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ON POWER SERIES, BELL POLYNOMIALS, HARDY–RAMANUJAN–RADEMACHER PROBLEM AND ITS STATISTICAL APPLICATIONS

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A possibly unknown approach to the problem of finding the common term of a power series is considered. A direct formula for evaluating this common term has been obtained. This formula provides useful expressions for direct evaluation of the number of partitions of a nonnegative integer and the partitions themselves. These expressions permit easily to work with power series, evaluate nth derivative of a composite function, calculate Bernoulli, Euler, Bell and other numbers, evaluate Bell polynomials, cycle index etc. Alternative expressions and some other new results for the truncated Bell polynomials have also been obtained. Some statistical applications of results under consideration and features of generalized probability generating functions are discussed.

The problem of solving in integers the Diophant's equations

$$x_1 + 2x_2 + ... + kx_k = n, \qquad k \leq n, \tag{1}$$

is very important for applications.

Let an insurance company expect that next year there will be n accidents among its clients. It is very interesting for the company to know the distribution of a number of the accidents, i. e. to know the number x_1 of people suffered in one accident, the number x_2 of people suffered in two accidents, etc. Thus, we are to solve the problem (1). We are able also to find the number of people suffered at least in k accidents and in no more than m accidents, $1 \leq k \leq m \leq n$, etc.

Let us study customers demands in a big department store. Suppose there were n purchases during a time T in a day and we are interested in the clients distribution by the number of purchases, i. e. we would like to know how many clients x_1 bought one thing, how many clients x_2 bought two things, etc. This might be useful when organizing a cashing, when distributing advertisements, etc.

Let us study a spreading of some decease, e. g. influenca, among a given group of citizens during some years. Let n be the number of deceases registered, then (1) gives the number x_1 of people who were ill once, the number x_2 of people who were ill twice, etc.

The integer n might be a number of authors who published their researches during a given period of time in scientific journals, then x_1 is the number of authors who published one paper, x_2 is that for authors who published two papers, etc.

Many important examples of applications of Diophant's equations in algebra, group theory, analysis, etc. may be found in references cited below.

1. INTRODUCTION

A usual way for deriving the common term of a power series which is an arbitrary complex power α of another power series consists of using the Faa di Bruno's formula for the n-th derivative of a compound function. It is this formula which defines the well known Bell polynomials (Riordan [17]). Schur [19] and Riordan [18] gave explicit formulas for the common terms of the above power series in a case when α is a positive or negative integer.

The need for evaluation of these common terms often arises in probability theory and mathematical statistics. Let one wants to generalize any distribution considering its probability generating function $h(t) = f(g(t))$, where $f(x) = x^{\pm n}$, n being an even integer and $g(t)$ being an analytic function. The problem of deriving the probabilities and moments of the corresponding generalized distribution is a problem of expanding $h(t)$ into a power series. Charalambides [3, 4] used Bell polynomials to derive probabilities of discrete distributions considering the probability generating function being known, see also Johnson and Kotz [10]. Philippou [13, 15, 16], Philippou et al. [14], Hirano [8], Ling [11], Hirano et al. [9] and others considered generalized discrete distributions of order k defining first the probabilities and then deriving desired generating functions, but, in each case, they used the Bell polynomials.

In our presentation we will consider the problem of finding the common term of a power series which is an arbitrary complex power of a polynomial. A direct formula for the common term of this series will be generalized, providing thus an alternative expression for Bell polynomials.

2. MAIN RESULT

Consider the problem

$$\left(\sum_{l=0}^{m} b_l z^l\right)^{\alpha} = \sum_{k=0}^{\infty} a_k(\alpha, m) z^k, \tag{1}$$

where α is an arbitrary complex. Here and in the sequel we consider all power series as being converged.

Lemma 1. Coefficients $a_k(\alpha, m)$ satisfy the recurrency

$$a_k(\alpha, m) = \sum_{s=0}^{[\frac{k}{m}]} \frac{(\alpha)_s}{s!} a_{k-ms}(\alpha - s, m-1) b_m^s, \tag{2}$$

where $(\alpha)_s = \alpha(\alpha-1), \dots (\alpha-s+1)$, $[x]$ is an integer part of x.

Applying the binom formula which is evidently valid for any complex α we obtain

$$[(b_0 + \cdots + b_{m-1}z^{m-1}) + b_m z^m]^\alpha = \sum_{k=0}^{\infty} a_k(\alpha, m) z^k$$

or

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} (b_m z^m)^k (b_0 + \cdots + b_{m-1}z^{m-1})^{\alpha-k} = \sum_{k=0}^{\infty} a_k(\alpha, m) z^k.$$

Since

$$(b_0 + \cdots + b_{m-1}z^{m-1})^{\alpha-k} = \sum_{l=0}^{\infty} a_l(\alpha-k, m-1) z^l,$$

we have

$$\sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \frac{(\alpha)_s}{s!} a_l(\alpha-s, m-1) b_m^s z^{ms+l} = \sum_{k=0}^{\infty} a_k(\alpha, m) z^k.$$

Assuming $l = k - ms$, summing over $0 \le s \le [\frac{k}{m}]$ and equating coefficients at z^k we arrive at (2).

For $m = 0$ formula (2) gives $b_0^\alpha = \sum_{k=0}^{\infty} a_k(\alpha, 0)$, since $a_0(\alpha, 0) = b_0^\alpha$, $a_k(\alpha, 0) = 0$, if $k \ge 1$.

For $m = 1$ we have $a_{k-s}(\alpha - s, 0) = 0$ if $0 \le s \le k-1$, and we obtain from (2) the binomial coefficients

$$a_k(\alpha, 1) = \frac{(\alpha)_k}{k!} b_0^{\alpha-k} b_1^k, \tag{3}$$

Using (3) formula (2) gives for $m = 2$

$$a_k(\alpha, 2) = \sum_{s=0}^{[\frac{k}{2}]} \frac{(\alpha)_{k-s}}{(k-s)!} \binom{k-s}{s} b_0^{\alpha-(k-s)} b_1^{k-2s} b_2^s. \tag{4}$$

Generalizing (4) for an arbitrary m we have the following

Theorem 1. If a subsequent power series exists then for any arbitrary complex α

$$\left(\sum_{l=0}^{m} b_l z^l \right)^\alpha = \sum_{k=0}^{\infty} a_k(\alpha, m) z^k,$$

where

$$a_k(\alpha, m) = \sum_{l_1=0}^{[\frac{(m-1)k}{m}]} \sum_{l_2=(2l_1-k)_+}^{[\frac{(m-2)l_1}{m-1}]} \cdots \sum_{l_{m-1}=(2l_{m-2}-l_{m-3})_+}^{[\frac{l_{m-2}}{2}]} \frac{(\alpha)_{k-l_1}}{(k-l_1)!} \cdot$$

$$\cdot \prod_{i=0}^{m-2} \binom{l_i - l_{i+1}}{l_{i+1} - l_{i+2}} b_0^{\alpha+l_1-k} \prod_{i=0}^{m-3} b_{i+1}^{l_i+l_{i+2}-2l_{i+1}} b_{m-1}^{l_{m-2}-2l_{m-1}} b_m^{l_{m-1}}, \tag{5}$$

$l_0 = k$, $l_m = 0$, $a_+ = max_+\{0, a\}$, $[x]$ being an integer part of x, or

$$a_k(\alpha, m) = \sum_{2l_1+\cdots+ml_{m-1}\le k} \frac{(\alpha)_{k-l_1-\cdots-(m-1)l_{m-1}}}{(k-2l_1-\cdots-ml_{m-1})!l_1!l_2!\cdots l_{m-1}!}\cdot$$

$$\cdot b_0^{\alpha-k+l_1+\cdots+(m-1)l_{m-1}} b_1^{k-2l_1-\cdots-ml_{m-1}} b_2^{l_1}\cdots b_m^{l_{m-1}}, \tag{6}$$

or

$$a_k(\alpha, m) = \sum_{l_1+\cdots+ml_m=k} \frac{(\alpha)_{l_1+\cdots+l_m}}{l_1!\cdots l_m!} b_0^{\alpha-l_1-\cdots-l_m} b_1^{l_1}\cdots b_m^{l_m}, \tag{7}$$

where $l_1, l_2, \cdots, l_m \ge 0$ are integers.

P r o o f. To prove the theorem we must show that coefficients $a_k(\alpha, m)$ defined by expression (5) satisfy recurrency (2) of Lemma 1. In other words we should show that

$$\sum_{l_1=0}^{[\frac{(m-1)k}{m}]} \sum_{l_2=0}^{[\frac{(m-2)l_1}{m-1}]} \cdots \sum_{l_{m-1}=0}^{[\frac{l_{m-2}}{2}]} \frac{(\alpha)_{k-l_1}}{(k-l_1)!}\binom{k-l_1}{l_1-l_2}\binom{l_1-l_2}{l_2-l_3}\cdots\binom{l_{m-2}-l_{m-1}}{l_{m-1}}\cdot$$

$$\cdot b_0^{\alpha+l_1-k} b_1^{k+l_2-2l_1} b_2^{l_1+l_3-2l_2}\cdots b_{m-2}^{l_{m-3}+l_{m-1}-2l_{m-2}} b_{m-1}^{l_{m-2}-2l_{m-1}} b_m^{l_{m-1}} =$$

$$= \sum_{s=0}^{[\frac{k}{m}]} \frac{(\alpha)_s}{s!} \sum_{t_1=0}^{[\frac{(m-2)(k-ms)}{m-1}]} \sum_{t_2=0}^{[\frac{(m-2)t_1}{m-1}]} \cdots \sum_{t_{m-2}=0}^{[\frac{t_{m-3}}{2}]} \frac{(\alpha-s)_{k-ms-t_1}}{(k-ms-t_1)!}\cdot$$

$$\cdot\binom{k-ms-t_1}{t_1-t_2}\binom{t_1-t_2}{t_2-t_3}\cdots\binom{t_{m-2}-t_{m-2}}{t_{m-2}}\cdot$$

$$\cdot b_0^{\alpha-s+t_1-k+ms} b_1^{k-ms+t_2-2t_1} b_2^{t_1+t_3-2t_2}\cdots b_{m-2}^{t_{m-3}-2t_{m-2}} b_{m-1}^{t_{m-2}} b_m^{s}. \tag{8}$$

Changing variables $s, t_1, t_2, \ldots, t_{m-2}$ on the rigth hand side of (8) by $s = l_{m-1}$, $t_1 = l_1 - (m-1)l_{m-1}$, $t_2 = l_2 - (m-2)l_{m-1}, \ldots, t_{m-2} = l_{m-2} - 2l_{m-1}$ we see that

$$\frac{(\alpha)_s}{s!}\frac{(\alpha-s)_{k-ms-t_1}}{(k-ms-t_1)!}\binom{k-ms-t_1}{t_1-t_2}\binom{t_1-t_2}{t_2-t_3}\cdots\binom{t_{m-3}-t_{m-2}}{t_{m-2}} =$$

$$= \frac{(\alpha)_{k-l_1}}{l_{m-1}!(k-l_1-l_{m-1})!}\binom{k-l_1-l_{m-1}}{l_1-l_2-l_{m-1}}\cdots\binom{l_{m-3}-l_{m-2}-l_{m-1}}{l_{m-2}-2l_{m-1}} =$$

$$= \frac{(\alpha)_{k-l_1}}{l_{m-1}!(k-2l_1+l_2)!\cdots(l_{m-3}-2l_{m-2}+l_{m-1})!(l_{m-2}-2l_{m-1})!} =$$

$$= \frac{(\alpha)_{k-l_1}}{(k-l_1)!}\cdot\binom{k-l_1}{l_1-l_2}\cdot\binom{l_1-l_2}{l_2-l_3}\cdots\binom{l_{m-2}-l_{m-1}}{l_{m-1}} \tag{9}$$

This is evidently the coefficient of the summand on the left hand side of (8). On the other hand

$$b_0^{\alpha-s+t_1-k+ms} b_1^{k-ms+t_2-2t_1} b_2^{t_1+t_3-2t_2} \cdots b_{m-2}^{t_{m-3}-2t_{m-2}} b_{m-1}^{t_{m-2}} b_m^s =$$

$$= b_0^{\alpha-k+l_1} b_1^{k+l_2-2l_1} b_2^{l_1+l_3-2l_2} \cdots b_{m-2}^{l_{m-3}+l_{m-1}-2l_{m-2}} b_{m-1}^{l_{m-2}-2l_{m-1}} b_m^{l_{m-1}}$$

and limits of summation on the right hand side of (8) become the same as on the left hand side of (8), i. e. (8) is the identity.

Owing to (9) we may write

$$a_k(\alpha, m) = \sum_{t_1} \sum_{t_2} \cdots \sum_{t_{m-1}} \frac{(\alpha)_{k-t_1}}{(k-2t_1+t_2)! \cdots (t_{m-2}-2t_{m-1})! t_{m-1}!} \cdot$$

$$\cdot b_0^{\alpha+t_1-k} b_1^{k+t_2-2t_1} \cdots b_{m-1}^{t_{m-2}-2t_{m-1}} b_m^{t_{m-1}}.$$

Changing in this expression t_1,t_2,...,t_{m-1} by

$$t_{m-1} = l_{m-1}, t_{m-2} = l_{m-2} + 2l_{m-1}, \ldots, t_1 = l_1 + 2l_2 + 3l_3 + \cdots + (m-1)l_{m-1}$$

we obtain formula (6).

Changing in (6) $(k - 2l_1 - 3l_2 - \cdots - ml_{m-1})$, $l_1, \ldots, l_{m-1}$ by $t_1, t_2, \ldots, t_m$ we obtain (7). This completes the proof of Theorem 1. □

Corollary. Putting $m = k$ in (5) we see that the upper limits of summation become

$$\left[\frac{(m-1)k}{m}\right] = \left[k - \frac{k}{m}\right] = [k-1] = k-1,$$

$$\left[\frac{(m-2)l_1}{m-1}\right] = \left[l_1 - \frac{l_1}{k-1}\right] = (l_1 - 1)_+ \le k-2,$$

$$\vdots$$

$$\left[\frac{l_{m-2}}{2}\right] = \left[l_{k-2} - \frac{l_{k-2}}{2}\right] = (l_{k-2} - 1)_+ \le 1.$$

From these it follows that coefficients $a_k(\alpha, m)$ do not contain terms depending on b_{k+1}.

In other words, considering the problem

$$\left(\sum_{l=0}^{\infty} b_l z^l\right)^{\alpha} = \sum_{k=0}^{\infty} a_k(\alpha) z^k$$

we may write $a_k(\alpha)$ as follows

$$a_k(\alpha) = \sum_{l_1=0}^{k-1} \sum_{l_2=(2l_1-k)_+}^{(l_1-1)_+} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{(l_{k-2}-1)_+} \frac{(\alpha)_{k-l_1}}{(k-l_1)!} \cdot$$

$$\cdot\binom{k-l_1}{l_1-l_2}\cdots\binom{l_{k-2}-l_{k-1}}{l_{k-1}}b_0^{\alpha-k+l_1}b_1^{k+l_2-2l_1}\cdots b_{k-1}^{l_{k-2}-2l_{k-1}}b_k^{l_{k-1}} \tag{10}$$

or

$$a_k(\alpha)=\sum_{2l_1+\cdots+kl_{k-1}\le k}\frac{(\alpha)_{k-l_1-\cdots-(k-1)l_{k-1}}}{(k-2l_1-\cdots-kl_{k-1})!l_1!l_2!\ldots l_{k-1}!}b_0^{\alpha-k+l_1+\cdots+(k-1)l_{k-1}}\cdot$$

$$\cdot b_1^{k-2l_1-\cdots-kl_{k-1}}b_2^{l_1}\cdots b_k^{l_{k-1}} \tag{11}$$

or

$$a_k(\alpha)=\sum_{l_1+\cdots+kl_k=k}\frac{(\alpha)_{l_1+\cdots+l_k}}{l_1!\cdots l_k!}b_0^{\alpha-l_1-\cdots-l_k}b_1^{l_1}\cdots b_k^{l_k}. \tag{12}$$

3. THE BELL POLYNOMIALS AND THE BEL NUMBERS

The complete Bell polynomials are by definition

$$Y_n(fg_1,fg_2,\ldots,fg_n)=\sum\frac{n!f_r}{r_1!r_2!\ldots r_n!}\left(\frac{g_1}{1!}\right)^{r_1}\cdots\left(\frac{g_n}{n!}\right)^{r_n}, \tag{13}$$

where $f^r=f_r$ and the summation is over all non negative integers satisfying the following conditions

$$r_1+2r_2+\cdots+nr_n=n,\qquad r_1+r_2+\cdots+r_n=r.$$

Consider the problem

$$\left(\sum_{k=0}^{\infty}b_kz^k\right)^{\alpha}=\sum_{k=0}^{\infty}a_k(\alpha)z^k,$$

where $b_0=1$. In accordance with (12) we obtain

$$a_k(\alpha)=\sum_{l_1+\cdots+kl_k=k}\frac{(\alpha)_{l_1+\cdots+l_k}}{l_1!\cdots l_k!}b_1^{l_1}\cdots b_k^{l_k}. \tag{14}$$

Denoting $g_m=m!b_m$, $m=1,2,\ldots,k$ and $f^j=f_j=(\alpha)_j$ we obtain from (13) and (14)

$$a_n(\alpha)=\frac{1}{n!}Y_n(fg_1,fg_2,\ldots,fg_n).$$

Bearing in mind (10) and (11) we obtain two alternative expressions for the Bell polynomials

$$Y_n(fg_1,\ldots,fg_n)=\sum_{k_1=0}^{n-1}\sum_{k_2=(2k_1-n)_+}^{(k_1-1)_+}\cdots\sum_{k_{n-1}=(2k_{n-2}-k_{n-3})_+}^{(k_{n-2}-1)_+}\cdot$$

$$\cdot\frac{n!f_{n-k_1}}{(n-2k_1+k_2)!(k_1-2k_2+k_3)!\cdots(k_{n-2}-2k_{n-1})!k_{n-1}!}\cdot$$

$$\cdot\left(\frac{g_1}{1!}\right)^{n-2k_1+k_2}\cdots\left(\frac{g_n}{n!}\right)^{k_{n-1}} \tag{15}$$

and

$$Y_n(fg_1,\dots,fg_n)=\sum_{2l_1+\dots nl_{n-1}\le n}\frac{n!f_{n-l_1-\dots-(n-1)l_{n-1}}}{(n-2l_1-\dots-nl_{n-1})!l_1!l_2!\cdots l_{n-1}!}\cdot$$

$$\cdot\left(\frac{g_1}{1!}\right)^{n-2l_1-\dots-nl_{n-1}}\left(\frac{g_2}{2!}\right)^{l_1}\cdots\left(\frac{g_n}{n!}\right)^{l_{n-1}}. \tag{16}$$

The Bell numbers B_n are obained from the Bell polynomials (13) if

$$fg_1=fg_2=\dots=fg_n=1,\quad \text{i.e. if}\quad f^j=f_j=1,\ g_j=1,\ j=1,2,\dots,n.$$

From (15) and (16) it follows that

$$B_n=\sum_{k_1=0}^{n-1}\sum_{k_2=(2k_1-n)_+}^{(k_1-1)_+}\cdots\sum_{k_{n-1}=(2k_{n-2}-k_{n-3})_+}^{(k_{n-2}-1)_+}\cdot$$

$$\cdot\frac{n!}{(n-2k_1+k_2)!(k_1-2k_2+k_3)!\cdots(k_{n-2}-2k_{n-1})!k_{n-1}!}\cdot$$

$$\cdot\left(\frac{1}{2!}\right)^{k_1-2k_2+k_3}\cdots\left(\frac{1}{(n-1)!}\right)^{k_{n-2}-2k_{n-1}}\left(\frac{1}{n!}\right)^{k_{n-1}} \tag{17}$$

or

$$B_n=\sum_{2l_1+\dots nl_{n-1}\le n}\frac{n!}{(n-2l_1-\dots-nl_{n-1})!l_1!l_2!\cdots l_{n-1}!}\cdot$$

$$\cdot\left(\frac{1}{2!}\right)^{l_1}\left(\frac{1}{3!}\right)^{l_2}\cdots\left(\frac{1}{n!}\right)^{l_{n-1}}. \tag{18}$$

The polynomials $T_{n;k}=T_{n;k}(fg_1,\dots,fg_m)$, $m=\min\{n,k\}$, defined by

$$T_{n;k}(fg_1,\dots,fg_m)=\begin{cases} Y_n(fg_1,\dots,fg_n), & n\le k,\\ Y_{n;k}(fg_1,\dots,fg_k), & n>k,\end{cases}$$

where

$$Y_{n;k}(fg_1,\dots,fg_k)=\sum\frac{n!f_r}{r_1!\cdots r_k!}\left(\frac{g_1}{1!}\right)^{r_1}\cdots\left(\frac{g_k}{k!}\right)^{r_k}$$

and the summation is extended over all $r_i\ge 0$ $(i=1,2,\dots,k)$ such that

$$r_1+2r_2+\dots+kr_k=n;\qquad r_1+r_2+\dots+r_k=r,$$

are called right truncated Bell polynomials (Charalambides [4]). In accordance with (5) and (6) we may write two following alternative expressions for the right truncated Bell polynomials $Y_{n;k}(fg_1, \ldots, fg_k)$:

$$Y_{n;k}(fg_1, \ldots, fg_k) = \sum_{l_1=0}^{[\frac{(k-1)n}{k}]} \sum_{l_2=(2l_1-n)_+}^{[\frac{(k-2)l_1}{k-1}]} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{[\frac{l_{k-2}}{2}]} \cdot$$

$$\cdot \frac{n! f_{n-l_1}}{(n-2l_1+l_2)!(l_1-2l_2+l_3)! \cdots (l_{k-2}-2l_{k-1})! l_{k-1}!} \cdot$$

$$\cdot \left(\frac{g_1}{1!}\right)^{n-2l_1+l_2} \cdots \left(\frac{g_k}{k!}\right)^{l_{k-1}}, \qquad n > k, \tag{19}$$

and

$$Y_{n;k}(fg_1, \ldots, fg_k) = \sum_{2l_1+\cdots+kl_{k-1} \le n} \frac{n! f_{n-l_1-\cdots-(k-1)l_{k-1}}}{(n-2l_1-\cdots-kl_{k-1})! l_1! l_2! \cdots l_{k-1}!} \cdot$$

$$\cdot \left(\frac{g_1}{1!}\right)^{n-2l_1-\cdots-kl_{k-1}} \left(\frac{g_2}{2!}\right)^{l_1} \cdots \left(\frac{g_k}{k!}\right)^{l_{k-1}}, \qquad n > k. \tag{20}$$

Charalambides [4] introduced the partial right truncated Bell polynomials by the following generating function

$$\sum_{n=0}^{\infty} T_{n,r;k}(g_1, g_2, \ldots, g_m) \frac{t^n}{n!} = \frac{[g_k(t)]^r}{r!}, \tag{21}$$

where $g_k(t) = \sum_{j=1}^{k} g_j \frac{t^j}{j!}$.

Using (2) and (21) we obtain the following recurrence relation for $T_{n,r;k}$ if $k \le n$

$$T_{n,r;k}(g_1, g_2, \ldots, g_k) = \sum_{s=0}^{[\frac{n}{k}]} \frac{n!}{(n-ks)! s! (k!)^s} T_{n-ks, r-s; k-1}(g_1, g_2, \ldots, g_{k-1}) g_k^s. \tag{22}$$

4. HARDY–RAMANUJAN–RADEMACHER PROBLEM

The partition means a set of nonnegative integers with the given sum $n \in \mathbf{N}$. In other words the partition is a representation of $n \in \mathbf{N}$ such that

$$n = x_1 + 2x_2 + \cdots + kx_k, \qquad k \in \mathbf{N},\ k \le n, \tag{23}$$

where $x_1, x_2, \ldots, x_k$ are nonnegative integers. We shall denote each such representation by $\{x_1, x_2, \ldots, x_k\}_n$. The problem is to find all these representations and to calculate their number $R_n^{(k)}$. It is well known (see e. g. Riordan [17]) that

$$\varphi(t,k) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^k)} = \sum_{n=0}^{\infty} R_n^{(k)} t^n, \tag{24}$$

i. e. $\varphi(t,k)$ is the generating function of $R_n^{(k)}$. Since finding the $R_n^{(k)}$ for moderate and large n is not simple, much work has been done to find ways of evaluating the $R_n^{(k)}$ (Euler, Cayley [2], Mac Mahon [12], Gupta H. [7] and others). All these ways are recurrent in their nature.

We see from (24) that

$$\varphi(t,k-1) = (1-t^k)\varphi(t,k) \tag{25}$$

which implies the recurrency

$$R_n^{(k)} - R_{n-k}^{(k)} = R_n^{(k-1)} \tag{26}$$

with the initial condition $R_n^{(1)} = 1$, $n > 0$.
Since

$$\frac{1}{1-t^k} = \sum_{n=0}^{\infty} t^{kn},$$

we have

$$\varphi(t,k) = \sum_{n=0}^{\infty} R_n^{(k)} t^n = \sum_{n=0}^{\infty} t^{kn} \sum_{n=0}^{\infty} R_n^{(k-1)} t^n = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\left[\frac{n}{k}\right]} R_{n-lk}^{(k-1)} \right) t^n$$

or

$$R_n^{(k)} = \sum_{l=0}^{\left[\frac{n}{k}\right]} R_{n-lk}^{(k-1)}. \tag{27}$$

The exact formula for the number of partitions of an nonnegative integer has been established by Hardy, Ramanujan and Rademacher (see e. g. Andrews [1]). But their formula gives only the number of partitions for any integer n, $n \in \mathbf{N}$. Theorem 1 exactly gives not only this number but all partitions themselves.

In this paper we give an expression for the direct evaluation of $R_n^{(k)}$ and a way to obtain all partitions $\{x_1, x_2, \dots, x_k\}_n$ for any $n \in \mathbf{N}$.

In Theorem 1 we have shown that the common term $a_n(\alpha,k)$ in

$$\left(\sum_{n=0}^{k} b_n z^n \right)^{\alpha} = \sum_{n=0}^{\infty} a_n(\alpha,k) z^n, \tag{28}$$

can be calculated by the expression

$$a_n(\alpha,k) = \sum_{l_1=0}^{\left[\frac{(k-1)n}{k}\right]} \sum_{l_2=(2l_1-n)_+}^{\left[\frac{(k-2)l_1}{k-1}\right]} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{\left[\frac{l_{k-2}}{2}\right]} \cdot$$

$$\cdot \frac{(\alpha)_{n-l_1}}{l_{k-1}!(n-2l_1+l_2)!(l_1-2l_2+l_3)!\cdots(l_{k-3}-2l_{k-2}+l_{k-1})!(l_{k-2}-2l_{k-1})!} \cdot$$

$$\cdot b_0^{\alpha+l_1-n} b_1^{n+l_2-2l_1} b_2^{l_1+l_3-2l_2} \cdots b_{k-1}^{l_{k-2}-2l_{k-1}} b_k^{l_{k-1}}, \tag{29}$$

where $l_1, l_2, \ldots, l_{k-1}$ are nonnegative integers, $l_0 = n$, $(a)_+ = \max\{a, 0\}$, $[x]$ is the integer part of x.

It is easily seen that the summation by $l_1, l_2, \ldots, l_{k-1}$ in (29) is such that

$$(n - 2l_1 + l_2) + 2(l_1 - 2l_2 + l_3) + \cdots$$

$$\cdots (k-2)(l_{k-3} - 2l_{k-2} + l_{k-1}) + (k-1)(l_{k-2} - 2l_{k-1}) + kl_{k-1} = n. \tag{30}$$

For $k \geq n$ (23) reduces to

$$x_1 + 2x_2 + \cdots + nx_n = n \tag{31}$$

and (29) becomes

$$a_n(\alpha) = \sum_{l_1=0}^{n-1} \sum_{l_2=(2l_1-n)_+}^{(l_1-1)_+} \cdots \sum_{l_{n-1}=(2l_{n-2}-l_{n-3})_+}^{(l_{n-2}-1)_+} \cdot$$

$$\cdot \frac{(\alpha)_{n-l_1}}{l_{n-1}!(n-2l_1+l_2)!(l_1-2l_2+l_3)!\cdots(l_{n-2}-2l_{n-1})!} \cdot$$

$$\cdot b_0^{\alpha+l_1-n} b_1^{n+l_2-2l_1} b_2^{l_1+l_3-2l_2} \cdots b_{n-1}^{l_{n-2}-2l_{n-1}} b_n^{l_{n-1}}, \tag{32}$$

where $l_1, l_2, \ldots, l_{n-1}$ are nonnegative integers, $l_0 = n$. The summation in (32) by $l_1, l_2, \ldots, l_{n-1}$ is such that

$$(n - 2l_1 + l_2) + 2(l_1 - 2l_2 + l_3) + \cdots + (n-1)(l_{n-2} - 2l_{n-1}) + nl_{n-1} = n. \tag{33}$$

Denoting $\{(n-2l_1+l_2), (l_1-2l_2+l_3), \ldots, (l_{k-2}-2l_{k-1}), l_{k-1}\}_n = \{x_1, x_2, \ldots, x_{k-1}, x_k\}_n$ and $\{(n-2l_1+l_2), (l_1-2l_2+l_3), \ldots, (l_{n-2}-2l_{n-1}), l_{n-1}\}_n = \{x_1, x_2, \ldots, x_{n-1}, x_n\}_n$ we thus have proved the following

Theorem 2. The numbers $R_n^{(k)}$ of solutions of equation

$$x_1 + 2x_2 + \cdots + kx_k = n, \qquad k \leq n,\ n \in \mathbf{N},$$

and the number R_n of solutions of equation

$$x_1 + 2x_2 + \cdots + nx_n = n, \qquad n \in \mathbf{N},$$

where $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n$ are nonnegative integers may be represented as

$$R_n^{(k)} = \sum_{l_1=0}^{[\frac{(k-1)n}{k}]} \sum_{l_2=(2l_1-k)_+}^{[\frac{(k-2)l_1}{k-1}]} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{[\frac{l_{k-2}}{2}]} 1, \qquad k \leq n, \tag{34}$$

and

$$R_n = \sum_{l_1=0}^{n-1} \sum_{l_2=(2l_1-n)_+}^{(l_1-1)_+} \cdots \sum_{l_{n-1}=(2l_{n-2}-l_{n-3})_+}^{(l_{n-2}-1)_+} 1. \tag{35}$$

Partitions $\{x_1, x_2, \ldots, x_k\}_n$, $k \le n$, are obtained from all sets $\{l_1, l_2, \ldots, l_{k-1}\}$, $k \le n$, in (34) or (35) by formula

$$\{(n - 2l_1 + l_2), (l_1 - 2l_2 + l_3), \ldots, (l_{k-2} - 2l_{k-1}), l_{k-1}\}_n, \qquad k \le n. \tag{36}$$

Formulas (34) and (35) interpret the well known expressions

$$R_n^{(k)} = \sum_{l_1+2l_2+\cdots+kl_k=n} 1 \tag{37}$$

and

$$R_n = \sum_{l_1+2l_2+\cdots+nl_n=n} 1, \tag{38}$$

which indicate only that summation in (37) and (38) is performed over all partitions like $n = l_1 + 2l_2 + \cdots + kl_k$, $k \le n$, but the problem of finding these partitions still remains.

5. APPLICATIONS

Formulas (10), (11) and (12) may be applied in mathematical statistics when estimating parameters of discrete probability distributions. Let, for example, $\mathbf{X} = (X_1, \ldots, X_n)$ be a sample where X_i, $i = 1, 2, \ldots, n$, are random variables with probability density function (for notations see Voinov and Nikulin [21])

$$f(x;\theta) = \begin{cases} \frac{a(x)\psi^x(\theta)}{B(\theta)}, & x \in \mathcal{X}, \\ 0 & \text{otherwise}, \end{cases} \tag{39}$$

where $\mathcal{X} = \{0, 1, 2, \ldots\}$, $a(x) > 0$ for any $x \in \mathcal{X}$, $\psi(\theta)$ and $B(\theta)$ are bounded, positive, differentiable functions on $\mathcal{X}$, Θ, $\Theta = \{\theta : 0 \le \theta < \rho\}$, ρ is a radius of convergence of a functional series $\sum_{x=0}^{\infty} a(x)\psi^x(\theta)$. The complete sufficient for θ statistic $T = \sum_{i=1}^{\infty} X_i$ is distributed as

$$g(t;\theta) = \begin{cases} \frac{b(t,n)\psi^t(\theta)}{B^n(\theta)}, & t = 0, 1, \ldots \\ 0, & \text{otherwise}, \end{cases} \tag{40}$$

where

$$B^n(\theta) = \left(\sum_{x=0}^{\infty} a(x)\psi^x(\theta) \right)^n = \sum_{t=0}^{\infty} b(t,n)\psi^t(\theta).$$

All minimum variance unbiased estimators of parameters of (39) depend on coefficients b(t,n). One may evidently use for their calculation formulas (10), (11) and (12).

Let $g(s)$ be an analytic function of s in the vicinity of $s=0$ such that $0<g(0)<1$, $g(1)=1$, i.e. $g(s)$ is the probability generating function.

If

$$\left(\frac{\partial}{\partial s}\right)^{x-1} (g(s))^x \,|_{s=0} \geq 0, \qquad x=1,2,\dots$$

then using Lagrange's expansion one may define a probability distribution as follows (Consul and Shenton [5, 6], Johnson and Kotz [10])

$$\mathsf{P}\{X=x\} = \frac{1}{x!}\left(\frac{\partial}{\partial s}\right)^{x-1} (g(s))^x \Bigg|_{s=0}, \qquad x=1,2,\dots \tag{41}$$

Since $g(s)$ is an analytic function in the vicinity of $s=0$, we may write

$$g(s) = \sum_{m=0}^{\infty} b_m s^m, \qquad b_0 > 0.$$

Then

$$(g(s))^x = \left(\sum_{m=0}^{\infty} b_m s^m\right)^x = \sum_{k=0}^{\infty} a_k(x) s^k. \tag{42}$$

Inserting (42) into (41) gives

$$\mathsf{P}\{X=x\} = \frac{a_{x-1}(x)}{x}, \qquad x=1,2,\dots \tag{43}$$

To calculate $a_{x-1}(x)$ defined by (42) one may use formulas (10), (11) and (12).

Formulas (10) – (12) together with Lagrange's expansion may also be used for quantiles evaluation as it was shown by Voinov, Neymann and Nikulin [20].

Let $y=g(z)$ be an analytic function on an open set G and for each $a \in G$ $b=g(a)$. Then Lagrange's theorem gives

$$z = a + \sum_{n=1}^{\infty} \frac{1}{n!}\left\{\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left[\left(\frac{z-a}{g(z)-b}\right)^n\right]\right\}_{z=a} (y-b)^n. \tag{44}$$

If $y=g(z)=\sum_{k=0}^{\infty} a_k(z-a)^k$ then

$$\left\{\frac{z-a}{[g(z)-b]}\right\}^n = \left(\sum_{k=1}^{\infty} a_k (z-a)^{k-1}\right)^{-n} = \sum_{k=0}^{\infty} b_k(-n)(z-a)^k.$$

The series (44) reduces to

$$z = a + \sum_{n=1}^{\infty} \frac{b_{n-1}(-n)}{n}(y-b)^n. \tag{45}$$

Let $F(x) = \sum_{k=0}^{\infty} p_k(x - x_0)^k$ be a distribution function and P be a given confidence level, $0 < P < 1$. Then solution x_p of equation $F(x) = P$ in accordance with (45) is

$$x_p = x_0 + \sum_{n=1}^{\infty} \frac{q_{n-1}(-n)}{n}(P - p_0)^n, \qquad p_0 = F(x_0), \tag{46}$$

where the coefficient $q_k(-n)$ is defined from relation

$$\left(\sum_{k=0}^{\infty} p_k(x - x_0)^{k-1}\right)^{-n} = \left(\sum_{k=0}^{\infty} p_{k+1}(x - x_0)^k\right)^{-n} = \sum_{k=0}^{\infty} q_k(-n)(x - x_0)^k. \tag{47}$$

Using formulas (10) – (12) with $\alpha = -n$ one may calculate quantiles x_p by formula (46).

Let a probability generating function $h(t) = f(g(t))$ be specified, where $f(x) = x^{\pm n}$ and a generaliser $g(t)$ is an analytic function. Then, as pointed out in the introduction, the problem of deriving the probabilities and moments of the corresponding generalized distribution is a problem of expanding $h(t)$ into a power series. When solving this problem one may evidently use formulas (5) – (7) or (10) – (12) of Section 2 of this paper. We would like to emphasize here that one may make an other generalization considering $f(x) = x^\alpha$, where α is an arbitrary complex. It is clear that probability generating functions like $f(x) = x^\alpha$ include $\alpha = \pm n$ as a particular case.

Another interesting application of this results is connected with the famous Faa di Bruno's formula for the nth derivative of a composite function $h(x) = f(g(x))$, where f and g are differentiable n times scalar functions of scalar variables, reads

$$h^{(n)}(x) = \sum \frac{n! f^{(k)}}{k_1! \cdots k_n!} \left(\frac{g'}{1!}\right)^{k_1} \cdots \left(\frac{g^{(n)}}{n!}\right)^{k_n}, \tag{48}$$

where $k_1 + 2k_2 + \cdots + nk_n = n$ and $k = k_1 + \cdots + k_n, k_1, k_2, \ldots, k_n$ are nonnegative integers. If $g^{(r)} \equiv g_r$ and $f^{(r)} \equiv f_r$, then (48) coincides with the expression for the Bell polynomial $Y_n(fg_1, fg_2, \ldots, fg_n)$, which can be represented by (15) as was shown in Section 3.

From (15) and (48) we obtain

$$h^{(n)}(x) = \sum_{k_1=0}^{n-1} \sum_{k_2=(2k_1-n)_+}^{(k_1-1)_+} \cdots \sum_{k_{n-1}=(2k_{n-2}-k_{n-3})_+}^{(k_{n-2}-1)_+} \cdot \frac{n! f^{(n-k_1)}}{(n - 2k_1 + k_2)!(k_1 - 2k_2 + k_3)! \cdots (k_{n-2} - 2k_{n-1})! k_{n-1}!} \cdot \left(\frac{g'}{1!}\right)^{n-2k_1+k_2} \cdots \left(\frac{g^{(n)}}{n!}\right)^{k_{n-1}}. \tag{49}$$

This formula makes it possible to evaluate $h^{(n)}(x)$ directly for any n. Formula (49) evidently solves the problem of the substitution a power series into a power series. Thus, formulas (7) and (10) easily permit work with power series. One may evaluate arbitrary powers of a power series, substitute a power series into a power series, derive the inverse function for any analytic function (see formula (45)).

Consider now the problem of directly evaluating the Bernoulli and Euler numbers.

Bernoulli numbers are defined by generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad | t |< 2\pi.$$

Since

$$\frac{e^t - 1}{t} = \frac{1}{t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1\right) = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!},$$

we have

$$\left[\sum_{k=0}^{\infty} \frac{t^k}{(k+1)!}\right]^{-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{k=0}^{\infty} a_k(-1)t^k. \tag{50}$$

Using formula (10) we obtain from (50)

$$B_k = \sum_{l_1=0}^{k-1} \sum_{l_2=(2l_1-k)_+}^{(l_1-1)_+} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{(l_{k-2}-1)_+} \cdot$$

$$\cdot \frac{(-1)^{k-l_1} k!(k-l_1)!}{(k-2l_1+l_2)!(l_1-2l_2+l_3)!\cdots(l_{k-2}-2l_{k-1})!l_{k-1}!} \cdot$$

$$\left(\frac{1}{2!}\right)^{k-2l_1+l_2} \left(\frac{1}{3!}\right)^{l_1-2l_2+l_3} \cdots \left(\frac{1}{(k+1)!}\right)^{l_{k-1}}. \tag{51}$$

Consider Euler numbers defined by the generating function

$$\frac{1}{\mathrm{Ch}t} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}, \qquad | t |< \pi/2.$$

Since

$$\mathrm{Ch}t = \sum_{m=0}^{\infty} \frac{t^{2k}}{(2k)!}, \tag{52}$$

we have

$$\left(\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right)^{-1} = \sum_{k=0}^{\infty} E_{2k} \frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} a_{2k}(-1)^k t^{2k}. \tag{53}$$

Using (10) and (52) we obtain from (53)

$$E_{2k} = \sum_{l_1=0}^{k-1} \sum_{l_2=(2l_1-k)_+}^{(l_1-1)_+} \cdots \sum_{l_{k-1}=(2l_{k-2}-l_{k-3})_+}^{(l_{k-2}-1)_+} \cdot$$

$$\cdot \frac{(-1)^{k-l_1}(2k)!(k-l_1)!}{(k-2l_1+l_2)!(l_1-2l_2+l_3)!\cdots(l_{k-2}-2l_{k-1})!l_{k-1}!} \cdot$$

$$\left(\frac{1}{2!}\right)^{k-2l_1+l_2} \cdots \left(\frac{1}{(2k)!}\right)^{l_{k-1}}. \tag{54}$$

Consider in conclusion the problem of evaluating the cycle index $C_n(t_1, t_2, \ldots, t_n)$ of a symmetric group. It is known (see e. g. Riordan [17]) that

$$C_n(t_1, t_2, \ldots, t_n) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{t_1}{1}\right)^{k_1} \left(\frac{t_2}{2}\right)^{k_2} \cdots \left(\frac{t_n}{n}\right)^{k_n}, \tag{55}$$

where $k_1, k_2, \ldots, k_n$ are nonnegative integers.

This problem is nothing new. Nevertheless, we consider it with a view to indicating another sphere of applications of partitions. Using (15) we may rewrite (55) as follows

$$C_n(t_1, \ldots, t_n) = \sum_{l_1=0}^{n-1} \sum_{l_2=(2l_1-n)_+}^{(l_1-1)_+} \cdots \sum_{l_{n-1}=(2l_{n-2}-l_{n-3})_+}^{(l_{n-2}-1)_+} \cdot$$

$$\cdot \frac{n!}{(n-2l_1+l_2)!(l_1-2l_2+l_3)!\cdots(l_{n-2}-2l_{n-1})!l_{n-1}!} \cdot$$

$$\left(\frac{t_1}{1}\right)^{n-2l_1+l_2} \left(\frac{t_2}{2}\right)^{l_1-2l_2+l_3} \cdots \left(\frac{t_n}{n}\right)^{l_{n-1}}. \tag{56}$$

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