

# Kybernetika

VOLUME 43 (2007), NUMBER 1

The Journal of the Czech Society for  
Cybernetics and Information Sciences

Published by:

Institute of Information Theory  
and Automation of the AS CR, v.v.i.

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The Journal has been monitored in the Science Citation Index since 1977 and it is abstracted/indexed in databases of Mathematical Reviews, Current Mathematical Publications, Current Contents ISI Engineering and Computing Technology.

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*Kybernetika*. Volume 43 (2007)

ISSN 0023-5954, MK ČR E 4902.

Published bimonthly by the Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. — Address of the Editor: P. O. Box 18, 182 08 Prague 8, e-mail: kybernetika@utia.cas.cz. — Printed by PV Press, Pod vrstevnicí 5, 140 00 Prague 4. — Orders and subscriptions should be placed with: MYRIS TRADE Ltd., P. O. Box 2, V Štíhlách 1311, 142 01 Prague 4, Czech Republic, e-mail: myris@myris.cz. — Sole agent for all “western” countries: Kubon & Sagner, P. O. Box 34 01 08, D-8 000 München 34, F.R.G.

Published in March 2007.

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## CONTROLLABILITY OF SEMILINEAR STOCHASTIC INTEGRODIFFERENTIAL SYSTEMS

K. BALACHANDRAN, S. KARTHIKEYAN AND J.-H. KIM

In this paper we study the approximate and complete controllability of stochastic integrodifferential system in finite dimensional spaces. Sufficient conditions are established for each of these types of controllability. The results are obtained by using the Picard iteration technique.

*Keywords:* Controllability, approximate controllability, stochastic integrodifferential system, Picard iteration

*AMS Subject Classification:* 93B05

### 1. INTRODUCTION

The problem of controllability of linear deterministic system is well documented. It is well known that controllability of deterministic equations are widely used in analysis and the design of control system. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. In many dynamical systems, it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls; that is there are systems which are completely controllable. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability.

The controllability of nonlinear deterministic systems in finite dimensional space has been extensively studied by several authors, see [1,5] and references therein. Controllability of linear stochastic systems in finite dimensional spaces has been studied by Dobov and Mordukhovich [3], Enrhardt and Kliemann [4], Mahmudov [9], Mahmudov and Denker [8] and Zabczyk [15]. There are very few works about controllability of nonlinear stochastic systems. In [14], the authors introduced the definitions of stochastic  $\epsilon$ -controllability and controllability with probability and established sufficient conditions for stochastic controllability of a class of nonlinear systems. In [6], using a stochastic Lyapunov-like approach, sufficient conditions for stochastic  $\epsilon$ -controllability are formulated. Balachandran and Dauer [2] and Mahmudov and Zorlu [12] studied the controllability of nonlinear stochastic systems.

The problem of controllability of a linear stochastic system of the form

$$\left. \begin{aligned} dx(t) &= [Ax(t) + Bu(t)] dt + \tilde{\sigma}(t) dw(t), \quad t \in [0, T] \\ x(0) &= x_0, \end{aligned} \right\} \quad (1)$$

has been studied by various authors [11,14], where  $\tilde{\sigma} : [0, T] \rightarrow R^{n \times n}$ .

Mahmudov [10, 11] studied approximate controllability of non-linear stochastic system when nonlinear  $f$  and  $\sigma$  are uniformly bounded and satisfy the Lipschitz condition. Recently, Mahmudov and Zorlu [13] investigated the approximate and complete controllability of the following semilinear stochastic system

$$dx(t) = [Ax(t) + Bu(t) + f(t, x(t), u(t))] dt + \sigma(t, x(t), u(t)) dw(t)$$

with non-Lipschitz coefficients when  $f$  and  $\sigma$  depends on control  $u$ . They established the results by using the Picard type approximation.

In this paper we shall study the approximate and complete controllability of the following semilinear stochastic integrodifferential system

$$\left. \begin{aligned} dx(t) &= \left[ Ax(t) + Bu(t) + f(t, x(t), u(t)) + \int_0^t g(t, s, x(s), u(s)) ds \right] dt \\ &\quad + \sigma(t, x(t), u(t)) dw(t), \quad t \in [0, T] \\ x(0) &= x_0, \end{aligned} \right\} \quad (2)$$

where  $A$  and  $B$  are matrices of dimensions  $n \times n, n \times m$  respectively,  $g : [0, T] \times [0, T] \times R^n \times R^m \rightarrow R^n, f : [0, T] \times R^n \times R^m \rightarrow R^n, \sigma : [0, T] \times R^n \times R^m \rightarrow R^{n \times n}$  and  $w$  is an  $n$ -dimensional Wiener process. The results generalize the results of [13].

## 2. PRELIMINARIES

In this paper we use the following notations:

- $(\Omega, \mathcal{F}, P) :=$  The probability space with probability measure  $P$  on  $\Omega$
- $\{\mathcal{F}_t | t \in [0, T]\} :=$  The filtration generated by  $\{w(s) : 0 \leq s \leq t\}$  and  $\mathcal{F} = \mathcal{F}_T$ .
- $L_2(\Omega, \mathcal{F}_T, R^n) :=$  The Hilbert space of all  $\mathcal{F}_T$ -measurable square integrable variables with values in  $R^n$ .
- $L_2^{\mathcal{F}}([0, T], R^n) :=$  The Hilbert space of all square integrable and  $\mathcal{F}_t$ -measurable processes with values in  $R^n$ .
- $C([0, T], L_2(\Omega, \mathcal{F}, P, X)) :=$  The Banach space of continuous maps from  $[0, T]$  into  $L_2(\Omega, \mathcal{F}, P, X)$  satisfying the condition  $\sup_{t \in [0, T]} E\|x(t)\|^2 < \infty$ .
- $X_s :=$  The Banach space with norm topology given by  $\|x\|_s^2 = \sup_{t \in [0, s]} E\|x(t)\|^2$  which is a closed subspace of  $C([0, T], L_2(\Omega, \mathcal{F}, P, X))$  consisting of measurable and  $\mathcal{F}_t$ -adapted processes  $x(t)$ .
- $U_s :=$  The Banach space with norm topology given by  $\|u\|_s^2 = \sup_{t \in [0, s]} E\|u(t)\|^2$  which is a closed subspace of  $C([0, T], L_2(\Omega, \mathcal{F}, P, U))$  consisting of measurable and  $\mathcal{F}_t$ -adapted processes  $u(t)$ .
- $\mathcal{L}(X, Y) :=$  The space of all linear bounded operators from a Banach space  $X$  to a Banach space  $Y$ .
- Denote  $S(t) = \exp(At)$ .

Now let us introduce the following operators and sets.

1. The operator  $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], R^m), L_2(\Omega, \mathcal{F}_T, R^n))$  is defined by

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds.$$

Clearly the adjoint  $(L_0^T)^* : L_2(\Omega, \mathcal{F}_T, R^n) \rightarrow L_2^{\mathcal{F}}([0, T], R^m)$  is defined by

$$(L_0^T)^* z = B^* S^*(T-t)E\{z|\mathcal{F}_t\}.$$

2. The controllability matrix  $\Gamma_s^T \in \mathcal{L}(R^n, R^n)$

$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)dt, \quad 0 \leq s < T$$

and the resolvent operator

$$R(\alpha, \Gamma_s^T) = (\alpha I + \Gamma_s^T)^{-1}, \quad 0 \leq s \leq T.$$

3. Set of all states attainable from  $x_0$  in time  $t > 0$

$$\mathcal{R}_t(x_0) = \{x(t; x_0, u) : u(\cdot) \in L_2(\Omega, \mathcal{F}_T, R^n)\}$$

where  $x(t, x_0, u)$  is the solution of (2) corresponding to  $x_0 \in R^n$ ,  $u(\cdot) \in L_2(\Omega, \mathcal{F}_T, R^n)$ .

Now for our convenience, let us introduce the following notations:

$$\begin{aligned} M_B &= \|B\|, & M_S &= \max\{\|S(t)\| : t \in [0, T]\}, \\ M_\Gamma &= \max\{\|\Gamma_s^T\| : s, t \in [0, T]\}. \end{aligned}$$

**Definition 2.1.** The stochastic system (2) is approximately controllable on  $[0, T]$  if

$$\overline{\mathcal{R}_T(x_0)} = L_2(\Omega, \mathcal{F}_T, R^n)$$

that is, if it is possible to steer the system from the initial point  $x_0$  to within a distance  $\epsilon > 0$  from all the final points in the state space  $L_2(\Omega, \mathcal{F}_T, R^n)$  at time  $T$ .

**Definition 2.2.** The stochastic system (2) is completely controllable on  $[0, T]$  if

$$\mathcal{R}_T(x_0) = L_2(\Omega, \mathcal{F}_T, R^n),$$

that is, if all the points in  $L_2(\Omega, \mathcal{F}_T, R^n)$  can be reached from the point  $x_0$  at time  $T$ .

We assume the following conditions on the problem:

- (H1) The functions  $f$ ,  $g$  and  $\sigma$  satisfies the Lipschitz condition and there exist constants  $L_1, L_2 > 0$  for  $x_1, x_2 \in X$ ,  $u_1, u_2 \in U$  and  $0 \leq s \leq t \leq T$

$$\begin{aligned} & \left\| f(t, x_1, u_1) - f(t, x_2, u_2) \right\|^2 + \left\| \sigma(t, x_1, u_1) - \sigma(t, x_2, u_2) \right\|^2 \\ & \leq L_1(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2) \\ & \quad \left\| \int_0^t g(t, s, x_1(s), u_1(s)) - g(t, s, x_2(s), u_2(s)) ds \right\|^2 \\ & \leq L_2(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2) \end{aligned}$$

(H2) The functions  $f, g$  and  $\sigma$  are continuous and there exists a constant  $L > 0$  such that,

$$\|f(t, x, u)\|^2 + \left\| \int_0^t g(t, s, x, u) ds \right\|^2 + \|\sigma(t, x, u)\|^2 \leq L(\|x\|^2 + \|u\|^2 + 1)$$

for all  $t \in [0, T]$  and all  $(x, u) \in X \times U$ .

(H2)' The functions  $f, g$  and  $\sigma$  are continuous and there exists a constant  $M_f > 0$  such that

$$\|f(t, x, u)\|^2 + \left\| \int_0^t g(t, s, x, u) ds \right\|^2 + \|\sigma(t, x, u)\|^2 \leq M_f$$

for all  $t, s \in [0, T]$  and all  $(x, u) \in X \times U$ .

(H3) The linear system (1) is approximately controllable.

(H4) The linear system (1) is completely controllable.

(H5)  $A$  is non-negative and self-adjoint.

(H6)  $BB^*$  is positive, that is there exists  $\gamma > 0$  such that  $\langle BB^*x, x \rangle \geq \gamma\|x\|^2$ .

(AC)  $\|\alpha R(\alpha, \Gamma_0^T)\| \rightarrow 0$  as  $\alpha \rightarrow 0^+$ .

Note that the assumptions (AC), (H3) and (H4) are equivalent, see [9]. The following lemmas whose proof can be found in [13] give a formula for a control steering the state  $x_0$  to some neighborhood of an arbitrary point  $h$ .

**Lemma 2.1.** For arbitrary  $f(\cdot) \in L_2^{\mathcal{F}}([0, T], R^n)$ ,  $\sigma(\cdot) \in L_2^{\mathcal{F}}([0, T], R^{n \times n})$ ,  $g(\cdot, t) \in L_2^{\mathcal{F}}([0, T], R^n)$ ,  $h \in L_2(\Omega, \mathcal{F}, R^n)$  the control

$$\begin{aligned} u^\alpha(t) &= B^*S^*(T-t)(\alpha I + \Gamma_0^T)^{-1}(Eh - S(T)x_0) \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r) dr \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}(S(T-r)\sigma(r) - \varphi(r)) dw(r) \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r) \left( \int_0^r g(r, s) ds \right) dr \end{aligned} \quad (3)$$

transfers the system

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)Bu(s) ds + \int_0^t S(t-s)f(s) ds \\ &\quad + \int_0^t S(t-s)\sigma(s) dw(s) + \int_0^t \int_0^s S(t-s)g(s, \tau) d\tau ds \end{aligned} \quad (4)$$

from  $x_0 \in R^n$  to some neighbourhood of  $h$  at time  $T$  and

$$\begin{aligned} x_\alpha(T) &= h - \alpha(\alpha I + \Gamma_0^T)^{-1}(\mathbf{E}h - S(T)x_0) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r) \, dr \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}(S(T-r)\sigma(r) - \varphi(r)) \, dw(r) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}S(T-r) \left( \int_0^r g(r,s) \, ds \right) \, dr \end{aligned}$$

where  $h$  has the following representation  $h = \mathbf{E}h + \int_0^T \varphi(r) \, dw(r)$ , see [7].

**Lemma 2.2.** Let Assumptions (H4), (H5) and (H6) hold. Then there exists  $C > 0$  such that for all  $g(\cdot) \in L_2^{\mathcal{F}}(0, T; R^n)$  the following inequality holds

$$\lim_{t \rightarrow T^-} \mathbf{E} \int_0^t \left\| \Gamma_r^T S^*(T-t)(\Gamma_r^T)^{-1}S(t-r)g(r) \right\|^2 \, dr \leq C \int_0^T \mathbf{E} \|g(r)\|^2 \, dr. \quad (5)$$

### 3. CONTROLLABILITY RESULTS

In this section we derive some controllability conditions for the semilinear stochastic integrodifferential system (2) by using the Picard approximation. In [8, 9] it is shown that complete controllability and approximate controllability of the linear system (1) coincide. But this may not always be true for semilinear stochastic integrodifferential systems.

In order to apply the Picard approximation we have to introduce the nonlinear operator  $\Phi_\alpha$ ,  $\alpha > 0$  from  $X_T \times U_T$  to  $X_T \times U_T$  which is defined by

$$\Phi_\alpha(x, u) = (z, w) \quad (6)$$

where

$$\begin{aligned} z(t) &= S(t)x_0 + \int_0^t S(t-r)Bw(r) \, dr + \int_0^t S(t-r)f(r, x(r), u(r)) \, dr \\ &\quad + \int_0^t S(t-r) \left[ \int_0^r g(r, \tau, x(\tau), u(\tau)) \, d\tau \right] \, dr \\ &\quad + \int_0^t S(t-r)\sigma(r, x(r), u(r)) \, dw(r), \\ w(t) &= B^*S^*(T-t) \left[ (\alpha I + \Gamma_0^T)^{-1}(\mathbf{E}h - S(T)x_0) + \int_0^t (\alpha I + \Gamma_r^T)^{-1}\varphi(r) \, dw(r) \right] \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r, x(r), u(r)) \, dr \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r) \left[ \int_0^r g(r, \tau, x(\tau), u(\tau)) \, d\tau \right] \, dr \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)\sigma(r, x(r), u(r)) \, dw(r), \end{aligned}$$

and  $\varphi \in L_2^{\mathcal{F}}([0, T], R^{n \times n})$  comes from the representation  $h = Eh + \int_0^T \varphi(r) dw(r)$  of  $h \in L_2(\Omega, \mathcal{F}, R^n)$ . It will be shown that the system (2) is approximately controllable if for all  $\alpha > 0$  there exists a fixed point of the operator  $\Phi_\alpha$ . To show this we employ the Picard type approximations to (6).

$$\begin{aligned} x_0(t) &= S(t)x_0 \\ x_{n+1}(t) &= S(t)x_0 + \int_0^t S(t-r)Bu_{n+1}(r)dr + \int_0^t S(t-r)f(r, x_n(r), u_n(r))dr \\ &\quad + \int_0^t S(t-r) \left[ \int_0^r g(r, \tau, x_n(\tau), u_n(\tau)) d\tau \right] dr \\ &\quad + \int_0^t S(t-r)\sigma(r, x_n(r), u_n(r)) dw(r) \end{aligned} \quad (7)$$

$$\begin{aligned} u_0(t) &= B^*S^*(T-t) \left[ (\alpha I + \Gamma_0^T)^{-1}(Eh - S(T)x_0) + \int_0^t (\alpha I + \Gamma_r^T)^{-1}\varphi(r) dw(r) \right] \\ u_{n+1}(t) &= B^*S^*(T-t) \left[ (\alpha I + \Gamma_0^T)^{-1}(Eh - S(T)x_0) + \int_0^t (\alpha I + \Gamma_r^T)^{-1}\varphi(r) dw(r) \right] \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r, x_n(r), u_n(r)) dr \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r) \left[ \int_0^r g(r, \tau, x_n(\tau), u_n(\tau)) d\tau \right] dr \\ &\quad - B^*S^*(T-t) \int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)\sigma(r, x_n(r), u_n(r)) dw(r). \end{aligned} \quad (8)$$

**Lemma 3.1.** Under the conditions (H1), (H2) the operator  $\Phi_\alpha$  is well defined and there exist  $M_T(\alpha), k_1(\alpha), k_2(\alpha) > 0$  such that if  $(x_1, u_1), (x_2, u_2) \in X_T \times U_T$  then

$$\|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_t^2 \leq M_T(\alpha)(L_1 + L_2) \left\{ \int_0^t \left( \sup_{0 \leq r \leq s} \mathbb{E}\|x_1(r) - x_2(r)\|^2 \right) ds \right.$$

$$\left. + \int_0^t \left( \sup_{0 \leq r \leq s} \mathbb{E}\|u_1(r) - u_2(r)\|^2 \right) ds \right\}$$

and

$$\|\Phi_\alpha(x_1, u_1)\|_t^2 \leq k_1(\alpha) + k_2(\alpha)LT \left\{ \sup_{0 \leq r \leq s} \mathbb{E}\|x_1(r)\|^2 + \sup_{0 \leq r \leq s} \mathbb{E}\|u_1(r)\|^2 + 1 \right\},$$

for each  $t \in [0, T]$ , where

$$\begin{aligned} M_T(\alpha) &= \max \left\{ 4M_S^2T + \frac{3}{\alpha^2}M_S^4M_B^2T + \frac{12}{\alpha^2}M_S^6M_B^4T^2, 4M_S^2 + \frac{3}{\alpha^2}M_S^4M_B^2 + \frac{12}{\alpha^2}M_S^6M_B^4T, \right. \\ &\quad \left. 4M_S^2T^2 + \frac{3}{\alpha^2}M_S^4M_B^2T^2 + \frac{12}{\alpha^2}M_S^6M_B^4T^3 \right\}, \end{aligned}$$

$$k_1(\alpha) = 5M_S^2\mathbb{E}\|x_0\|^2 + \frac{40}{\alpha^2}M_S^4M_B^4T \left\{ 2\|Eh\|^2 + 2M_S^2\mathbb{E}\|x_0\| + \int_0^T \mathbb{E}\|\varphi(r)\|^2 dr \right\},$$

$$k_2(\alpha) = \left( 5M_S^2 + \frac{4}{\alpha^2}M_S^4M_B^2 + \frac{20}{\alpha^2}M_S^6M_B^4T \right) \max\{T^2, T, 1\}.$$

Proof. Let us consider

$$\begin{aligned}
& \|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_t^2 = \sup_{0 \leq s \leq t} \mathbb{E} \|z_1(s) - z_2(s)\|^2 + \sup_{0 \leq s \leq t} \mathbb{E} \|w_1(s) - w_2(s)\|^2 \\
& \leq 4M_S^2 M_B^2 \int_0^t \mathbb{E} \|w_1(s) - w_2(s)\|^2 ds \\
& \quad + 4M_S^2 T \mathbb{E} \int_0^t \|f(r, x_1(r), u_1(r)) - f(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + 4M_S^2 \mathbb{E} \int_0^t \|\sigma(r, x_1(r), u_1(r)) - \sigma(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + 4M_S^2 T^2 \mathbb{E} \left\| \int_0^r (g(r, \tau, x_1(\tau), u_1(\tau)) - g(r, \tau, x_2(\tau), u_2(\tau))) d\tau \right\|^2 \\
& \quad + \frac{3}{\alpha^2} M_S^4 M_B^2 T^2 \mathbb{E} \int_0^t \|f(r, x_1(r), u_1(r)) - f(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + \frac{3}{\alpha^2} M_S^4 M_B^2 \mathbb{E} \int_0^t \|\sigma(r, x_1(r), u_1(r)) - \sigma(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + \frac{3}{\alpha^2} M_S^4 M_B^2 T^2 \mathbb{E} \left\| \int_0^r (g(r, \tau, x_1(\tau), u_1(\tau)) - g(r, \tau, x_2(\tau), u_2(\tau))) d\tau \right\|^2 \\
& \leq \left( 4M_S^2 T + \frac{3}{\alpha^2} M_S^4 M_B^2 T + \frac{12}{\alpha^2} M_S^6 M_B^4 T^2 \right) \\
& \quad \times \int_0^t \mathbb{E} \|f(r, x_1(r), u_1(r)) - f(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + \left( 4M_S^2 + \frac{3}{\alpha^2} M_S^4 M_B^2 + \frac{12}{\alpha^2} M_S^6 M_B^4 T \right) \\
& \quad \times \int_0^t \mathbb{E} \|\sigma(r, x_1(r), u_1(r)) - \sigma(r, x_2(r), u_2(r))\|^2 dr \\
& \quad + \left( 4M_S^2 T^2 + \frac{3}{\alpha^2} M_S^4 M_B^2 T^2 + \frac{12}{\alpha^2} M_S^6 M_B^4 T^3 \right) \\
& \quad \times \mathbb{E} \left\| \int_0^r (g(r, \tau, x_1(\tau), u_1(\tau)) - g(r, \tau, x_2(\tau), u_2(\tau))) d\tau \right\|^2 \\
& \leq M_T(\alpha) \left\{ \int_0^t \mathbb{E} \|f(r, x_1(r), u_1(r)) - f(r, x_2(r), u_2(r))\|^2 dr \right. \\
& \quad + \int_0^t \mathbb{E} \|\sigma(r, x_1(r), u_1(r)) - \sigma(r, x_2(r), u_2(r))\|^2 dr \\
& \quad \left. + \mathbb{E} \left\| \int_0^r (g(r, \tau, x_1(\tau), u_1(\tau)) - g(r, \tau, x_2(\tau), u_2(\tau))) d\tau \right\|^2 \right\} \\
& \leq M_T(\alpha)(L_1 + L_2) \left\{ \int_0^t \left( \sup_{0 \leq r \leq s} \mathbb{E} \|x_1(r) - x_2(r)\|^2 \right) ds \right. \\
& \quad \left. + \int_0^t \left( \sup_{0 \leq r \leq s} \mathbb{E} \|u_1(r) - u_2(r)\|^2 \right) ds \right\}.
\end{aligned}$$



Observe that standard computations yield,

$$\begin{aligned}
& \|\Phi_\alpha(x_1, u_1)\|_t^2 = \sup_{0 \leq s \leq t} \mathbb{E} \|z_1(s)\|^2 + \sup_{0 \leq s \leq t} \mathbb{E} \|w_1(s)\|^2 \\
& \leq 5M_S^2 \mathbb{E} \|x_0\|^2 + 5M_S^2 M_B^2 \int_0^t \mathbb{E} \|w(s)\|^2 ds + 5M_S^2 T \mathbb{E} \int_0^t \|f(r, x_1(r), u_1(r))\|^2 dr \\
& \quad + 5M_S^2 \mathbb{E} \int_0^t \|\sigma(r, x_1(r), u_1(r))\|^2 dr + 5M_S^2 T^2 \mathbb{E} \left\| \int_0^r g(r, \tau, x_1(\tau), u_1(\tau)) d\tau \right\|^2 \\
& \quad + 4M_S^2 M_B^2 \left\{ \frac{2}{\alpha^2} (2\|\mathbb{E}h\|^2 + 2M_S^2 \mathbb{E} \|x_0\|^2) + \frac{2}{\alpha^2} \int_0^t \mathbb{E} \|\varphi(r)\|^2 dr \right\} \\
& \quad + \frac{4}{\alpha^2} M_S^4 M_B^2 T \int_0^t \mathbb{E} \|f(r, x_1(r), u_1(r))\|^2 dr + \frac{4}{\alpha^2} M_S^4 M_B^2 \int_0^t \mathbb{E} \|\sigma(r, x_1(r), u_1(r))\|^2 dr \\
& \quad + \frac{4}{\alpha^2} M_S^4 M_B^2 T^2 \mathbb{E} \left\| \int_0^r g(r, \tau, x_1(\tau), u_1(\tau)) d\tau \right\|^2 \\
& \leq 5M_S^2 \mathbb{E} \|x_0\|^2 + \frac{40}{\alpha^2} M_S^4 M_B^4 T \left\{ 2\|\mathbb{E}h\|^2 + 2M_S^2 \mathbb{E} \|x_0\| + \int_0^T \mathbb{E} \|\varphi(r)\|^2 dr \right\} \\
& \quad + \left( 5M_S^2 T + \frac{4}{\alpha^2} M_S^4 M_B^2 T + \frac{20}{\alpha^2} M_S^6 M_B^4 T^2 \right) \mathbb{E} \int_0^t \|f(r, x_1(r), u_1(r))\|^2 dr \\
& \quad + \left( 5M_S^2 + \frac{4}{\alpha^2} M_S^4 M_B^2 + \frac{20}{\alpha^2} M_S^6 M_B^4 T \right) \mathbb{E} \int_0^t \|\sigma(r, x_1(r), u_1(r))\|^2 dr \\
& \quad + \left( 5M_S^2 T^2 + \frac{4}{\alpha^2} M_S^4 M_B^2 T^2 + \frac{20}{\alpha^2} M_S^6 M_B^4 T^3 \right) \mathbb{E} \left\| \int_0^r g(r, \tau, x_1(\tau), u_1(\tau)) d\tau \right\|^2 \\
& \leq k_1(\alpha) + k_2(\alpha)LT \left\{ \sup_{0 \leq r \leq s} \mathbb{E} \|x_1(r)\|^2 + \sup_{0 \leq r \leq s} \mathbb{E} \|u_1(r)\|^2 + 1 \right\}. \quad \square
\end{aligned}$$

**Lemma 3.2.** Under the conditions (H1), (H2) the sequence  $(x_n, u_n)$  is bounded in  $X_T \times U_T$ .

*Proof.* By Lemma 3.1 for any  $n \geq 0$  we have

$$\begin{aligned}
\|(x_{n+1}, u_{n+1})\|^2 &= \sup_{0 \leq s \leq t} \mathbb{E} \|x_{n+1}(s)\|^2 + \sup_{0 \leq s \leq t} \mathbb{E} \|u_{n+1}(s)\|^2 \\
&\leq k_1 + k_2LT \left\{ \sup_{0 \leq r \leq s} \mathbb{E} \|x_n(r)\|^2 + \sup_{0 \leq r \leq s} \mathbb{E} \|u_n(r)\|^2 + 1 \right\} \quad (9)
\end{aligned}$$

where  $k_1, k_2$  are positive constants independent of  $n$ . Then by (9) and successive approximation, we obtain that

$$\begin{aligned}
\|(x_{n+1}, u_{n+1})\|^2 &\leq (k_1 + k_2LT) \left[ 1 + k_2LT + \cdots + k_2^n L^n T^n \right] \\
&\quad + (k_2LT)^{n+1} \{ \mathbb{E} \|x_0(t)\|^2 + \mathbb{E} \|u_0(t)\|^2 + 1 \} \\
&\leq (k_1 + k_2LT) \left[ 1 + k_2LT + k_2^2 L^2 T^2 + \cdots + k_2^n L^n T^n \right] + (k_2LT)^{n+1} C_0
\end{aligned}$$

where  $C_0 = 1 + M_S^2 \mathbb{E} \|x_0\|^2 + \frac{2}{\alpha^2} M_S^2 M_B^2 \left\{ 2\|Eh\|^2 + 2M_S^2 \mathbb{E} \|x_0\|^2 + \int_0^T \mathbb{E} \|\varphi(r)\|^2 dr \right\}$ .

Thus, we have

$$\|(x_{n+1}, u_{n+1})\|^2 \leq (k_1 + k_2 LT) \frac{1 - (k_2^2 LT)^n}{1 - (k_2^2 LT)} + (k_2 LT)^{n+1} C_0. \quad \square$$

**Lemma 3.3.** Under the conditions (H1), (H2) the sequence  $(x_n, u_n)$  is a Cauchy sequence in  $X_T \times U_T$ .

*Proof.* Let us take

$$\begin{aligned} r_n(t) &= \sup_{m \geq n} \|(x_m, u_m) - (x_n, u_n)\|_t^2, \\ p_n(t) &= \sup_{m \geq n} \|x_m - x_n\|_t^2, \\ q_n(t) &= \sup_{m \geq n} \|u_m - u_n\|_t^2. \end{aligned}$$

The functions  $r_n, p_n, q_n, n \geq 0$ , are well defined, uniformly bounded and evidently monotone non-decreasing. Since  $\{r_n(t) : n \geq 0\}, \{p_n(t) : n \geq 0\}, \{q_n(t) : n \geq 0\}$  are monotone non-increasing sequences for each  $t \in [0, T]$ , there exists a monotone non-decreasing function  $(r(t), p(t), q(t))$  such that

$$\lim_{n \rightarrow \infty} (r_n(t), p_n(t), q_n(t)) = (r(t), p(t), q(t)).$$

By Lemma 3.1 we obtain that

$$\begin{aligned} \|\Phi_\alpha(x_m, u_m) - \Phi_\alpha(x_n, u_n)\|_t^2 &\leq M_T(\alpha)(L_1 + L_2) \int_0^t \left\{ \sup_{0 \leq r \leq s} \mathbb{E} \|x_{m-1}(r) - x_{n-1}(r)\|^2 \right. \\ &\quad \left. + \sup_{0 \leq r \leq s} \mathbb{E} \|u_{m-1}(r) - u_{n-1}(r)\|^2 \right\} ds \end{aligned}$$

from which it follows that

$$\begin{aligned} r(t) &\leq r_n(t) = p_n(t) + q_n(t) \\ &\leq M_T(\alpha)(L_1 + L_2) \int_0^t \left\{ \sup_{0 \leq r \leq s} \mathbb{E} \|x_{m-1}(r) - x_{n-1}(r)\|^2 \right. \\ &\quad \left. + \sup_{0 \leq r \leq s} \mathbb{E} \|u_{m-1}(r) - u_{n-1}(r)\|^2 \right\} ds \\ &= M_T(\alpha)(L_1 + L_2) \int_0^t [p_{n-1}(s) + q_{n-1}(s)] ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$r(t) \leq p(t) + q(t) \leq M_T(\alpha)(L_1 + L_2) \int_0^t [p(s) + q(s)] ds.$$

Now if  $w = p + q$ , then

$$w'(t) \leq M_T(\alpha)(L_1 + L_2)[p(t) + q(t)] \leq 2M_T(\alpha)(L_1 + L_2)w(t)$$

and also we see that  $w(0) = 0$ . Then, by Grownwall's inequality it follows that  $w(t) = 0$  for all  $t \in [0, T]$ . But

$$\|(x_m, u_m) - (x_n, u_n)\|_T^2 \leq p_n(T) + q_n(T) \rightarrow w(T) = 0.$$

Therefore  $\|(x_m, u_m) - (x_n, u_n)\|_T^2 \rightarrow 0$  as  $n, m \rightarrow \infty$ .  $\square$

**Theorem 3.1.** Under the conditions (H1), (H2) the operator (6) has a unique fixed point.

*Proof.* By Lemma 3.3 the sequence  $(x_n, u_n)$  is Cauchy in  $X_T \times U_T$ . The completeness of  $X_T \times U_T$  implies the existence of a process  $(x, u) \in X_T \times U_T$  such that

$$\lim_{n \rightarrow \infty} \|(x_n, u_n) - (x, u)\|_T^2 = 0.$$

Hence taking the limit in (7) we see that  $(x, u)$  is a fixed point of  $\Phi_\alpha$ . Further, if  $(x_1, u_1), (x_2, u_2) \in X_T \times U_T$  are two fixed points of  $\Phi_\alpha$ , then Lemma 3.1 would imply that

$$\begin{aligned} \|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_t^2 &\leq M_T(\alpha)(L_1 + L_2) \left\{ \int_0^t \left( \sup_{0 \leq r \leq s} \mathbb{E} \|x_1(r) - x_2(r)\|^2 \right. \right. \\ &\quad \left. \left. + \sup_{0 \leq r \leq s} \mathbb{E} \|u_1(r) - u_2(r)\|^2 \right) ds \right\} \end{aligned}$$

So as in the proof Lemma 3.3 we obtain that

$$\|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_T^2 = 0$$

Consequently  $(x_1, u_1) = (x_2, u_2)$  in  $X_T \times U_T$ . Hence  $\Phi_\alpha$  has a unique fixed point.  $\square$

If  $\alpha = 0$  the nonlinear operator  $\Phi_0$  is defined by

$$\Phi_0(x, u) = (z, w) \tag{10}$$

where

$$\begin{aligned} z(t) &= S(t)x_0 + \int_0^t S(t-r)Bw(r)dr + \int_0^t S(t-r)f(r, x(r), u(r))dr \\ &\quad + \int_0^t S(t-r)\sigma(r, x(r), u(r))dw(r) + \int_0^t S(t-r) \left[ \int_0^r g(r, \tau, x(\tau), u(\tau))d\tau \right] dr, \\ w(t) &= B^*S^*(T-t) \left[ (\Gamma_0^T)^{-1}(\mathbb{E}h - S(T)x_0) + \int_r^t (\Gamma_0^T)^{-1}\varphi(r)dw(r) \right] \\ &\quad - B^*S^*(T-t) \int_0^t (\Gamma_r^T)^{-1}S(T-r)f(r, x(r), u(r))dr \\ &\quad - B^*S^*(T-t) \int_0^t (\Gamma_r^T)^{-1}S(T-r)\sigma(r, x(r), u(r))dw(r) \\ &\quad - B^*S^*(T-t) \int_0^t (\Gamma_r^T)^{-1}S(T-r) \left[ \int_0^r g(r, \tau, x(\tau), u(\tau))d\tau \right] dr. \end{aligned}$$

**Theorem 3.2.** Assume hypotheses (H1), (H2) and (H4) hold. Then the operator  $\Phi_0$  has a fixed point.

*Proof.* The proof is similar to that of Theorem 3.1. Note that here we need to use estimation (5) from Lemma 2.2.  $\square$

**Theorem 3.3.** Assume hypotheses (H1), (H2)' and (H3) are satisfied. Then the system (2) is approximately controllable.

*Proof.* Let  $(x^\alpha, u^\alpha)$  be a fixed point of  $\Phi_\alpha$  in  $X_T \times U_T$ . By Lemma 2.1,  $x^\alpha$  satisfies the following equality

$$\begin{aligned} x^\alpha(T) &= h - \alpha(\alpha I + \Gamma_0^T)^{-1}(\mathbb{E}h - S(T)x_0) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r, x^\alpha(r), u^\alpha(r)) \, dr \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}(S(T-r)\sigma(r, x^\alpha(r), u^\alpha(r)) - \varphi(r)) \, dw(r) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1}S(T-r) \left[ \int_0^r g(r, \tau, x^\alpha(\tau), u^\alpha(\tau)) \, d\tau \right] \, dr. \end{aligned} \quad (11)$$

By (11) and the assumption (H2),

$$\begin{aligned} \mathbb{E}\|x^\alpha(T) - h\|^2 &\leq 5\|\alpha R(\alpha, \Gamma_0^T)(\mathbb{E}h - S(T)x_0)\|^2 \\ &\quad + 5T \int_0^T \mathbb{E}\|\alpha R(\alpha, \Gamma_r^T)S(T-r)f(r, x^\alpha(r), u^\alpha(r))\|^2 \, dr \\ &\quad + 5 \int_0^T \mathbb{E}\|\alpha R(\alpha, \Gamma_r^T)S(T-r)\sigma(r, x^\alpha(r), u^\alpha(r))\|^2 \, dr \\ &\quad + 5 \int_0^T \mathbb{E}\|\alpha R(\alpha, \Gamma_r^T)\varphi(r)\|^2 \, dr \\ &\quad + 5T \int_0^T \mathbb{E}\|\alpha R(\alpha, \Gamma_r^T)S(T-r) \left[ \int_0^r g(r, \tau, x^\alpha(\tau), u^\alpha(\tau)) \, d\tau \right]\|^2 \, dr \\ &\leq 5\|\alpha R(\alpha, \Gamma_r^T)\|^2 \|\mathbb{E}h - S(T)x_0\|^2 \\ &\quad + 5T \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 \mathbb{E}\|S(T-r)f(r, x^\alpha(r), u^\alpha(r))\|^2 \, dr \\ &\quad + 5 \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 \mathbb{E}\|S(T-r)\sigma(r, x^\alpha(r), u^\alpha(r))\|^2 \, dr \\ &\quad + 5 \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 \mathbb{E}\|\varphi(r)\|^2 \, dr \\ &\quad + 5T \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 \mathbb{E}\left\|S(T-r) \left[ \int_0^r g(r, \tau, x^\alpha(\tau), u^\alpha(\tau)) \, d\tau \right]\right\|^2 \, dr \end{aligned}$$

$$\begin{aligned} &\leq 5\|\alpha R(\alpha, \Gamma_r^T)\|^2 \|\mathbf{E}h - S(T)x_0\|^2 + 5M_S^2 M_f(2T+1) \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 dr \\ &\quad + 5 \int_0^T \|\alpha R(\alpha, \Gamma_r^T)\|^2 \mathbf{E}\|\varphi(r)\|^2 dr. \end{aligned}$$

Since  $\|\alpha R(\alpha, \Gamma_r^T)\|^2 \leq 1$ ,  $\|\alpha R(\alpha, \Gamma_r^T)\|^2 \rightarrow 0$  as  $\alpha \rightarrow 0^+$  for all  $0 \leq r \leq T$ , by the Lebesgue dominated convergence theorem  $\mathbf{E}\|x^\alpha(T) - h\|^2 \rightarrow 0$  as  $\alpha \rightarrow 0^+$ . This gives the approximate controllability.  $\square$

**Theorem 3.4.** Assume hypotheses (H1)–(H6) are satisfied. Then the system (2) is completely controllable.

*Proof.* By Theorem 3.2, the operator  $\Phi_0$  has a fixed point. So, the control

$$\begin{aligned} u_0(t) &= B^* S^*(T-t)(\Gamma_0^T)^{-1}(\mathbf{E}h - S(T)x_0) \\ &\quad - B^* S^*(T-t) \int_0^t (\Gamma_r^T)^{-1} S(T-r) f(r, x(r), u(r)) dr \\ &\quad - B^* S^*(T-t) \int_0^t (\Gamma_r^T)^{-1} (S(T-r)\sigma(r, x(r), u(r)) - \varphi(r)) dw(r) \\ &\quad - B^* S^*(T-t) \int_0^t (\Gamma_r^T)^{-1} S(T-r) \left[ \int_0^r g(r, \tau, x(\tau), u(\tau)) d\tau \right] dr \end{aligned}$$

transfers the system (2) from  $x_0$  to  $h$ . Hence, the theorem is proved.  $\square$

#### 4. EXAMPLE

Consider the following semilinear stochastic integrodifferential system

$$\left. \begin{aligned} dx(t) &= \left[ Ax(t) + Bu(t) + f(t, x(t), u(t)) + \int_0^t g(t, s, x(s), u(s)) ds \right] dt \\ &\quad + \sigma(t, x(t), u(t)) dw(t), \quad t \in [0, T] \\ x(0) &= x_0, \end{aligned} \right\} \quad (12)$$

where  $w(t)$  is one-dimensional Brownian motion and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f(t, x(t), u(t)) = \begin{bmatrix} (2 + \cos x_2(t))x_1(t) + 3x_2(t) + u_1(t) \\ (3 + \sin x_1(t))x_2(t) + 2x_1(t) + u_2(t) \end{bmatrix}$$

$$\int_0^t g(t, s, x(s), u(s)) ds = \begin{bmatrix} \int_0^t (e^{-x_1(s)} + u_1(s)) ds \\ \int_0^t e^{-s} (5x_1(s) + 3x_2(s) + u_2(s)) ds \end{bmatrix}$$

$$\sigma(t, x(t), u(t)) = \begin{bmatrix} \frac{(2t^2+1)e^{-t}}{(1+x_1(t)+u_1(t))} \\ \frac{\sin t \cos t e^{-t}}{(1+x_2(t)+u_2(t))} \end{bmatrix}$$

The corresponding iterative scheme for (12) is

$$\begin{aligned} x_{n+1}(t) &= S(t)x_0 + \int_0^t S(t-r)Bu_{n+1}(r)dr + \int_0^t S(t-r)f(r, x_n(r), u_n(r))dr \\ &\quad + \int_0^t S(t-r) \left[ \int_0^r g(r, \tau, x_n(\tau), u_n(\tau))d\tau \right] dr \\ &\quad + \int_0^t S(t-r)\sigma(r, x_n(r), u_n(r))dw(r) \end{aligned} \quad (13)$$

where the fundamental matrix  $S(t)$  is given by

$$S(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

The controllability matrix is given by

$$\Gamma_0^T = \int_0^T S(T-t)BB^*S^*(T-t)dt = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = TI$$

and it is nonsingular for  $T > 0$ . Moreover, it is easy to show that for all  $(x, u) \in R^2 \times R^2$ ,  $|f(t, x(t), u(t))|^2 \leq 75(|x|^2 + |u|^2 + 1)$ ,  $\left| \int_0^t g(t, s, x(s), u(s))ds \right|^2 \leq 40(T+1)(1 + |x|^2 + |u|^2)$ ,  $|\sigma(t, x(t), u(t))| \leq 2(2t^2 + 1)e^{-t}$ . By defining a suitable control (8) and by applying the Picard iteration technique to (13), one can establish the approximate and complete controllability of the stochastic system (12).

#### ACKNOWLEDGEMENT

The work of first and third author was supported by Korea Research Foundation Grant (KRF-2004-015-C00054).

(Received October 10, 2005.)

#### REFERENCES

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- [1] K. Balachandran and J. Dauer: Controllability of nonlinear systems via fixed point theorems. *J. Optim. Theory Appl.* 53 (1987), 345–352.
  - [2] J. Dauer and K. Balachandran: Sample controllability of general nonlinear stochastic systems. *Libertas Math.* 17 (1997), 143–153.
  - [3] M. A. Dobov and B. S. Mordukhovich: Theory of controllability of linear stochastic systems. *Differential Equations* 14 (1978), 1609–1612.
  - [4] M. Enrhardt and W. Kliemann: Controllability of stochastic linear systems. *Systems Control Lett.* 2 (1982), 45–153.

- [5] J. Klamka: Schauder's fixed point theorem in nonlinear controllability problems. *Control Cybernet.* **29** (2000), 153–165.
- [6] J. Klamka and L. Socha: Some remarks about stochastic controllability. *IEEE Trans. Automat. Control* **22** (1977), 880–881.
- [7] R. S. Lipster and A. N. Shiryaev: *Statistics of Random Processes*. Springer, New York 1977.
- [8] N. I. Mahmudov and A. Denker: On controllability of linear stochastic systems. *Internat. J. Control* **73** (2000), 144–151.
- [9] N. I. Mahmudov: Controllability of linear stochastic systems. *IEEE Trans. Automat. Control* **46** (2001), 724–731.
- [10] N. I. Mahmudov: On controllability of semilinear stochastic systems in Hilbert spaces. *IMA J. Math. Control Inform.* **19** (2002), 363–376.
- [11] N. I. Mahmudov: Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. *SIAM J. Control Optim.* **42** (2004), 1604–1622.
- [12] N. I. Mahmudov and S. Zorlu: Controllability of nonlinear stochastic systems. *Internat. J. Control* **76** (2003), 95–104.
- [13] N. I. Mahmudov and S. Zorlu: Controllability of semilinear stochastic systems. *Internat. J. Control* **78** (2005), 997–1004.
- [14] Y. Sunahara, T. Kabeuchi, S. Asada, and K. Kishino: On stochastic controllability for nonlinear systems. *IEEE Trans. Automat. Control* **19** (1974), 49–54.
- [15] J. Zabczyk: Controllability of stochastic linear systems. *Systems Control Lett.* **1** (1987), 25–31.

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