, \nu_N \to +\infty$ as $(N \to \infty)$ and one dimensional distribution function $G(\cdot)$ fulfilling the relation

$$\liminf_{N\to\infty} P\{\omega : \nu_N \|\hat{a}(N, \omega) - a\| < c\} \geq G(c)$$

for every $c \in E_1$,
then

$$\liminf_{N\to\infty} P\left\{\omega : \nu_N \left| \max_{x\in X} \mathsf{E}_{\hat{a}(N,\omega)}\, \overline{g}(x, \xi(\omega)) - \max_{x\in X} \mathsf{E}_a\, \overline{g}(x, \xi(\omega)) \right| < c \right\} \geq$$
$$\geq \quad G\left(\frac{c}{\overline{L}}\right) \quad \text{for every } c \in E_1.$$

Moreover, if the assumption iii) is fulfilled, then also

$$\liminf_{N\to\infty} P\left\{\omega : \nu_N \|\underline{x}_{\hat{a}(N,\omega)} - \underline{x}_a\|^2 < c\right\} \geq G\left(\frac{\rho \cdot c}{4 \cdot \overline{L}}\right) \quad \text{for every } c \in E_1.$$

P r o o f. The assertion of Theorem 5 follows immediately from Theorem 1, the assumptions of Theorem 5 and the elementary properties of the probability measure. □

Theorem 6. Let $X = E_n^+$, $A \subset E_s$ be nonempty set, $\alpha \in (0, 1)$ and a finite $\mathsf{E}_a\, g(x, \xi(\omega))$ exists for $a \in A$, $x \in X$. If the assumptions i'), ii'), iii') are fulfilled and if

1. $\hat{a}(N) = \hat{a}(N, \omega)$, $N = 1, \ldots$ is a sequence of statistical estimates of the parameter

value $a \in \operatorname{int} A$ for which

a) there exists a neighbourhood $U(a)$ such that $X_{a'}(\alpha) \neq \emptyset$ for all $a' \in U(a)$,

b) there exists a real-valued sequence ν_N, $N = 1, 2 \ldots,$ such that $\lim_{N \to \infty} \nu_N = +\infty$, and one-dimensional distribution function $G(\cdot)$ fulfilling the relation

$$\liminf_{N \to +\infty} P\left\{\omega : \nu_N \|\hat{a}(N, \omega) - a\| < c\right\} \geq G(c)$$

for every $c \in E_1$,

then

$$\liminf_{N \to \infty} P\left\{\omega : \nu_N \left| \max_{X_{\hat{a}(N,\omega)}} \mathsf{E}_{\hat{a}(N,\omega)}\, g(x, \xi(\omega)) - \max_{X_a(\alpha)} \mathsf{E}_a\, g(x, \xi(\omega)) \right| < c\right\} \geq$$

$$\geq \quad G\left(c / \left(L + \frac{L' \sqrt{n}}{\gamma_1}\right)\right) \quad \text{for every } c \in E_1.$$

Proof. The assertion of Theorem 6 follows immediately from Theorem 2, the assumptions and elementary properties of the probability measure. □

Remarks.

1. It follows from Theorem 1 and Theorem 2 that the optimal value is a Lipschitz function of the parameter a, in both cases under considered assumptions. Consequently, we can obtain the first part of the assertion of Theorem 5 and the assertion of Theorem 6 immediately from Theorem 15 in [11], too.

2. If a) $f_i(x)$, $i = 1 \ldots, \ell$, are convex functions on E_n,

b) the probability measure, corresponding to the distribution function $F_0(\cdot)$ is logaritmic concave,

then it follows from [12] that $X_0(\alpha)$ is a convex set. Consequently, the approximative sets are convex, too. (The definition of logaritmic concave probability measure is given for example in [12].)

3. It happens rather often that the estimate of the unknown parameter a can be introduced as a sample average. Then it is easy to see that to obtain a converge rate we can utilize the method of large deviations in the case of independent random sample [10]. The case of dependent sample is discussed in [10], too.

4. Theorem 5 and Theorem 6 present some convergence results. It is easy to see that some similar results can be also introduced for finite natural numbers N.

5. APPLICATIONS TO SEQUENCES OF STOCHASTIC OPTIMIZATION PROBLEMS

It is well-known that many practical problems repeat in time. It is also well-known that if we solve such optimization problems with respect to time dependence, we often obtain rather better results than by solving the corresponding separated problems. In particular, this appears in the case of stochastic optimization problems. Namely, there often exists a stochastic dependence of random elements.

Let $\xi^j(\omega) = \xi^j = (\xi_1^j(\omega), \dots, \xi_s^j(\omega))$, $j = 1, 2, \dots$, be s-dimensional random vectors defined on (Ω, S, P),
$g^1(x, z)$ be a real-valued, continuous function defined on $E_n \times E_s$,
$g^j(x^j, z^{j-1}, z^j)$, $j = 2, \dots$, be real-valued, continuous functions defined on $E_n \times E_s \times E_s$,
$X^j(z^{j-1}) = X^j$, $j = 2, \dots$, be mappings of E_s into the space of non-empty, compact subsets of E_n, and $X^1 \subset E_n$ be a non-empty, compact set.

We shall introduce the stochastic optimization problem (w.r.t. the discrete time interval $1 \div N$) as a problem of finding $(x^1, x^2, \dots, x^N)$, $x^1 \in X^1$, $x^j = x^j(\xi^{j-1}(\omega)) \in X^j(\xi^{j-1}(\omega))$, $j = 2, \dots, N$, for which

$$\mathsf{E}\left\{g^1(x^1, \xi^1(\omega)) + \sum_{j=2}^{N} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega))\right\} \tag{22}$$

is maximal.

The aim of this section is to utilize the former results to obtain some estimates of the optimal value and the optimal solution of (22) under very special conditions. In detail, we shall consider the case when there exist s-dimensional random vectors $\eta^j(\omega) = \eta^j, j = 1, 2, \dots$, defined on (Ω, S, P) such that

$$\xi^j(\omega) = \sum_{i=1}^{j} \eta^i(\omega), \quad j = 1, 2, \dots. \tag{23}$$

In what follows
$F_{a^j}^{\eta^j}(\cdot)$ denotes the distribution function of the random vector $\eta^j(\omega)$, $j = 1, 2, \dots$,
$F_{b^j}^{\xi^j}(\cdot)$ denotes the distribution function of the random vector $\xi^j(\omega)$, $j = 1, 2, \dots$,
$F_{b^j, b^{j-1}}^{\xi^j|\xi^{j-1}}(\cdot)$ and $\mathsf{E}_{b_j, b_{j-1}}^{\xi^j|\xi^{j-1}}$ denote the conditional distribution function and the conditional mathematical expectation of the random vectors $\xi^j(\omega)$ by $\xi^{j-1}(\omega)$, $j = 1, 2, \dots$, respectively,
$a^j \in A$, $b^j \in A$ are parameters, $j = 1, 2, \dots$,
$F_{b^1, \dots, b^N}^{\xi^1, \dots, \xi^N}(\cdot)$ denotes the common distribution function of $\xi^1(\omega), \xi^2(\omega), \dots, \xi^N(\omega)$,
$\mathsf{E}_{b^1, \dots, b^N}^{\xi^1, \dots, \xi^N}$ denotes the operator of mathematical expectation corresponding to the distribution function $F_{b^1, \dots, b^N}^{\xi^1, \dots, \xi^N}$.
Moreover, we shall assume that there exist s-dimensional distribution functions $F_0^{\eta^j}(\cdot)$, $F_0^{\xi^j}(\cdot)$, $j = 1, 2, \dots$, such that

$$\begin{aligned} F_{a^j}^{\eta^j}(z) &= F_0^{\eta^j}(z - a^j), \quad j = 1, 2, \dots, \\ F_{b^j}^{\xi^j}(z) &= F_0^{\xi^j}(z - b^j), \quad b^j = \sum_{i=1}^{j} a^i. \end{aligned} \tag{24}$$

It is easy to see that under our assumptions it holds

$$F_{b^j, b^{j-1}}^{\xi^j|\xi^{j-1}}(z^j) = F_{a^j}^{\eta^j}(z^j - \xi^{j-1}(\omega)). \tag{25}$$

Remark. If for example, $\eta^j(\omega),\ j = 1, 2 \ldots$, are independent normal distributed random vectors with the average $a^j,\ j = 1, 2, \ldots$, then relations (24), (25) are fulfilled.

First, the following lemma is proved in [8].

Lemma 7. Let there exist a finite

$$\mathsf{E}\left\{g^1(x^1, \xi^1(\omega)) + \sum_{j=2}^{N} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega))\right\} \text{ for every } x^1, x^2, \ldots, x^N \in E_n.$$

If for every $i = 2, \ldots, N,\ b^1, \ldots, b^N \in A$

1. $\overline{x}^i(\xi^{i-1}(\omega), b^{i-1}) = \overline{x}^i$ is a solution of the problem to find

$$\max_{x^i \in X^i(\xi^{i-1}(\omega))} \mathsf{E}_{b^i, b^{i-1}}^{\xi^i | \xi^{i-1}} g^i(x^i, \xi^{i-1}(\omega), \xi^i \omega)),$$

2. $\mathsf{E}_{b^i, b^{i-1}}^{\xi^i | \xi^{i-1}}\ g^i(\overline{x}^i, \xi^{i-1}(\omega), \xi^i \omega))$ is a measurable function w.r.t. the σ-algebra given by $\xi^1(\omega), \ldots, \xi^{i-1}(\omega)$ and if $\overline{x}^1$ is a solution of the problem to find

$$\max_{x^1 \in X^1} \mathsf{E}\, g^1(x_1^1, \xi^1(\omega)),$$

then $(\overline{x}^1, \overline{x}^2(\xi^1(\omega)), \ldots, \overline{x}^N(\xi^{N-1}(\omega))$ is a solution of the problem given by (22).

(We have omitted somewhere the index $\xi^j, b^j, j = 1, 2, \ldots$, at the symbol of mathematical expectation. The same shorthand notation will be used also in the sequel.)

Now we are in a position to present the following result on the stability.

Theorem 7. Let $A \subset E_s$ be a nonempty set. If

1. for every $x^1 \in E_n,\ g^1(x^1, z^1)$ is a Lipschitz function of $z^1 \in E_s$ with Lipschitz constant L_1 independent of $x^1 \in E_n$,

2. for every $x^j \in E_n,\ z^{j-1} \in E_s,\ g^j(x^j, z^{j-1}, z^j),\ j = 1, 2 \ldots$, are Lipschitz functions of z^j with Lipschitz constant L_1 independent of $x^j \in E_n,\ z^{j-1} \in E_s$,

3. for every $z^j \in E_s$, $g^j(x^j, z^{j-1}, z^j)$, $j = 1, 2 \ldots$, are Lipschitz functions of x^j, z^{j-1} with Lipschitz constant L_1' independent of $z^j \in E_s$,

4. there exist finite $\mathsf{E}_{a^1}^{\xi^1} g^1(x^1, \xi^1(\omega)),\ \mathsf{E}_{b^j, b^{j-1}}^{\xi^j, \xi^{j-1}} g^j(x^j,\ \xi^{j-1}(\omega),\ \xi^j(\omega))$, for every $a^j \in A, b^j = \sum\limits_{i=1}^{j} a^i,\ b^j \in A,\ x^j \in E_n,\ j = 1, 2, \ldots,$

5. there exists a real-valued constant C such that

$$\triangle[X^j(z^{j-1}(1)), X^j(z^{j-1}(2))] \le C\|z^{j-1}(1) - z^{j-1}(2)\|$$
$$\text{for every } z^{j-1}(1), z^{j-1}(2) \in E_s, \quad j = 2, \ldots,$$

6. $\max_{x^i \in X^i(\xi^{i-1})} \mathsf{E}_{b^i, b^{i-1}}^{\xi^i|\xi^{i-1}} g^i(x^i, \xi^{i-1}(\omega), \xi^i(\omega)),\ \ i = 2, \ldots, N,\ b^i,\ b^{i-1} \in A$ are measurable functions,

then

$$\Bigg| \max_K \mathsf{E}_{b^1(1),\ldots,b^N(1)}^{\xi^1,\ldots\xi^N} \left[g^1(x^1, \xi^1(\omega)) + \sum_{j=2}^N g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega))] \right] -$$

$$\max_K \mathsf{E}_{b^1(2),\ldots,b^N(2)}^{\xi^1,\ldots\xi^N} \left[g^1(x^1, \xi^1(\omega)) + \sum_{j=2}^N g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right] \Bigg|$$

$$\leq \sum_{j=1}^N \left[L_1 \|a^j(1) - a^j(2)\| + \sum_{i=1}^{j-1} L_1' (C+1) \|a^i(1) - a^i(2)\| \right],$$

for $K = \{x^1, x^2, \ldots, x^N :\ x^1 \in X^1,\ x^2 \in X^2(\xi^1(\omega)), \ldots, x^N \in X^N(\xi^{N-1}(\omega))\}$,
$a^i(r) \in A,\ \sum_{i=1}^{j} a^i(r) = b^j(r) \in A,\ i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, N,\ r = 1, 2\ (\sum_{i=1}^{0} \equiv 0)$.

P r o o f. First, according to Lemma 7, it is easy to see that

$$\max_K \mathsf{E}_{b^1,\ldots,b^N}^{\xi^1,\ldots,\xi^N} \left[g^1(x^1, \xi^1(\omega)) + \sum_{j=2}^N g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right] =$$

$$= \max_{x^1 \in X^1} \mathsf{E}_{a^1}^{\xi^1} g^1(x^1, \xi^1(\omega)) + \tag{26}$$

$$+ \sum_{j=2}^N \mathsf{E}_{b^{j-1}}^{\xi^{j-1}} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}_{b^j, b^{j-1}}^{\xi^j|\xi^{j-1}} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega))$$

for every $a^j \in A$, $b^j = \sum_{i=1}^{j} a^j \in A$.

Further, since it follows from Lemma 1 that

$$\left| \mathsf{E}_{a^1(1)}^{\xi^1} g^1(x^1, \xi^1(\omega)) - \mathsf{E}_{a^1(2)}^{\xi^1} g^1(x^1, \xi^1(\omega)) \right| \leq L_1 \|a^1(1) - a^1(2)\|$$

and simultaneously

$$\left| \mathsf{E}_{b^j(1), b^{j-1}(1)}^{\xi^j|\xi^{j-1}} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) - \mathsf{E}_{b^j(2), b^{j-1}(2)}^{\xi^j|\xi^{j-1}} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right| \leq$$

$$\leq L_1 \|a^j(1) - a^j(2)\|$$

for every

$$x^1 \in X^1, x^j \in X^j(\xi^{j-1}(\omega)), \quad j = 2, \ldots, N,\ \omega \in \Omega,$$

it is easy to see that also

$$\left| \max_{x^1 \in X^1} \mathsf{E}_{a^1(1)}^{\xi^1} g^1(x^1, \xi^1(\omega)) - \max_{x^1 \in X^1} \mathsf{E}_{a^1(2)}^{\xi^1} g^1(x^1, \xi^1(\omega)) \right| \leq L_1 \|a^1(1) - a^1(2)\| \tag{27}$$

and

$$\begin{aligned}
&\left| \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{b^j(2),b^{j-1}(2)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) - \right.\\
- &\left. \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{b^j(1),b^{j-1}(1)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right| \qquad (28)\\
\le\ & L_1\|a^j(1) - a^j(2)\|
\end{aligned}$$

for every $a^1(1)$, $a^1(2)$, $b^j(1)$, $b^j(2) \in A$, $j = 1, 2, \ldots, N$.

However, employing the triangular inequality, we obtain for $j = 2, \ldots, N$,

$$\begin{aligned}
&\left| \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(2)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(2)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right.\\
- &\left. \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(1)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(1)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right| \le\\
\le &\left| \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(2)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(2)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) - \right. \qquad (29)\\
- &\left. \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(2)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(1)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right| +\\
+ &\left| \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(2)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(1)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) - \right.\\
&\left. \mathsf{E}^{\xi^{j-1}}_{b^{j-1}(1)} \max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j(1)}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)) \right|.
\end{aligned}$$

Since it follows from Lemma 2 of [7] that

$$\max_{x^j \in X^j(\xi^{j-1}(\omega))} \mathsf{E}^{\xi^j|\xi^{j-1}}_{a^j}\, g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)), \quad j = 2, \ldots, N,$$

is (for every $a^j \in A$) a Lipschitz function of $z^{j-1} \in E_s$ with Lipschitz constant $L'_1(C+1)$ we get, utilizing relations (24), (26), (27), (28) and Lemma 1, the validity of the assertion of Theorem 7. □

Further, we shall pay attention to estimates problems. To this end we shall restrict our consideration to the case $a^j = a$, $a \in A$, $j = 1, 2, \ldots$. A specific situation arises in this case. Namely, if relation (23) is satisfied, then the random sequence $\{\xi^j(\omega)\}_{j=1}^{\infty}$ is fully determined by the random sequence $\{\eta^j(\omega)\}_{j=1}^{\infty}$. However, then it is obvious that an estimate of the parameter a can be obtained from one realization of the random sequence $\{\xi^j(\omega)\}_{j=1}^{\infty}$, under some additional assumptions, of course. More precisely, we can obtain an estimate on the realizations bases of the first N members of the random sequence $\{\xi^N(\omega)\}_{N=1}^{\infty}$.

Theorem 8. Let $A \subset E_s$ be a nonempty set. If the assumptions $1, 2, 4, 6$ of Theorem 7 are fulfilled and if $\hat{a}(N, \omega) = \hat{a}(N)$, $N = 1, 2 \ldots$, are statistical estimates of the parameter $a \in \operatorname{int} A$ obtained from the first $N-1$ members of the random sequence $\{\xi^j(\omega)\}_{j=1}^{\infty}$, then

$$\begin{aligned}
&\hat{a}(N,\omega) \to a \quad \text{a.s.} \quad \Longrightarrow \\
&\lim_{N\to\infty} \max_{x^N \in X^N(\xi^{N-1}(\omega))} \mathsf{E}_{\hat{a}(N,\omega)}^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)) = \\
&\max_{x^N \in X^N(\xi^{N-1}(\omega))} \mathsf{E}_{a}^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)) \quad \text{a.s.}
\end{aligned} \tag{30}$$

If, moreover, for every $z^{j-1}, z^j \in E_s$, $j = 1, 2, \ldots$,

a) $X^j(z^{j-1})$, $j = 1, \ldots$, are convex sets,

b) for every z^{j-1}, $z^j \in E_s$, $g^j(x^j, z^{j-1}, z^j)$ are strongly concave functions of $x^j \in E_n$ with a parameter $\rho > 0$,

c)

$$x_{\hat{a}(N,\omega)}^N = \arg \max_{x^N \in X^N(\xi^{N-1}(\omega))} \mathsf{E}_{\hat{a}(N,\omega)}^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)), \quad N = 1, 2, \ldots,$$

$$x_a^N = \arg \max_{x^N \in X^N(\xi^{N-1}(\omega))} \mathsf{E}_a^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)),$$

are measurable functions,
then also

$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \;\text{ a.s.} \Longrightarrow \lim_{N\to\infty} \|x_{\hat{a}(N,\omega)}^N - x_a^N\|^2 = 0 \;\text{ a.s.} \tag{31}$$

$(\mathsf{E}_a^{\xi^N|\xi^{N-1}} := \mathsf{E}_{b^N, b^{N-1}}^{\xi^N|\xi^{N-1}}, \quad b^N = b^{N-1} + a)$

The validity of the assertion of Theorem 8 follows immediately from the assertion of Theorem 1.

The next corollary follows immediately from Theorem 8.

Corollary 1. Let $A \subset E_s$ be a nonempty set. If the assumptions $1, 2, 4, 6$ of Theorem 7 are fulfilled and if $\hat{a}(N,\omega) = \hat{a}(N), N = 1, 2\ldots$, are statistical estimates (defined in Theorem 8) of the parameter $a \in \operatorname{int} A$, then

$$\begin{aligned}
&\lim_{N\to\infty} \hat{a}(N,\omega) = a \quad \text{a.s.} \quad \Longrightarrow \\
&\lim_{M\to\infty} \frac{1}{M} \sum_{N=1}^{M} \Bigg| \max_{x^N \in X^N(\xi^{N-1})} \mathsf{E}_{\hat{a}(N,\omega)}^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)) \\
&- \max_{x^N \in X^N(\xi^{N-1})} \mathsf{E}_a^{\xi^N|\xi^{N-1}} g^N(x^N, \xi^{N-1}(\omega), \xi^N(\omega)) \Bigg| = \; 0 \;\text{ a.s.}
\end{aligned}$$

If, moreover, for every z^{j-1}, $z^j \in E_s$, $j = 1, 2\ldots$, assumptions a), b), c) of Theorem 8 are satisfied, then also

$$\lim_{N\to\infty} \hat{a}(N,\omega) = a \;\text{ a.s.} \quad \Longrightarrow \lim_{M\to\infty} \frac{1}{M} \sum_{N=1}^{M} \left\| x_{\hat{a}(N)(\omega)}^N - x_a^N \right\|^2 = 0 \;\text{ a.s.}$$

Remark. Sufficient assumptions under which

$$\max_{X^j(\xi^{j-1}(\omega))} \mathsf{E}_a^{\xi^j|\xi^{i-1}} g^j(x^j, \xi^{j-1}(\omega), \xi^j(\omega)), \quad j = 2, 3, \ldots,$$

are measurable functions follow for example from Lemma 1 of [16].

6. CONCLUSION

In this paper we have dealt with stability of one very special problem in stochastic programming with unknown parameters. However, it is well-known that real-life problems satisfy not seldom only our assumptions. Moreover, the assumptions under which even the approximative problems are concave ones follow from Remark in Section 4 in the chance constrained case, too.

It is evident that the obtained results can be in many other ways applied to time depedent stochastic optimization problems. The aim of this paper was only to turn attention to this possibility.

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RNDr. Vlasta Kaňková, CSc., Ústav teorie informace a automatizace AV ČR (Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic), Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.