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EDITORIAL TO THE SPECIAL ISSUE ON "RANDOM VARIABLES, JOINT DISTRIBUTION FUNCTIONS, AND COPULAS"

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In 1973 Abe Sklar published in this journal the paper [18], which he had also presented at the Sixth Prague Conference on Information Theory, Statistical Decision Functions and Random Processes (Prague, September 19 – 25, 1971). In a previous, now historic, paper [17], he had introduced the concept of an n-dimensional copula, as a function C from $[0,1]^n$ onto $[0,1]$ such that (1) $C(1,\dots,1,x_j,1,\dots,1)=x_j$ for every $j\in\{1,2,\dots,n\}$ and every $x_j\in[0,1]$, (2) $C(x_1,\dots,x_n)=0$ if $x_j=0$ for at least one index j, and (3) all n-dimensional differences of C are nonnegative. There he further announced what we now call *Sklar's theorem*.

In the *Kybernetika* paper he sketched the proof of the above statement, developed some of its consequences, and discussed various connections between copulas and random variables, associative copulas, binary operations on spaces of one-dimensional distribution functions induced by copulas, and applications to probabilistic metric spaces and probabilistic information spaces. At the time copulas were used almost exclusively in the theory of Probabilistic Metric Spaces [15].

Only some years later, after the publication of the paper by Schweizer and Wolff [16], did the statistical community "discover" copulas. This had two effects: on the one hand, it was quickly realized that copulas were of paramount importance in dealing with questions concerning the dependence of random variables, and, on the other hand the number of papers dealing with copulas, under both the theoretical and the applied aspects, grew enormously. For a brief history of the early years of copula theory we refer the reader to Schweizer's synthesis [14].

After nearly 50 years from their introduction, copulas can be considered as a standard tool for the constructions of stochastic models, as clearly stated in the books by Joe [9] and Nelsen [12]. As a consequence, they are frequently used in finance: see, just to make few examples, the books by Cherubini et al. [3] and Mc Neil et al. [11], and the recent survey papers by Embrechts [4] and Genest et al. [6]. Moreover, copulas were also found to be of great interest in environmental sciences, especially hydrology: see the book by Salvadori et al. [13] and the paper by Genest and Favre [5]. Outside the field of applied probability and statistics, they appear, for example, in the theory of functional equations and inequalities [1] and fuzzy set theory [8, 10]. Today, copulas may therefore be considered as a well-established field of mathematical research with a great impact on several applied sciences.

This special issue of *Kybernetika* is devoted to "Random variables, joint distribution functions, and copulas". This seems to us to be a fitting way of celebrating the thirty-fifth anniversary of Sklar's paper on this journal. As a response of copula community to our call for this special issue, we have obtained 17 submissions covering several interesting aspects of the copula theory and its applications. Each of them has been submitted to a careful peer-review process that selected ten contributions presented here. Special thanks are due to all the anonymous Reviewers of this special issue for their careful reading, to the editorial office of Kybernetika for their help in handling the manuscripts, and to the editorial board of the journal for their kind hospitality. Moreover, we are grateful to all the authors of these papers for submitting us their remarkable contributions.

We hope that this special issue may stimulate further investigations in this fascinating field.

SUMMARY OF THE CONTRIBUTIONS

Here, we summarize shortly the content of this special issue.

Mayor, Mesiar and Torrens discuss the quasi-homogeneity of copulas, which they characterize by the convexity and strict monotonicity of their diagonal section. As a by-product, they introduce a new construction method for copulas when only their diagonal section is known.

Jaworsky and Rychlik focus on the properties of copulas of order statistics and characterize the marginal distribution functions of order statistics that may correspond to absolute continuous and possibly exchangeable copulas.

Charpentier proposes a general framework to compare the strength of the dependence in survival models at the changing of the time, i.e., to compare the dependence between a random vector of lifetimes $\boldsymbol{X}$ and the random vector $\boldsymbol{X}_t$ of the residual lifetimes $\boldsymbol{X}$ given $\boldsymbol{X} > \boldsymbol{X}_t$ as the time t elapses. In particular, he examines in detail the cases when the survival copula of $\boldsymbol{X}$ is Archimedean with a factor representation or is a distorted copula.

Pellerey studies a pair (X, Y) of exchangeable lifetimes whose dependence structure is described by an Archimedean survival copula and provides sufficient conditions for the comparison in usual stochastic and lower orthant orders of (X, Y) and the corresponding pair (X_t, Y_t) of residual lifetimes after time $t > 0$. Some of the results and examples he presents here are hence applied to the study of the relationships between univariate and bivariate aging.

Mesiar, Jágr, Juráňová and Komorníková study the univariate conditioning of copulas and thus provide a construction method for copulas based on an *a priori* given copula. Relying on the recent gluing method, these authors introduce the g-ordinal sum of copulas and a representation of copulas by means of g-ordinal sums. They also show that starting from any Ali–Mikhail–Haq copula with a given parameter $\lambda > 0$ one can construct via (univariate) conditioning any Ali–Mikhail–Haq copula with parameter $\mu \in [0, \lambda]$.

Úbeda-Flores studies properties of the distribution function of the random variable $C(X, Y)$ when the copula of the random pair (X, Y) is either M or W, the copulas for which each of X and Y is almost surely an increasing, or, respectively, a decreasing function of the other one. He also studies the distribution functions of $M(X, Y)$ and $W(X, Y)$ given that the joint distribution function of the random variables X and Y is any copula.

Erdely and González-Barrios analyze some properties of the empirical diagonal of a copula and obtain its exact distribution under independence for the two and three-dimensional cases and further notice the ideas proposed in this paper can be carried out to higher dimensions. The results obtained are useful in designing a nonparametric test for independence, therefore solving an open problem proposed in [1].

Erdely, González-Barrios and Nelsen analyze some properties of the discrete copulas in terms of permutations and observe the connection between discrete copulas and the empirical copulas; they then analyze a statistic that indicates when the discrete copula is symmetric and obtain its main statistical properties under independence. The results obtained are useful in designing a nonparametric test for symmetry of copulas.

Michiels and De Schepper tackle an important and usually forgotten problem in the applications of copulas, namely that of avoiding the wrong choice of copula in setting up a statistical model. To this end, they introduce and discuss the "test space problem" as a part of the whole copula fitting process. In particular, they explain how an efficient copula test space can be constructed by taking into account information about the existing dependence and present a complete overview of bivariate test spaces for all possible situations. The practical use is illustrated by means of a numerical application based on an illustrative portfolio.

Rodríguez-Lallena and Úbeda-Flores consider the concept of quasi-copula, a generalization of a copula used especially in finding bounds for distribution functions [2, 7]. Specifically, they introduce and characterize the class of multivariate quasi-copulas with quadratic sections in one variable. Some interesting examples are illustrated.

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