BALANCED REDUCTION
OF LINEAR PERIODIC SYSTEMS[1]

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For linear periodic discrete-time systems the analysis of the model error introduced by a truncation on the balanced minimal realization is performed, and a bound for the infinity norm of the model error is introduced. The results represent an extension to the periodic systems of the well known results on the balanced truncation for time-invariant systems. The general case of periodically time-varying state-space dimension has been considered.

1. INTRODUCTION

The approximation of high-order plants and controllers models by models of lower-order is a central aspect of control system design. There are many model reduction methods based on model analysis or frequency domain concepts (see, e. g., [9, 16]). Recently, methods based on the truncation of the balanced realization have been analyzed in the framework of linear continuous and discrete time-invariant systems [1, 10, 18, 20] and of linear continuous time-varying systems [22, 23].

On the other hand, a wide interest has been devoted to the analysis and design of periodic discrete-time systems (see e. g. [2, 12] and references therein). For this class of systems the model reduction has been analyzed using an Hankel-norm approximation and under the hypothesis of time-invariant dimension of the state-space and of time reversibility [25]. The aim of this paper is to provide a method for the model reduction of a periodic system without the assumption of time reversibility and with periodically time-varying dimensions of the state space. In fact, for the class of discrete-time periodic systems the minimal (reachable and observable) realization is generally described by periodic difference equations whose matrices have periodically time-varying dimensions [6, 11]. Therefore, the notion of balanced minimal realization has to be necessarily introduced in the context of minimal periodic systems with time-varying dimensions, where the time reversibility is not guaranteed.

The paper is organized as follows. In Section 2 some preliminaries about periodic systems will be recalled. The existence of the balanced minimal realization of an asymptotically stable periodic system with time-varying dimensions will be provided

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in Section 3. In the same section the analysis of the model error introduced by a truncation on the balanced realization will be performed. In particular, a bound for the infinity norm of the model error is found. Numerical examples and some concluding remarks end the paper.

2. PRELIMINARIES

Consider a linear periodic discrete-time system S described by the following equations:

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)\,u(k) & (1) \\
y(k) &= C(k)x(k) + D(k)\,u(k) & (2)
\end{aligned}
$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^{n(k)}$ is the state, with $n(k+\omega) = n(k)$, $u(k) \in \mathbb{R}^p$ is the input, $y(k) \in \mathbb{R}^q$ is the output and $A(\cdot), B(\cdot),\ C(\cdot), D(\cdot)$ are periodic matrices of period ω (briefly, ω-periodic).

Denote by $\Phi(k, k_0)$ the transition matrix:

$$
\begin{aligned}
\Phi(k, k_0) &:= A(k-1)A(k-2)\cdots A(k_0),\ \ k > k_0,\ k, k_0 \in \mathbb{Z}, & (3) \\
\Phi(k, k) &:= I_{n(k)}, \quad k \in \mathbb{Z}, & (4)
\end{aligned}
$$

where $I_{n(k)}$ denotes the identity matrix of dimension $n(k)$. Given a time instant k and an m-dimensional discrete-time signal $v(\cdot) \in \mathbb{R}^m$, denote with $v_k^L(h) \in \mathbb{R}^{m\omega}$ the lifted discrete time signal associated to $v(\cdot)$ and defined by:

$$
v_k^L(h) := \begin{bmatrix} v'(h\omega + k) & v'(h\omega + k + 1) & \cdots & v'(h\omega + k + \omega - 1) \end{bmatrix}', \quad h \in \mathbb{Z}^+. \quad (5)
$$

In the following, two time-invariant representations of the ω-periodic system S are recalled and some related results are analyzed.

For an arbitrary integer k, let $u_k^L(h)$ and $y_k^L(h)$ the lifted signals associated to $u(\cdot)$ and $y(\cdot)$, respectively, $\xi_k(h) := x(h\omega + k)$ and consider the time-invariant system $S^L(k)$ described by:

$$
\begin{aligned}
\xi_k(h+1) &= E_k \xi_k(h) + J_k u_k^L(h) & (6) \\
y_k^L(h) &= L_k \xi_k(h) + M_k u_k^L(h) & (7)
\end{aligned}
$$

where

$$
\begin{align}
E_k \quad &:= \quad \Phi(\omega + k, k), \tag{8}\\
J_k \quad &:= \quad \begin{bmatrix} \Phi(\omega + k, k + 1)B(k) & \Phi(\omega + k, k + 2)B(k+1) & \cdots & B(k + \omega - 1) \end{bmatrix}, \tag{9}
\end{align}
$$

$$
L_k \quad := \quad \begin{bmatrix} C(k) \\ C(k+1)\Phi(k+1, k) \\ \vdots \\ C(k + \omega - 1)\Phi(k + \omega - 1, k) \end{bmatrix}, \tag{10}
$$

$$
M_k \quad := \quad \begin{bmatrix} D(k) & 0 & 0 & \cdots & 0 \\ H_{21}(k) & D(k+1) & 0 & \cdots & 0 \\ H_{31}(k) & H_{32}(k) & D(k+2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{\omega,1}(k) & H_{\omega,2}(k) & H_{\omega,\omega-1}(k) & \cdots & D(k + \omega - 1) \end{bmatrix} \tag{11}
$$

with

$$
H_{ij}(k) := C(k + i - 1)\,\Phi(k + i - 1, k + j)\,B(k + j - 1), \tag{12}
$$
$$
i = 2, 3, \ldots, \omega, \; j = 1, 2, \ldots, \omega - 1, \; i > j.
$$

System $S^L(k)$ will be referred to as the lifted representation at time k of S (briefly, lifted system). Obviously, $S^L(k + \omega) = S^L(k)$ for all integer k and system $S^L(k)$ is equivalent to the ω-periodic system S [17]. The time instant k can be considered as the initial time of ω-rate sampling for the state of system S.

The transfer matrix $W_k(z) := L_k(zI_{n(k)} - E_k)^{-1}J_k + M_k$ associated to the lifted representation of S depends on the initial sampling time k as is stated in the following result.

Lemma 2.1. [13] For all the integer k, the transfer matrix $W_k(z)$ satisfies the following relation:

$$
W_{k+1}(z) = \begin{bmatrix} 0 & I_{q(\omega-1)} \\ zI_q & 0 \end{bmatrix} W_k(z) \begin{bmatrix} 0 & z^{-1}I_p \\ I_{p(\omega-1)} & 0 \end{bmatrix}. \tag{13}
$$

The infinity norm of the associated transfer matrix $W_k(z)$, i.e. $||W_k(z)||_\infty := \sup_\theta \bar{\sigma}(W_k(e^{j\theta}))$, is independent of the initial sampling instant as stated by the following lemma, whose proof is reported in the Appendix.

Lemma 2.2. The infinity norm of the transfer matrix $W_k(z)$ satisfies the following relation:
$$
||W_{k+1}(z)||_\infty = ||W_k(z)||_\infty, \quad \forall\, k \in \mathbb{Z}. \tag{14}
$$

The notion of lifted system at time k allows to analyze structural and stability properties and pole-zero structures of periodic systems [2, 4, 13, 14]. For example,

the subspace of reachable (unobservable) states of system S at time k is readily seen to coincide with that of system $S^L(k)$ if it is expressed in terms of matrices E_k, J_k, L_k and M_k. Therefore, system S is reachable (observable) at time k if and only if system $S^L(k)$ is reachable (observable).

Moreover, it is well known that the characteristic polynomial of E_k (the monodromy matrix of $A(\cdot)$) is independent of k, whence it characterizes the stability of S [7]. Also the solutions of ω-periodic Lyapunov equations can be found making use of the lifted representation of the ω-periodic system S. Computation algorithms and related applications of these equations can be also found in [24].

Lemma 2.3. [25], [3] For an arbitrary integer k, let M_k and N_k solutions of the following Lyapunov equations:

$$E_k M_k E_k' + J_k J_k' = M_k, \tag{15}$$
$$E_k' N_k E_k + L_k' L_k = N_k. \tag{16}$$

Let $M(\cdot)$ and $N(\cdot)$ ω-periodic matrices satisfying the following ω-periodic Lyapunov equations:

$$A(k)\,M(k)\,A(k)' + B(k)\,B(k)' = M(k+1) \quad \forall\, k \in \mathbb{Z} \tag{17}$$
$$A(k)'N(k+1)\,A(k) + C(k)'C(k) = N(k) \quad \forall\, k \in \mathbb{Z}. \tag{18}$$

Then, $M(k) = M_k$ and $N(k) = N_k$, for all the integer k.

A similar set of results can be also stated by a different time-invariant representation which is now recalled. Given a time instant k and an m-dimensional discrete-time signal $v(\cdot) \in \mathbb{R}^m$, denote with $v_k^C(i) \in \mathbb{R}^{m\omega}$ the cyclic discrete time signal associated to $v(\cdot)$ and defined by:

$$v_k^C(i) := \begin{bmatrix} v_0^{k,C}(i)' & v_1^{k,C}(i)' & \cdots & v_{\omega-1}^{k,C}(i)' \end{bmatrix}', \quad i \geq k, \tag{19}$$

with:

$$v_j^{k,C}(i) := \begin{cases} v(i) & i = k+j+h\omega \\ * & i \neq k+j+h\omega \end{cases}, \quad h \in \mathbb{Z}^+, j = 0, \ldots, \omega-1, \tag{20}$$

where $*$ means that $v_j^{k,C}(i)$ can be freely assigned for $i \neq k+j+h\omega$ [8].

For an arbitrary integer k, let $u_k^C(i)$, $y_k^C(i)$ and $x_k^C(i)$ the cyclic signals associated to $u(\cdot)$, $y(\cdot)$ and $x(\cdot)$, respectively, and consider the time-invariant system $S^C(k)$ defined by

$$x_k^C(i+1) = \mathcal{A}_k x_k^C(i) + \mathcal{B}_k u_k^C(i) \tag{21}$$
$$y_k^C(i) = \mathcal{C}_k x_k^C(i) + \mathcal{D}_k u_k^C(i) \tag{22}$$

where

$$\mathcal{A}_k = \begin{bmatrix} 0 & \cdots & 0 & A(\omega-1+k) \\ A(k) & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A(\omega-2+k) & 0 \end{bmatrix},$$

$$\mathcal{B}_k = \begin{bmatrix} 0 & \cdots & 0 & B(\omega - 1 + k) \\ B(k) & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & B(\omega - 2 + k) & 0 \end{bmatrix},$$

$$\mathcal{C}_k = \mathrm{diag}\{C(k), \quad C(k+1), \quad \ldots, \quad C(\omega - 1 + k)\},$$

$$\mathcal{D}_k = \mathrm{diag}\{D(k), \quad D(k+1), \quad \ldots, \quad D(\omega - 1 + k)\}.$$

System $S^C(k)$ will be referred to as the cyclic representation at time k of S (briefly, cyclic system) [8], [19].

With the cyclic representation $S^C(k)$ the following transfer matrix can be associated:

$$\mathcal{W}_k(z) = \mathcal{C}_k(zI_\nu - \mathcal{A}_k)^{-1}\mathcal{B}_k + \mathcal{D}_k, \tag{23}$$

where $\nu := \sum_{i=0}^{\omega-1} n(i)$.

The relationships between systems $S^L(k)$ and $S^C(k)$ in terms of their transfer matrices $W_k(z)$ and $\mathcal{W}_k(z)$ are precisely stated by the following lemma.

Lemma 2.4. [5] For any integer k, the following relation is satisfied:

$$\mathcal{W}_k(z) = \mathrm{diag}\{I_q, z^{-1}I_q, \cdots, z^{-\omega+1}I_q\}W_k(z^\omega)\,\mathrm{diag}\{I_p, zI_p, \cdots, z^{\omega-1}I_p\}. \tag{24}$$

The infinity norms of the transfer matrices $W_k(z)$ and $\mathcal{W}_k(z)$ coincides as stated in the following lemma whose proof is reported in the Appendix.

Lemma 2.5. For an arbitrary integer k, the infinity norms of the transfer matrices $W_k(z)$ and $\mathcal{W}_k(z)$ satisfy the following relation:

$$||W_k(z)||_\infty = ||\mathcal{W}_k(z)||_\infty. \tag{25}$$

Also the notion of cyclic system allows to analyze structural and stability properties of periodic systems. System S is reachable (observable) at time k if and only if system $S^C(k)$ is reachable (observable) [5, 15, 19]. System S is asymptotically stable if and only if for an arbitrary integer k system $S^C(k)$ is asymptotically stable [19].

3. BALANCED REALIZATION AND MODEL REDUCTION

An ω-periodic coordinate transformation on the state space is described by:

$$\hat{x}(k) = T(k)\,x(k), \tag{26}$$

where $T(k) \in \mathbb{R}^{n(k) \times n(k)}$ is an ω-periodic non singular matrix. In the new base the realization $(A(\cdot),\ B(\cdot),\ C(\cdot),\ D(\cdot))$ of the ω-periodic system S has the following form:

$$\hat{A}(k) \quad = \quad T(k+1)\,A(k)\,T(k)^{-1}, \quad \forall\,k \in \mathbb{Z} \tag{27}$$

$$\hat{B}(k) \quad = \quad T(k+1)\,B(k-), \qquad \forall\,k \in \mathbb{Z} \tag{28}$$

$$\hat{C}(k) \quad = \quad C(k)\,T(k)^{-1}, \qquad\quad \forall\,k \in \mathbb{Z}. \tag{29}$$

Assume system S to be *reachable and observable at all times* (the ω-periodic realization $(A(\cdot),\ B(\cdot),\ C(\cdot),\ D(\cdot))$ is minimal) and *asymptotically stable* (the eigenvalues of the monodromy matrix E_k lie inside the open unit disk). Under these assumptions it is possible to show the following result whose proof is given in the Appendix.

Lemma 3.1. Assume that system S is reachable and observable at all times and asymptotically stable. Then, for all the integer k, there exists an ω-periodic coordinate transformation on the state-space described by (26) such that the following relations hold:

$$\hat{A}(k)\,\Sigma(k)\hat{A}(k)' + \hat{B}(k)\hat{B}(k)' = \Sigma(k+1), \tag{30}$$

$$\hat{A}(k)'\Sigma(k+1)\hat{A}(k) + \hat{C}(k)'\hat{C}(k) = \Sigma(k), \tag{31}$$

where

$$\Sigma(k) = \mathrm{diag}\{\sigma_1(k), \sigma_2(k), \cdots, \sigma_{n(k)}(k)\}, \tag{32}$$

with $\sigma_i(k+\omega) = \sigma_i(k) > 0$, $i = 1 \ldots n(k)$.

Analogously to the time-invariant case an ω-periodic asymptotically stable minimal realization $(A(\cdot),\ B(\cdot),\ C(\cdot),\ D(\cdot))$ which satisfies relations (30), (31) with $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$ substituted with $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ is called *balanced realization*.

Lemma 3.2. For any integer k, the lifted representation $S^L(k)$ and the cyclic representation $S^C(k)$ of the ω-periodic S with balanced realization are characterized by time-invariant balanced realizations.

Also the proof of this lemma is given in the Appendix.

Now assume that the ω-periodic asymptotically stable minimal realization of S is in the *balanced form* and consider the following compatible partition with ω-periodic time-varying dimensions:

$$A(k) \quad = \quad \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix}, \quad B(k) \quad = \quad \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix}, \tag{33}$$

$$C(k) \quad = \quad \begin{bmatrix} C_1(k) & C_2(k) \end{bmatrix}, \tag{34}$$

$$\Sigma(k) \quad = \quad \begin{bmatrix} \Sigma_1(k) & 0 \\ 0 & \Sigma_2(k) \end{bmatrix}, \qquad x(k) \quad = \quad \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{35}$$

where the dimensions of all the blocks are suitably chosen, in particular
$A_{11}(k) \in \mathbb{R}^{n_1(k+1) \times n_1(k)}$, $A_{12}(k) \in \mathbb{R}^{n_1(k+1) \times n_2(k)}$, $A_{21}(k) \in \mathbb{R}^{n_2(k+1) \times n_1(k)}$,
$A_{22}(k) \in \mathbb{R}^{n_2(k+1) \times n_2(k)}$, $B_1(k) \in \mathbb{R}^{n_1(k+1) \times p}$, $B_2(k) \in \mathbb{R}^{n_2(k+1) \times p}$, $C_1(k) \in \mathbb{R}^{q \times n_1(k)}$,
$C_2(k) \in \mathbb{R}^{q \times n_2(k)}$, $\Sigma_1(k) = \text{diag}\{\sigma_1(k), \sigma_2(k), \ldots, \sigma_{n_1(k)}(k)\} \in \mathbb{R}^{n_1(k) \times n_1(k)}$,
$\Sigma_2(k) = \text{diag}\{\sigma_{n_1(k)+1}(k), \sigma_{n_1(k)+2}(k), \ldots, \sigma_{n(k)}(k)\} \in \mathbb{R}^{n_2(k) \times n_2(k)}$, $x_1(k) \in \mathbb{R}^{n_1(k)}$,
$x_2(k) \in \mathbb{R}^{n_2(k)}$, and $n_1(k) \le n(k)$, $n_1(k+\omega) = n_1(k)$ and $n_2(k) := n(k) - n_1(k)$ for
all $k \in \mathbb{Z}$.

If the truncation operation is applied to the ω-periodic minimal balanced realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of the system S, the following reduced order ω-periodic model S^r is obtained:

$$x_1(k+1) = A_{11}(k)x_1(k) + B_1(k)\,u(k) \tag{36}$$

$$y(k) = C_1(k)x_1(k) + D(k)\,u(k) \tag{37}$$

whose time-invariant cyclic representation at time k has $(\mathcal{A}_k^{11}, \mathcal{B}_k^1, \mathcal{C}_k^1, \mathcal{D}_k)$ as realization, where $\mathcal{A}_k^{11} \in \mathbb{R}^{\nu_1 \times \nu_1}$, $\nu_1 := \sum_{i=0}^{\omega-1} n_1(i)$, and $\mathcal{B}_k^1 \in \mathbb{R}^{\nu_1 \times \omega p}$ have the same cyclic structure of $\mathcal{A}_k$ and $\mathcal{B}_k$ with the ω-periodic matrix $A(\cdot)$ and $B(\cdot)$ substituted by $A_{11}(\cdot)$ and $B_1(\cdot)$, respectively, and $\mathcal{C}_k^1 \in \mathbb{R}^{\omega p \times \nu_1}$ has the same diagonal structure of $\mathcal{C}_k$ with the ω-periodic matrix $C(\cdot)$ substituted by $C_1(\cdot)$. Denote by $\mathcal{W}_k^r(z)$ the transfer matrix of the cyclic representation at time k of the reduced order ω-periodic model S^r:

$$\mathcal{W}_k^r(z) := \mathcal{C}_k^1(zI_{\nu_1} - \mathcal{A}_k^{11})^{-1}\mathcal{B}_k^1 + \mathcal{D}_k. \tag{38}$$

Theorem 3.1. Assume that system S is reachable and observable at all times and asymptotically stable. Let $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ an ω-periodic balanced realization of system S. Let $(A_{11}(\cdot), B_1(\cdot), C_1(\cdot), D(\cdot))$ the ω-periodic realization of the reduced order ω-periodic model S^r, whose state-space dimension is $n_1(k) \le n(k)$, with $n_1(k+\omega) = n_1(k)$. Let $\mathcal{W}_k^r(z)$ the transfer matrix of the cyclic representation at time k of the reduced model S^r. Then, the ω-periodic model S^r is asymptotically stable and for an arbitrary integer k:

$$||\mathcal{W}_k(z) - \mathcal{W}_k^r(z)||_\infty \le 2 \sum_{i=0}^{\omega-1} \sum_{\ell=n_1(i)+1}^{n(i)} \sigma_\ell(i), \tag{39}$$

where $\sigma_\ell(i) > 0$ are the entries of the diagonal matrix $\Sigma_2(i)$, for $i = 0, 1, \ldots, \omega - 1$.

P r o o f. Consider the following orthogonal matrix:

$$\mathcal{U}_k := \begin{bmatrix} \mathcal{U}_k^a & \mathcal{U}_k^b \end{bmatrix} \in \mathbb{R}^{\nu \times \nu}, \tag{40}$$

where

$$\mathcal{U}_k^a := \operatorname{diag}\left\{\begin{bmatrix} I_{n_1(k)} \\ 0_{n_2(k),n_1(k)} \end{bmatrix}, \begin{bmatrix} I_{n_1(k+1)} \\ 0_{n_2(k+1),n_1(k+1)} \end{bmatrix}, \cdots, \begin{bmatrix} I_{n_1(k+\omega-1)} \\ 0_{n_2(k+\omega-1),n_1(k+\omega-1)} \end{bmatrix}\right\} \\ \in \mathbb{R}^{\nu\times\nu_1},$$

$$\mathcal{U}_k^b := \operatorname{diag}\left\{\begin{bmatrix} 0_{n_1(k),n_2(k)} \\ I_{n_2(k)} \end{bmatrix}, \begin{bmatrix} 0_{n_1(k+1),n_2(k+1)} \\ I_{n_2(k+1)} \end{bmatrix}, \cdots, \begin{bmatrix} 0_{n_1(k+\omega-1),n_2(k+\omega-1)} \\ I_{n_2(k+\omega-1)} \end{bmatrix}\right\} \\ \in \mathbb{R}^{\nu\times\nu_2},$$

$0_{n,m}$ is the zero matrix of dimension $n \times m$, $\nu_1 = \sum_{i=0}^{\omega-1} n_1(i)$, $\nu_2 = \sum_{i=0}^{\omega-1} n_2(i)$ and $\nu_1 + \nu_2 = \nu$. Moreover, define $\Sigma_k := \operatorname{diag}\{\Sigma(k), \Sigma(k+1), \cdots, \Sigma(k+\omega-1)\}$. Taking into account the structure of the orthogonal matrix $\mathcal{U}_k$ and the partition of the balanced realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ and of $\Sigma(k)$ described by (33)–(35), the following relations can be derived:

$$\bar{\mathcal{A}}_k := \mathcal{U}_k' \mathcal{A}_k \mathcal{U}_k = \begin{bmatrix} \mathcal{A}_k^{11} & \mathcal{A}_k^{12} \\ \mathcal{A}_k^{21} & \mathcal{A}_k^{22} \end{bmatrix} \tag{41}$$

$$\bar{\mathcal{B}}_k := \mathcal{U}_k' \mathcal{B}_k = \begin{bmatrix} \mathcal{B}_k^1 \\ \mathcal{B}_k^2 \end{bmatrix}, \quad \bar{\mathcal{C}}_k := \mathcal{C}_k \mathcal{U}_k = \begin{bmatrix} \mathcal{C}_k^1 & \mathcal{C}_k^2 \end{bmatrix} \tag{42}$$

$$\bar{\Sigma}_k := \mathcal{U}_k' \Sigma_k \mathcal{U}_k = \begin{bmatrix} \Sigma_k^1 & 0 \\ 0 & \Sigma_k^2 \end{bmatrix}, \tag{43}$$

where $\mathcal{A}_k^{11} \in \mathbb{R}^{\nu_1\times\nu_1}$, $\mathcal{B}_k^1 \in \mathbb{R}^{\nu_1\times\omega p}$ and $\mathcal{C}_k^1 \in \mathbb{R}^{\omega p\times\nu_1}$ have been above introduced and $\mathcal{A}_k^{12} \in \mathbb{R}^{\nu_1\times\nu_2}$, $\mathcal{A}_k^{21} \in \mathbb{R}^{\nu_2\times\nu_1}$ and $\mathcal{A}_k^{22} \in \mathbb{R}^{\nu_2\times\nu_2}$ have the same cyclic structure of $\mathcal{A}_k$ with the ω-periodic matrix $A(\cdot)$ substituted by $A_{12}(\cdot)$, $A_{21}(\cdot)$ and $A_{22}(\cdot)$, respectively; $\mathcal{B}_k^2 \in \mathbb{R}^{\nu_2\times\omega p}$ has the same cyclic structure of $\mathcal{B}_k$ with the ω-periodic matrix $B(\cdot)$ substituted by $B_2(\cdot)$; $\mathcal{C}_k^2 \in \mathbb{R}^{\omega p\times\nu_2}$ has the same diagonal structure of $\mathcal{C}_k$ with the ω-periodic matrix $C(\cdot)$ substituted by $C_2(\cdot)$; $\Sigma_k^1 \in \mathbb{R}^{\nu_1\times\nu_1}$ and $\Sigma_k^2 \in \mathbb{R}^{\nu_2\times\nu_2}$ have the same diagonal structure of Σ_k with the ω-periodic matrix $\Sigma(\cdot)$ substituted by $\Sigma_1(\cdot)$ and $\Sigma_2(\cdot)$, respectively.

The orthogonal matrix $\mathcal{U}_k$ describes a state-space transformation of the time-invariant cyclic representation $S^C(k)$. Under the assumption of system S in a balanced realization, by Lemma 3.2, the cyclic representation $S^C(k)$ is balanced. Then, it is easy to see that also in the new base the time-invariant cyclic representation $S^C(k)$ with realization $(\bar{\mathcal{A}}_k, \bar{\mathcal{B}}_k, \bar{\mathcal{C}}_k, \mathcal{B}_k)$ is balanced.

Now, by well known results on the balanced realization (Theorem 4.2 in [20]) and on the balanced truncation of discrete-time time-invariant system (Theorem 2 in [1]) applied to the time-invariant cyclic representation $S^C(k)$ with realization $(\bar{\mathcal{A}}_k, \bar{\mathcal{B}}_k, \bar{\mathcal{C}}_k, \mathcal{D}_k)$ the asymptotic stability of the cyclic representation of S^r (equivalent to the asymptotic stability of S^r [19]) and the relation (39) are derived. The double summation in the right hand side of (39) is a consequence of the structure of $\Sigma_k^2 = \operatorname{diag}\{\Sigma_2(k), \Sigma_2(k+1), \ldots, \Sigma_2(k+\omega-1)\}$. $\qquad\square$

This result introduces an upper bound on the approximation error measured by $||\mathcal{W}_k(z) - \mathcal{W}_k^r(z)||_\infty$. Note that, the approximation error can be also expressed

by $||W_k(z) - W_k^r(z)||_\infty$ where $W_k(z)$ is the transfer matrix of the lifted system at time k and $W_k^r(z) = L_k^1(zI_{n_1(k)} - E_k^{11})^{-1}J_k^1 + M_k$ where E_k^{11}, J_k^1 and L_k^1 are defined as E_k, J_k and L_k with matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ substituted by $A_{11}(\cdot)$, $B_1(\cdot)$ and $C_1(\cdot)$, respectively. In fact, a relation similar to (24) holds for $\mathcal{W}_k^r(z)$ and $W_k^r(z^\omega)$ and arguing as in the proof of Lemma 2.5 it is possible to show that $||W_k(z) - W_k^r(z)||_\infty = ||\mathcal{W}_k(z) - \mathcal{W}_k^r(z)||_\infty$.

Note that, if $\omega = 1$ (the time-invariant case) the upper bound of the model error stated in the Theorem 3.1 reduces to the well-known time-invariant upper bound [1].

A model reduction of a periodic system with the desired accuracy can be obtained by the following algorithm.

Algorithm 3.1.

Step 0. An ω-periodic system S of the form (1), (2) is given. Verify that S is reachable and observable at all times and asymptotically stable. Set a threshold positive value γ.

Step 1. By Lemma 2.3, compute the ω-periodic solution $M(k)$ and $N(k)$ of the ω-periodic Lyapunov equations (17) and (18) respectively. Then, compute a Cholesky factorization of $M(k) = R(k)R(k)'$ and a singular value decomposition of $R(k)'Q(k)R(k) = U(k)\Sigma(k)^2U(k)'$ where $\Sigma(k)=\text{diag}\{\sigma_1(k), \sigma_2(k), \cdots \cdots, \sigma_{n(k)}(k)\}$ with $\sigma_i(k) \geq \sigma_{i+1}(k)$, for all $k \in \mathbb{Z}$.

Step 2. For $k = 0, 1, \ldots, \omega$, find the positive integer $n_1(k)$ such that $\sigma_{n_1(k)}(k) > \gamma$.

Step 3. Compute the upper bound of the truncation error $\epsilon_\gamma := 2\sum_{i=0}^{\omega-1}\sum_{\ell=n_1(i)+1}^{n(i)} \sigma_\ell(i)$. If the upper bound ϵ_γ is acceptable go to Step 4, else set a lower threshold positive value γ and go to Step 2.

Step 4. Apply to system S an ω-periodic state-space transformation described by (26) with $T(k) = \Sigma(k)^{\frac{1}{2}}U(k)'R(k)^{-1}$ where $\Sigma(k)^{\frac{1}{2}} = \text{diag}\{\sqrt{\sigma_1(k)}, \sqrt{\sigma_2(k)}, \cdots, \sqrt{\sigma_{n(k)}(k)}\}$. In the new base deduce the block partition of the balanced realization of S as described by (33), (34) with $n_1(k)$ specified at the Step 2. The corresponding matrices $A_{11}(\cdot)$, $B_1(\cdot)$ and $C_1(\cdot)$ are a realization of the reduced order model S^r described by (36), (37) and the upper bound of the infinity norm of the model error is lower or equal to ϵ_γ.

4. NUMERICAL EXAMPLES

Example 4.1. Consider the 2-periodic asymptotically stable, reachable and observable system S described by the following matrices:

$$A(0) = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C(0) = 1,$$

$$A(1) = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \quad B(1) = 1, \quad C(1) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

where the state-space dimension is $n(0) = 1$, $n(1) = 2$. The lifted representation $S^L(k)$ has the following transfer matrix:

$$W_0(z) = \begin{bmatrix} 0 & \frac{1}{z-0.25} \\ 1 & 0 \end{bmatrix}, \quad W_1(z) = \begin{bmatrix} 0 & \frac{1}{z} \\ \frac{z}{z-0.25} & 0 \end{bmatrix}.$$

It can be verified by means of Lemma 3.1 that the system S is in balanced form, with:

$$P(0) = Q(0) = \Sigma(0) = \frac{16}{15}, \qquad P(1) = Q(1) = \Sigma(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{4}{15} \end{bmatrix},$$

and the Step 1 of the Algorithm 3.1 can be omitted. Choosing a positive threshold value $\gamma = 0.3$, the positive integer $n_1(k)$ at the Step 2 is equal to one for each integer k, and the upper bound $\epsilon_\gamma = 2 \sum_{i=0}^{1} \sum_{\ell=n_1(i)+1}^{n(i)} \sigma_\ell(i) = \frac{8}{15}$.

The state-space transformation at the Step 4 is not necessary and a realization of the reduced order model has the following form:

$$A_{11}(0) = A_{11}(1) = 0, \quad B_1(0) = B_1(1) = 1, \quad C_1(0) = C_1(1) = 1,$$

with the following transfer matrix :

$$W_0^r(z) = W_1^r(z) = \begin{bmatrix} 0 & \frac{1}{z} \\ 1 & 0 \end{bmatrix}.$$

Note that the infinity norm of the model errors is $||W_0(z) - W_0^r(z)||_\infty = ||W_1(z) - W_1^r(z)||_\infty = \frac{1}{3}$ and it verifies Theorem 3.1, where the upper bound given by (39) is $\frac{8}{15}$.

Example 4.2. Consider the 2-periodic asymptotically stable, reachable and observable system S described by the following matrices:

$$A(0) = \begin{bmatrix} 0.6 & 0.19 & -0.7 & 0.54 \\ 0.72 & 0.91 & -1.17 & 0.68 \\ 0.29 & 0.56 & -0.56 & 0.28 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0.17 \\ 0.46 \\ 0.28 \end{bmatrix},$$

$$C(0) = \begin{bmatrix} 1.13 & -1.67 \ 10^{-1} & -1.07 & 1.01 \end{bmatrix},$$

$$A(1) = \begin{bmatrix} -4.28 & 0.46 & 5.28 \\ -3.16 & -2 & 6.71 \\ -5.66 & -3.53 & 12 \\ -5.22 & -3.38 & 11.2 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -0.19 \\ -0.07 \\ -0.96 \\ -0.24 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} -2.8 \ 10^{-2} & 7.73 \ 10^{-3} & 3.12 \ 10^{-2} \end{bmatrix},$$

where the state-space dimension is $n(0) = 4$, $n(1) = 3$. At the Step 1 the 2-periodic diagonal matrix $\Sigma(k)$ has the following form:

$$\Sigma(0) = \text{diag}\{1.38, 0.75, 1.61 \ 10^{-3}, 4.16 \ 10^{-9}\},$$

$$\Sigma(1) = \text{diag}\{1.31, 7.54 \ 10^{-1}, 6.9 \ 10^{-5}\}.$$

With a positive threshold value $\gamma = 10^{-4}$, at the Step 2 the dimension $n_1(k)$ of the reduced model is $n_1(0) = 3$, $n_1(1) = 2$, and the upper bound ϵ_γ is

$$\epsilon_\gamma = 2 \sum_{i=0}^{1} \sum_{\ell=n_1(i)+1}^{n(i)} \sigma_\ell(i) = 2(\sigma_4(0) + \sigma_3(1)) = 1.38 \ 10^{-4}.$$

Applying the state-space transformation at the Step 4, a balanced realization of system S is obtained and the reduced model S^r of dimension $n_1(k)$ described by (36), (37) has the following form:

$$A_{11}(0) = \begin{bmatrix} -1.03 \ 10^{-4} & -7.58 \ 10^{-1} & -9.07 \ 10^{-6} \\ -7.40 \ 10^{-1} & -1.62 \ 10^{-4} & 3.86 \ 10^{-2} \end{bmatrix}$$

$$B_1(0) = \begin{bmatrix} 9.38 \ 10^{-1} \\ -2.10 \ 10^{-4} \end{bmatrix},$$

$$C_1(0) = \begin{bmatrix} 9.82 \ 10^{-1} & -1.96 \ 10^{-4} & 2.2 \ 10^{-2} \end{bmatrix},$$

$$A_{11}(1) = \begin{bmatrix} 9.77 \ 10^{-1} & 2.96 \ 10^{-5} \\ 5.27 \ 10^{-5} & -1 \\ 1.05 \ 10^{-2} & 7.49 \ 10^{-6} \end{bmatrix}, \ B_1(1) = \begin{bmatrix} -3.52 \ 10^{-1} \\ 1.29 \ 10^{-4} \\ 3.83 \ 10^{-2} \end{bmatrix},$$

$$C_1(1) = \begin{bmatrix} 8.08 \ 10^{-3} & -8.13 \ 10^{-6} \end{bmatrix},$$

and the infinity norm of the model error is $||W_k(z) - W_k^r(z)||_\infty = ||\mathcal{W}_k(z) - \mathcal{W}_k^r(z)||_\infty = 6.9 \ 10^{-5}$ and it satisfies Theorem 3.1, where the upper bound ϵ_γ is equal to $1.38 \ 10^{-4}$.

With a different choice for the positive threshold value $\gamma = 2 \ 10^{-3}$, at the Step 2 the dimension $n_1(k)$ of the reduced model is $n_1(0) = n_1(1) = 2$, and the upper bound ϵ_γ has the following value:

$$\begin{aligned} \epsilon_\gamma &= 2 \sum_{i=0}^{1} \sum_{\ell=n_1(i)+1}^{n(i)} \sigma_\ell(i) = 2(\sigma_3(0) + \sigma_4(0) + \sigma_3(1)) \\ &= 3.36 \ 10^{-3}. \end{aligned}$$

In this second case the reduced model S^r described by (36), (37) has a constant state-space dimension $n_1(k) = 2$ and

$$A_{11}(0) = \begin{bmatrix} -1.03 \ 10^{-4} & -7.58 \ 10^{-1} \\ -7.40 \ 10^{-1} & -1.62 \ 10^{-4} \end{bmatrix}$$

$$B_1(0) = \begin{bmatrix} 9.38 \ 10^{-1} \\ -2.10 \ 10^{-4} \end{bmatrix}$$

$$C_1(0) = \begin{bmatrix} 9.82 \ 10^{-1} & -1.96 \ 10^{-4} \end{bmatrix},$$

$$A_{11}(1) = \begin{bmatrix} 9.77 \ 10^{-1} & 2.96 \ 10^{-5} \\ 5.27 \ 10^{-5} & -1.0000 \end{bmatrix}, \ B_1(1) = \begin{bmatrix} -3.52 \ 10^{-1} \\ 1.29 \ 10^{-4} \end{bmatrix},$$

$$C_1(1) = \begin{bmatrix} 8.08 \ 10^{-3} & -8.13 \ 10^{-6} \end{bmatrix}.$$

The infinity norm of the model error is $||W_k(z)-W_k^r(z)||_\infty = ||\mathcal{W}_k(z)-\mathcal{W}_k^r(z)||_\infty =$ 1.61 10^{-3} and it satisfies Theorem 3.1, where the upper bound ϵ_γ is equal to 3.36 10^{-3}.

5. CONCLUDING REMARKS

An analysis of model error introduced by a simple truncation of a periodic system in a balanced realization is performed. The general case of periodically time-varying dimension of the state-space is considered. The results represent an extension to the periodic systems of the well known results on the balanced truncation for time-invariant systems. When the period ω is equal to one (the time-invariant case), the introduced bound for the infinity norm of the model error reduces to the one of time-invariant systems.

For time-invariant systems with no poles on the imaginary axis the balanced reduction has been solved also for unstable systems [21]. The possible extension of this result also to the class of discrete-time periodic systems is under investigation.

APPENDIX A

P r o o f o f t h e L e m m a 2.2. The infinity norm of $\begin{bmatrix} 0 & I_{q(\omega-1)} \\ zI_q & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & z^{-1}I_p \\ I_{p(\omega-1)} & 0 \end{bmatrix}$ are equal to one. Then, by Lemma 2.1 and the submultiplicative property of the infinity norm the following relation holds:

$$||W_{k+1}(z)||_\infty \le ||W_k(z)||_\infty \quad \forall \, k \in \mathbb{Z}. \tag{44}$$

Moreover, by Lemma 2.1 it easy to verify the following relation:

$$W_k(z) = \begin{bmatrix} 0 & z^{-1}I_q \\ I_{q(\omega-1)} & 0 \end{bmatrix} W_{k+1}(z) \begin{bmatrix} 0 & I_{p(\omega-1)} \\ zI_p & 0 \end{bmatrix}, \quad \forall \, k \in \mathbb{Z}.$$

This relation together with the submultiplicative property of the infinity norm implies:

$$||W_k(z)||_\infty \le ||W_{k+1}(z)||_\infty \quad \forall \, k \in \mathbb{Z}. \tag{45}$$

Relations (44) and (45) prove the lemma.   □

P r o o f o f t h e L e m m a 2.5. The infinity norm of $\mathrm{diag}\{I_q, z^{-1}I_q, \cdots, z^{-\omega+1}I_q\}$ and of $\mathrm{diag}\{I_p, zI_p, \cdots, z^{\omega-1}I_p\}$ are equal to one. Then, by Lemma 2.4 and the submultiplicative property of the infinity norm the following relation holds:

$$||\mathcal{W}_{k+1}(z)||_\infty \le ||W_k(z^\omega)||_\infty \quad \forall \, k \in \mathbb{Z}. \tag{46}$$

Moreover, by Lemma 2.4 it easy to verify the following relation:

$$W_k(z^\omega) = \mathrm{diag}\{I_q, zI_q, \cdots, z^{\omega-1}I_q\}\mathcal{W}_k(z)\,\mathrm{diag}\{I_p, z^{-1}I_p, \cdots, z^{-\omega+1}I_p\}, \ \forall\, k \in \mathbb{Z}.$$

This relation together with the submultiplicative property of the infinity norm implies:

$$||W_k(z^\omega)||_\infty \leq ||\mathcal{W}_k(z)||_\infty \quad \forall\, k \in \mathbb{Z}. \tag{47}$$

Being $||W_k(z^\omega)||_\infty = ||W_k(z)||_\infty$, relations (46) and (47) prove the lemma. $\square$

Proof of the Lemma 3.1. The stability assumption on the minimal realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ implies for any arbitrary integer k the asymptotic stability of the reachable and observable time-invariant lifted system $S^L(k)$. This fact implies that the reachability and observability gramians of $S^L(k)$, expressed by $P_k = \sum_{i=0}^\infty (E_k)^i J_k J_k'(E_k')^i$ and $Q_k = \sum_{i=0}^\infty (E_k')^i L_k' L_k(E_k)^i$, satisfy the following Lyapunov equations [20]:

$$E_k P_k E_k' + J_k J_k' = P_k \tag{48}$$
$$E_k' Q_k E_k + L_k' L_k = Q_k. \tag{49}$$

By Lemma 2.3 the above relations imply the existence of ω-periodic matrices $P(k) = P_k$ and $Q(k) = Q_k$ which satisfies the following ω-periodic Lyapunov equations:

$$A(k)P(k)A(k)' + B(k)\,B(k)' = P(k+1), \quad \forall\, k \in \mathbb{Z}, \tag{50}$$
$$A(k)'Q(k+1)A(k) + C(k)'B(k) = Q(k), \quad \forall\, k \in \mathbb{Z}. \tag{51}$$

Consider a Cholesky factorization of $P(k) = R(k)R(k)'$ and a singular value decomposition of $R(k)'Q(k)R(k) = U(k)\Sigma(k)^2 U(k)'$ where $\Sigma(k) = \mathrm{diag}\{\sigma_1(k), \sigma_2(k), \cdots, \sigma_{n(k)}(k)\}$. Introduce an ω-periodic state-space transformation described by (26) with $T(k) = \Sigma(k)^{\frac{1}{2}}U(k)'R(k)^{-1}$ where $\Sigma(k)^{\frac{1}{2}} = \mathrm{diag}\{\sqrt{\sigma_1(k)}, \sqrt{\sigma_2(k)}, \cdots, \sqrt{\sigma_{n(k)}(k)}\}$. By relations (27) – (29), the equations (50) and (51) imply:

$$\hat{A}(k)\,T(k)P(k)\,T(k)'\hat{A}(k)' + \hat{B}(k)\hat{B}(k)' \tag{52}$$
$$= T(k+1)P(k+1)\,T(k+1)',$$

$$\hat{A}(k)'(T(k+1)')^{-1}Q(k+1)\,T(k+1)^{-1}\hat{A}(k) + \hat{C}(k)'\hat{C}(k) \tag{53}$$
$$= (T(k)')^{-1}Q(k)\,T(k)^{-1},$$

that are equivalent to the relations (30) and (31). $\square$

Proof of the Lemma 3.2. For any integer k, the asymptotic stability and minimality of S implies the asymptotic stability and minimality of the lifted representation $S^L(k)$ and of the cyclic representation $S^C(k)$. Moreover, relation (30) implies that:

$$\hat{A}(k+1)\hat{A}(k)\Sigma(k)\hat{A}(k)'\hat{A}(k+1)' + \hat{A}(k+1)\hat{B}(k)\hat{B}(k)'\hat{A}(k+1)' \tag{54}$$
$$+\hat{B}(k+1)\hat{B}(k+1)' = \Sigma(k+2),$$

and by induction it follows:

$$\hat{E}_k \Sigma(k) \hat{E}'_k + \hat{J}_k \hat{J}'_k = \Sigma(k), \tag{55}$$

where $\hat{E}_k$ and $\hat{J}_k$ are defined as E_k and J_k with matrices $A(\cdot)$ and $B(\cdot)$ substituted by $\hat{A}(\cdot)$ and $\hat{B}(\cdot)$, respectively.

In the same way, relation (31) implies that:

$$\hat{E}'_k \Sigma(k) \hat{E}_k + \hat{L}'_k \hat{L}_k = \Sigma(k), \tag{56}$$

where $\hat{L}_k$ is defined as L_k with matrices $A(\cdot)$ and $C(\cdot)$ substituted by $\hat{A}(\cdot)$ and $\hat{C}(\cdot)$, respectively.

Denoting with $\hat{\mathcal{A}}_k$, $\hat{\mathcal{B}}_k$ and $\hat{\mathcal{C}}_k$ matrices defined as matrices $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{C}_k$ with $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ substituted by $\hat{A}(\cdot)$, $\hat{B}(\cdot)$ and $\hat{C}(\cdot)$, respectively, and with $\Sigma_k = \text{diag}\{\Sigma(k), \Sigma(k+1), \cdots, \Sigma(k+\omega-1)\}$, relations (30) and (31) imply that:

$$\hat{\mathcal{A}}_k \Sigma_k \hat{\mathcal{A}}'_k + \hat{\mathcal{B}}_k \hat{\mathcal{B}}'_k = \Sigma_k, \tag{57}$$

$$\hat{\mathcal{A}}'_k \Sigma_k \hat{\mathcal{A}}_k + \hat{\mathcal{C}}'_k \hat{\mathcal{C}}_k = \Sigma_k. \tag{58}$$

Then, relations (55) – (58) prove that $(\hat{E}_k, \hat{J}_k, \hat{L}_k, \hat{M}_k)$ and $(\hat{\mathcal{A}}_k, \hat{\mathcal{B}}_k, \hat{\mathcal{C}}_k, \mathcal{D}_k)$ are time-invariant balanced realizations of $S^L(k)$ and $S^C(k)$, respectively [20].   □

REFERENCES

[1] U. M. Al–Saggaf and G. F. Franklin: An error bound for a discrete reduced order model of a linear multivariable system. IEEE Trans. Automat. Control *AC–32* (1987), 9, 815–819.

[2] S. Bittanti: Deterministic and stochastic linear periodic systems. In: Time Series and Linear Systems (S. Bittanti, ed.), Springer–Verlag, Berlin 1986.

[3] P. Bolzern and P. Colaneri: Existence and uniqueness conditions for the periodic solutions of the discrete–time periodic Lyapunov equations. In: Proc. of 25th Conference on Decision and Control, Athens 1986, pp. 1439–1443.

[4] P. Bolzern, P. Colaneri and R. Scattolini: Zeros of discrete–time linear periodic systems. IEEE Trans. Automat. Control *AC–31* (1986), 1057–1059.

[5] P. Colaneri and S. Longhi: The lifted and cyclic reformulations in the minimal realization of linear discrete–time periodic systems. In: Proc. of the 1st IFAC Workshop on New Trends in Design of Control Systems, Smolenice 1994, pp. 329–334.

[6] P. Colaneri and S. Longhi: The realization problem for linear periodic systems. Automatica *31* (1995), 775–779.

[7] D. S. Evans: Finite–dimensional realizations of discrete–time weighting patterns. SIAM J. Appl. Math. *22* (1972), 45–67.

[8] D. S. Flamm: A new shift–invariant representation for periodic linear systems. Systems Control Lett. *17* (1991), 9–14.

[9] L. Fortuna, G. Nunnari and A. Gallo: Model Order Reduction Techniques with Applications in Electrical Engineering. Springer–Verlag, Berlin 1992.

[10] K. Glover: All optimal Hankel–norm approximation of linear multivariable systems and their L^∞–errors bounds. Internat. J. Control *39* (1984), 6, 1115–1193.

[11] I. Gohberg, M. A. Kaashoek and L. Lerer: Minimality and realization of discrete time–varying systems. Oper. Theory: Adv. Appl. *56* (1992), 261–296.

[12] O. M. Grasselli and S. Longhi: Disturbance localization by measurements feedback for linear periodic discrete–time systems. Automatica *24* (1988), 375–385.

[13] O. M. Grasselli and S. Longhi: Zeros and poles of linear periodic discrete–time systems. Circuits Systems Signal Process. *7* (1988), 361–380.

[14] O. M. Grasselli and S. Longhi: Robust tracking and regulation of linear periodic discrete–time systems. Internat. J. Control *54* (1991), 613–633.

[15] O. M. Grasselli and S. Longhi: The geometric approach for linear periodic discrete–time systems. Linear Algebra Appl. *158* (1991), 27–60.

[16] M. Gree and D. J. N. Limebeer: Linear Robust Control. Prentice Hall, Englewood Cliffs, N. J. 1995.

[17] R. A. Mayer and C. S. Burrus: A unified analysis of multirate and periodically time–varying digital filters. Trans. Ccts Syst. *CSA–22* (1975), 162–168.

[18] B. C. Moore: Principal component analysis in linear systems: controllability, observability and modal reduction. Trans. Automat. Control *AC–26* (1981), 17–32.

[19] B. Park and E. I. Verriest: Canonical forms on discrete linear periodically time–varying systems and a control application. In: Proc. of the 28th IEEE Conference on Decision and Control, Tampa 1989, pp. 1220–1225.

[20] L. Pernebo and L. M. Silverman: Model reduction via balanced state space representations. IEEE Trans. Automat. Control *AC–27* (1982), 2, 382–387.

[21] G. Salomon, K. Zhou and E. Wu: A new balanced realization and model reduction method for unstable systems. In: Proc. of 14th IFAC World Congress, Beijing 1999, vol. D, pp. 123–128.

[22] S. Shokoohi, L. M. Silverman and P. Van Dooren: Linear time–variable systems: balancing and model reduction. Trans. Automat. Control *AC-28* (1983), 8, 810–822.

[23] S. Shokoohi, L. M. Silverman and P. Van Dooren: Linear time–variable systems: stability of reduced models. Automatica *20* (1984), 1, 59–67.

[24] A. Varga: Periodic Lyapunov equations: some applications and new algorithms. Internat. J. Control *67* (1997), 69–87.

[25] B. Xie, R. K. A. V. Aripirala and V. L. Syrmos: Model reduction of linear discrete–time periodic systems using Hankel–norm approximation. In: Proc. of IFAC 13th World Congress, San Francisco, 1996, pp. 245–250.

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