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TRANSFER FUNCTION EQUIVALENCE OF FEEDBACK/FEEDFORWARD COMPENSATORS[1]

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Equivalence of several feedback and/or feedforward compensation schemes in linear systems is investigated. The classes of compensators that are realizable using static or dynamic, state or output feedback are characterized. Stability of the compensated system is studied. Applications to model matching are included.

1. INTRODUCTION

This is a *tutorial* which presents a study of equivalence, from the transfer function point of view, of several commonly used feedback and/or feedforward compensation schemes. Two compensators will be called transfer-function equivalent if their application to the given system results in systems that have the same transfer function. It is shown that a cascade compensator is transfer-function equivalent to a two-degree-of-freedom compensator as well as to a static feedback applied to a dynamic extension of the system.

The subclasses of these compensators that are equivalent to a standard static or dynamic, state or output feedback are identified. The proofs are constructive and provide simple design procedures.

Two transfer-function equivalent compensators can have different internal properties. That is why an original result on the stability of the overall closed-loop system is included.

These results are important *per se* in linear system theory. They are also useful in applications. A typical application area is the model matching problem. The results presented allow splitting the problem in two linear subproblems: first a cascade compensator is determined to achieve the match and then realized in terms of the configuration desired.

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2. CLASSES OF COMPENSATORS

We shall study several common feedback and/or feedforward configurations with an eye on the equivalence of various compensation schemes.

Consider a linear system governed by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) \tag{1}$$

where $u \in R^m$ is the input, $x \in R^n$ is the state, and $y \in R^p$ is the output. The system gives rise to the tranfer functions

$$T(s) = (sI - A)^{-1}B \tag{2}$$

$$T'(s) = C(sI - A)^{-1}B \tag{3}$$

which are rational, strictly proper $n \times m$ matrices.

A common compensation scheme used to modify (1) is the *static state feedback* defined by

$$u(s) = Fx(s) + Gv(s) \tag{4}$$

where $v \in R^m$ is an external input and F, G are constant matrices.

A more general compensator is one which involves a *dynamic state feedback* according to the equation

$$u(s) = F(s)\,x(s) + Gv(s) \tag{5}$$

where F is a proper rational matrix and G is constant.

A set of p integrators

$$\dot{x}'(t) = u'(t)$$

can be adjoined to system (1) to give an extended system. A *static state feedback applied to the extended system* according to the equations

$$\begin{aligned} u(s) &= F_{11}x(s) + F_{12}x'(s) + G_1v(s) \\ u'(s) &= F_{21}x(s) + F_{22}x'(s) + G_2v(s) \end{aligned} \tag{6}$$

will result in a dynamic compensation relative to the original system (1).

One can define a compensator of the form

$$u(s) = F(s)\,x(s) + G(s)\,v(s) \tag{7}$$

which makes explicit the presence of a dynamic state feedback as well as a dynamic feedforward, the so-called *two-degree-of-freedom compensator*. Here F and G are proper rational matrices of appropriate sizes.

The equation

$$u(s) = K(s)\,v(s) \tag{8}$$

where K is a proper rational matrix, defines a pure feedforward dynamic compensator, or *cascade compensator*, which is frequently used in the classical control theory.

Output feedback can be used in lieu of state feedback. In particular, *static output feedback* is defined by

$$u(s) = F'y(s) + G'v(s) \tag{9}$$

where F' and G' are constant matrices, while

$$u(s) = F'(s)\,y(s) + G'v(s) \tag{10}$$

is a *dynamic output feedback* when F' is a proper rational matrix and G' is constant.

Similarly, one can consider a *static output feedback applied to the extended system* according to the equations

$$\begin{aligned} u(s) &= F'_{11}y(s) + F'_{12}x'(s) + G'_1v(s) \\ u'(s) &= F'_{21}y(s) + F'_{22}x'(s) + G'_2v(s) \end{aligned} \tag{11}$$

or a *two-degree-of-freedom compensator* of the form

$$u(s) = F'(s)\,y(s) + G'(s)\,v(s) \tag{12}$$

where F' and G' are proper rational matrices, or again a *cascade compensator*

$$u(s) = K'(s)\,v(s) \tag{13}$$

where K' is a proper rational matrix.

3. TRANSFER FUNCTION EQUIVALENCE

Consider the classes of compensators defined by (4) – (13). Each class is obtained by allowing F, G or F', G' or K, K' to vary within the specified limits.

Two compensator classes are said to be *transfer function equivalent* if, for any compensator of one class, one can find a compensator in the other class such that their application to the given system (1) will result in systems that have the same transfer function.

This kind of equivalence reflects just the ability of two compensators to produce the same input-output behaviour. In particular this equivalence says nothing about the dynamical order, stability, or other properties of the systems which depend on a particular realization. This problem will be addressed later.

Our first goal is to investigate which classes are transfer function equivalent.

Lemma 1. [4], [7] The compensator classes (6), (7), and (8) are transfer function equivalent.

P r o o f . We shall establish the following chain of implications.

We first show that each compensator (6) can be represented in the form (7). To this end we apply (6) to the extended system to obtain the overall system equations

$$\begin{aligned} \dot{x}(t) &= (A + BF_{11})\,x(t) + BF_{12}x'(t) + BG_1v(t) \\ \dot{x}'(t) &= F_{21}x(t) + F_{22}x'(t) + G_2v(t) \\ u(t) &= F_{11}x(t) + F_{12}x'(t) + G_1v(t) \end{aligned}$$

and calculate the transfer functions from x and v to u. On identifying with (7), one obtains

$$F(s) = F_{11} + F_{12}(sI - F_{22})^{-1}F_{21}$$
$$G(s) = G_1 + F_{12}(sI - F_{22})^{-1}G_2.$$

Since $sI - F_{22}$ has a strictly proper inverse, both F and G are proper rational matrices.

We now show that any compensator (7) can be realized in the form (8). To see this, we apply (7) to equation (1) in the transfer function form (2),

$$x(s) = T(s)\,u(s)$$

and calculate the transfer function from v to u. Comparing with (8), one obtains

$$K(s) = [I - F(s)\,T(s)]^{-1}\,G(s).$$

Since T is strictly proper, and F is proper, $I - FT$ is bi-proper. Hence K is proper.

Finally let us show that each compensator (8) can be represented in the form (6). Given a proper rational K, let

$$K(s) = \overline{C}(sI - \overline{A})^{-1}\overline{B} + \overline{D}$$

for some state-space ralization $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$. Then

$$\begin{array}{lll} F_{11} = 0 & F_{12} = \overline{C} & G_1 = \overline{D} \\ F_{21} = 0 & F_{22} = \overline{A} & G_2 = \overline{B} \end{array}$$

define a state feedback of the form (6). □

Lemma 2. [8] The compensator classes (11), (12), and (13) are transfer function equivalent.

P r o o f . Following the pattern of Lemma 1, we shall prove the following chain of implications.

We first show that each compensator (11) can be represented in the from (12). To see this, we apply (11) to the extended system to obtain the overall system equations

$$\begin{array}{rcl} \dot{x}(t) & = & (A + BF'_{11}C)\,x(t) + BF'_{12}x'(t) + BG'_1v(t) \\ \dot{x}'(t) & = & F'_{21}Cx(t) + F'_{22}x'(t) + G'_2v(t) \\ y(t) & = & Cx(t) \\ u(t) & = & F'_{11}y(t) + F'_{12}x'(t) + G'_1v(t) \end{array}$$

and calculate the transfer functions from y and v to u. On identifying with (12), one obtains

$$\begin{array}{rcl} F'(s) & = & F'_{11} + F'_{12}(sI - F'_{22})^{-1}F'_{21} \\ G'(s) & = & G'_{11} + F'_{12}(sI - F'_{22})^{-1}G'_2. \end{array}$$

Since $sI - F'_{22}$ has a strictly proper inverse, both F' and G' are proper rational matrices.

We now show that any compensator (12) can be represented in the form (13). To this end we apply (12) to equations (1) in the transfer function form (3),

$$y(s) = T'(s)\,u(s)$$

and calculate the transfer function from v to u. Comparing with (12), one obtains

$$K'(s) = [I - F'(s)\,T'(s)]^{-1}G'(s).$$

Since T' is strictly proper and F' is proper, $I - F'T'$ is biproper. Hence K' is proper.

Finally let us show that any compensator (13) can be realized in the form (11). Given a proper rational K', let

$$K'(s) = \overline{C}'(sI - \overline{A}')^{-1}\overline{B}' + \overline{D}'$$

for some state-space realization $(\overline{A}', \overline{B}', \overline{C}', \overline{D}')$. Then

$$\begin{array}{lll} F'_{11} = 0 & F'_{12} = \overline{C}' & G'_1 = \overline{D}' \\ F'_{21} = 0 & F'_{22} = \overline{A}' & G'_2 = \overline{B}' \end{array}$$

define an output feedback of the form (11). □

Note that the pure feedforward compensators (8) and (13) can be equally realized with state or output feedback. Therefore Lemma 1 and Lemma 2 can be combined to give the following result.

Theorem 1. The compensator classes (6), (7), (8) and (11), (12), (13) are transfer function equivalent.

In view of this equivalence, and the special role played by (8) or (13), the cascade compensator (8) will be used to represent any of the above feedback/feedforward compensators.

The class of static/dynamic state feedback compensators (4) and (5) as well as the class of static/dynamic output feedback compensators (9) and (10) is less general than (8) and will be studied in the sections to follow.

4. DYNAMIC STATE FEEDBACK

Dynamic state feedback (5) is a special case of (6), hence of (8). It is interesting to identify the subclass of cascade compensators K which are transfer function equivalent to dynamic state feedback.

These compensators satisfy

$$K(s) = [I - F(s)\,T(s)]^{-1}G. \tag{14}$$

We impose a restrictive assumption that G is non-singular; this will greatly simplify the analysis [3].

Theorem 2. [1], [7] Given a proper rational $m \times m$ matrix K, there exist a proper rational F and a constant non-singular G such that (8) holds if and only if K is bi-proper.

Proof. Since T is strictly proper and F is proper, $I - FT$ is bi-proper. Since G is non-singular, K is bi-proper as well.

Conversely, suppose that K is bi-proper. Let G be defined by

$$G = K(\infty).$$

Then $V(s) = K^{-1}(s) - G^{-1}$ is a strictly proper rational matrix. The equation

$$V(s) = X(s)\,T(s) \tag{15}$$

has a proper rational solution X if and only if the infinite zero structure of T coincides with that of $\begin{bmatrix} T \\ V \end{bmatrix}$. The infinite zero structure of T is given by $(s^{-1}, \dots, s^{-1})$, see [8]. Since V is strictly proper, the solvability condition is verified and a proper rational X exists that satisfy (15). Let F be defined by

$$F(s) = -GX(s).$$

Then

$$K^{-1}(s) = G^{-1} - G^{-1}F(s)\,T(s)$$

and (14) holds. □

5. STATIC STATE FEEDBACK

This is a further specialization in which both F and G are constant. Which cascade compensators K are transfer function equivalent to static state feedback (4)? Those which satisfy

$$K(s) = [I - FT(s)]^{-1}G. \tag{16}$$

We again assume that G is non-singular and write T in the form

$$T(s) = N(s)\,D^{-1}(s) \tag{17}$$

where N and D are right coprime polynomial matrices.

Theorem 3. [2], [7] Given a proper rational $m \times m$ matrix K, there exist constant matrices F and G with G non-singular, such that (16) holds if and only if

(a) K is bi-proper

(b) $K^{-1}D$ is polynomial.

Proof. Condition (a) follows from Theorem 2. Then

$$K^{-1}(s)\,D(s) = G^{-1}D(s) - G^{-1}F\,N(s)$$

is a polynomial matrix, which is (b).

Conversely, let K satisfy (a) and define G by

$$G = K(\infty).$$

Then $V(s) = K^{-1}(s) - G^{-1}$ is a strictly proper rational matrix. Furthermore, let K satisfy (b). Then

$$V(s) = M(s)\, D^{-1}(s)$$

for a polynomial matrix M. Polynomial row vectors $w(s)$ such that $w(s)\, D^{-1}(s)$ is strictly proper form an $R-$linear space $\mathcal{V}$. Using (17), we have

$$T(s) = N(s)\, D^{-1}(s)$$

and note that the rows of N span $\mathcal{V}$. Therefore the equation

$$V(s) = XT(s) \tag{18}$$

has a constant solution X and

$$F = -GX$$

makes (16) hold. □

If system (1) is controllable, then the rows of N form a basis for $\mathcal{V}$ and the matrices F, G that realize K are unique.

6. DYNAMIC OUTPUT FEEDBACK

Dynamic output feedback (10) is a special case of (12), hence also of (8). It is of interest to identify the subclass of cascade compensators K which are transfer function equivalent to a dynamic output feedback.

These compensators satisfy

$$K(s) = [I - F'(s)\, T'(s)]^{-1} G'. \tag{19}$$

We impose a restrictive assumption that G' is non-singular. This will simplify the analysis [3].

Theorem 4. [8] Given a proper rational $m \times m$ matrix K, there exist a proper rational F' and a constant non-singular G' such that (19) holds if and only if

(a) K is bi-proper

(b) T' and $\begin{bmatrix} T' \\ K_{SP}^{-1} \end{bmatrix}$ have identical infinite zero structure, where K_{SP}^{-1} denotes the strictly proper part of K^{-1}.

P r o o f . Since T' is strictly proper and F' is proper, $I - F'T'$ is bi-proper. Since G' is non-singular, K is bi-proper as well. This is (a).

Write

$$K^{-1}(s) = G'^{-1} - G'^{-1}F'(s)\,T'(s)\,.$$

Then

$$K_{SP}^{-1}(s) = -G'^{-1}F'(s)\,T'(s)$$

and

$$\begin{bmatrix} T'(s) \\ K_{SP}^{-1}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G'^{-1}F'(s) & I \end{bmatrix} \begin{bmatrix} T'(s) \\ 0 \end{bmatrix}.$$

This proves (b), for the two matrices are related by a bi-proper transformation.

Conversely, suppose that K satisfies (a) and define G' by

$$G' = K(\infty).$$

Then $V(s) = K^{-1}(s) - G'^{-1} = K_{SP}^{-1}(s)$, the strictly proper part of $K^{-1}(s)$. In view of (b), the equation

$$V(s) = X'(s)\,T'(s) \tag{20}$$

has a proper rational solution X', see [8]. Define

$$F'(s) = -G'X'(s).$$

Then

$$K^{-1}(s) = G'^{-1} - G'^{-1}F'(s)\,T'(s)$$

and (19) holds. □

A comparison of Theorem 2 and Theorem 4 reveals that the class of cascade compensators that can be realized via dynamic output feedback is a subclass of those that are realizable using a dynamic state feedback. It is the condition (b) of Theorem 4 that makes the difference. This conditon is needed to solve equation (20). Its state feedback counterpart, equation (15), has a guaranteed solution thanks to a special infinite zero structure of the input-state transfer function T. This property is not shared by T', hence solvability of (20) must be ensured by an assumption.

7. STATIC OUTPUT FEEDBACK

This is a further restriction which requires both F' and G' to be constant. Which cascade compensators K are transfer function equivalent to static output feedback (9)? Those which satisfy

$$K(s) = [I - F'T'(s)]^{-1}G'. \tag{21}$$

We again assume that G' is non-singular. Using (17), write T' in the form

$$\begin{aligned} T'(s) &= CT(s) \\ &= CN(s)\,D^{-1}(s) \\ &= N'(s)\,D^{-1}(s) \end{aligned} \tag{22}$$

where N' and D are polynomial matrices, not necessarily right coprime.

Theorem 5. [8] Given a proper rational $m \times m$ matrix K, there exist constant matrices F' and G' with G' non-singular, such that (20) holds if and only if

(a) K is bi-proper

(b) $K^{-1}D$ is polynomial

(c) N' and $\begin{bmatrix} N' \\ K_{SP}^{-1}D \end{bmatrix}$ have identical row span in $\mathcal{V}$.

Proof. Condition (a) follows from Theorem 4. Then

$$K^{-1}(s)\,D(s) = G'^{-1}D(s) - G'^{-1}F'N'(s)$$

is a polynomial matrix, which is (b). Furthermore,

$$K_{SP}^{-1}(s)\,D(s) = -G'^{-1}F'N'(s).$$

This shows that the row span of $K_{SP}^{-1}D$ is included in that of N'. Consequently (c) holds.

Conversely, let K satisfy (a) and define G' by

$$G' = K(\infty).$$

Then $V(s) = K^{-1}(s) - G'^{-1} = K_{SP}^{-1}(s)$, the strictly proper part of $K^{-1}(s)$. Furthermore, let K satisfy (b). Then $V(s)\,D(s)$ is a polynomial matrix. In view of (c), the equation

$$V(s)\,D(s) = X'N'(s) \tag{23}$$

has a constant solution X'. Letting

$$F' = -G'X'$$

we obtain (21), which completes the proof. □

Comparing Theorem 3 with Theorem 5 we observe that the class of cascade compesansotors that can be realized via static output feedback is a subclass of those that are realizable using a static state feedback. The additional property needed is the conditon (c) of Theorem 5. This condition ensures that equation (23) has a constant solution. Its state feedback counterpart, equation (18), has a guaranteed solution as the rows of N span the $R-$linear space $\mathcal{V}$. The rows of N' span only a subspace of $\mathcal{V}$, hence solvability of (23) must be secured by an assumption.

8. STABILITY

Transfer function equivalent compensators can have different internal properties, those which depend on a particular realization.

Stability is the most important design specification of this sort. That is why it is natural to ask when a compensator, in one of of the forms (4) – (7) or (9) – (12), which is transfer function equivalent to a cascade compensator (8) or (13), stabilizes the system.

The requirement of stability will mean that the states of the system and of the compensator go to zero from all initial values. A necessary requisite is of course that system (1) is stabilizable and, in the case of output feedback, also detectable.

For the general configuration of the compensator, namely (6), (7) or (11), (12), only general stability checks are available. Thus, for static state feedback (6) applied to an extended system, the state-transition matrix

$$\begin{bmatrix} A + BF_{11} & BF_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

should be a stability matrix. Similarly, for static output feedback (11) applied to an extended system, the matrix

$$\begin{bmatrix} A + BF'_{11}C & BF'_{12} \\ F'_{21}C & F'_{22} \end{bmatrix}$$

should be a stability matrix. In the case of a two-degree-of-freedom compensator (7) based on state feedback, we write as in (17)

$$T(s) = N(s)\, D^{-1}(s)$$

where N, D is a pair of right coprime polynomial matrices and

$$F(s) = -P^{-1}(s)\, Q(s), \quad G(s) = P^{-1}(s)\, R(s) \tag{24}$$

where P, Q, R is a triple of left coprime polynomial matrices. Then [5] the matrix

$$(PD + QN)^{-1}(s)$$

should be a stable (i.e., analytic in $Re\, s \geq 0$) rational matrix. Similarly, when a two-degree-of-freedom compensator (12) based on output feedback is used, we write as in (22)

$$T'(s) = N'(s)\, D^{-1}(s)$$

(the polynomial matrices N' and D may not be right coprime, but their common right divisors are stable by the assumption of stabilizability and detectability) and

$$F'(s) = -P'^{-1}(s)\, Q'(s),\ G'(s) = P'^{-1}(s)\, R'(s) \tag{25}$$

where P', Q', R' is a triple of left coprime polynomial matrices. Then [5]

$$(P'D + Q'N')^{-1}(s)$$

should be a stable rational matrix.

When the compensator is realized as a dynamic state or output feedback, see (5) and (10), then G and G' are constant non-singular matrices and simplified stability checks are available which make use of the underlying transfer-function equivalent precompensator (8) or (13). Indeed, write

$$G^{-1}F(s) = -\overline{P}^{-1}(s)\, \overline{Q}(s)$$

where $\overline{P}$ and $\overline{Q}$ is a pair of left coprime polynomial matrices. Then $\overline{P}$, $\overline{Q}$ is related with P, Q, R defined in (24) as

$$P = \overline{P}G^{-1}, \quad Q = \overline{Q}, \quad R = \overline{P}$$

and

$$\begin{aligned}(PD + QN)^{-1}(s) &= D^{-1}(s)[I - F(s)\,T(s)]^{-1}P^{-1}(s) \\ &= D^{-1}(s)\,K(s)\,\overline{P}^{-1}(s)\end{aligned}$$

on using (14). Thus a dynamic state feedback (5) will stabilize system (1) if and only if $D^{-1}K\overline{P}^{-1}$ is a stable rational matrix, where K is the transfer-function equivalent cascade compensator (14). In the case of dynamic output feedback (10), write

$$G'^{-1}F'(s) = -\overline{P}'^{-1}(s)\overline{Q}'(s)$$

where $\overline{P}'$ and $\overline{Q}'$ is a pair of left coprime polynomial matrices. Then $\overline{P}', \overline{Q}'$ is related with P', Q', R' defined in (25) as

$$P' = \overline{P}'G'^{-1}, \quad Q' = \overline{Q}', \quad R' = \overline{P}'$$

and

$$\begin{aligned}(P'D + Q'N')^{-1}(s) &= D^{-1}(s)\,[I - F'(s)\,T'(s)]^{-1}P'^{-1}(s) \\ &= D^{-1}(s)\,K(s)\overline{P}'^{-1}(s)\end{aligned}$$

on using (19). Thus a dynamic output feedback (10) will stabilize system (1) if and only if $D^{-1}K\overline{P}'^{-1}$ is a stable rational matrix, where K is the transfer-function equaivalent cascade compensator (19).

These results are particularly useful when K is realized using static state or output feedback, see (4) and (9). Then a further simplification occurs: F and F' are constant as well, which entails that $\overline{P}$ and $\overline{P}'$ are constant matrices. Then one can tell whether the static state or output feedback will stabilize system (1) from $D^{-1}K$, where K is the underlying transfer-function equivalent cascade sompensator given by (16) or (21), depending on the type of feedback in question. In fact, $K^{-1}D$ is a polynomial matrix in these cases and its determinant is the pole polynomial of the closed-loop system [6].

9. MODEL MATCHING

A typical application of the above results is the problem of *model matching* [7], [9], [10]. Given a plant

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with a strictly proper, rational $l \times m$ transfer function matrix T_P of rank m and a model transfer function matrix T_M, which is assumed to be also strictly proper, rational, and of size $l \times m$ and rank m. We seek to find a compensator, specified in one of the forms (4) – (7) and (9) – (12), such that the closed-loop system is stable and has transfer matrix T_M.

To make contact with the preceding sections, we recall (2) and (3) and identify T_P with T'. Then the model matching equation, namely

$$T_P(s)[I - F(s)\,T(s)]^{-1}G(s) = T_M(s)$$

relevant for compensators (4) – (7), or

$$T_P(s)\,[I - F'(s)\,T'(s)]^{-1}G'(s) = T_M(s)$$

in the case of compensators (9) – (12), immediately suggests the following two-step solution: determine a matching cascade compensator K from the equation

$$T_P(s)\,K(s) = T_M(s) \tag{26}$$

and then realize K in one of the forms (4) – (7) desired,

$$K(s) = [I - F(s)\,T(s)]^{-1}G(s)$$

where F and G are either proper rational or constant matrices, or in one of the forms (9) – (12),

$$K(s) = [I - F'(s)\,T'(s)]^{-1}G'(s),$$

where F' and G' are either proper rational or constant matrices.

The assumptions that T_P and T_M have full column rank m secure that the model matching equation (26) has at most one rational matrix solution K.

The matching equation (26) has a proper rational solution K if and only if the matrices $[T_P \;\; T_M]$ and T_P have identical infinite zero structure [8]. In the scalar case, this means that the relative degree of T_P does not exceed that of T_M.

Using the equivalence result provided by Theorem 1, the above condition is necessary and sufficient to achieve the match via any of the two-degree-of-freedom compensation schemes (6), (7) or (11), (12).

Suppose we want to implement dynamic state feedback (5). Theorem 2 requires that K be bi-proper. Thus the equation

$$T_M(s)\,K^{-1}(s) = T_P(s)$$

should have a proper rational solution $K^{-1}(s)$. This is the case if and only if the matrices $[T_P \;\; T_M]$ and T_M have identical infinite zero structure [8]. Combining the two conditions, a match via (5) is possible if and only if T_P and T_M have identical infinite zero structure. This reduces to identical relative degrees in the scalar case.

Finally, let us realize the match using static state feedback (4). Theorem 3 imposes a further condition that $K^{-1}D$ be polynomial. Writing T_P and T_M in terms of their right coprime polynomial factorizations,

$$T_P(s) = N_P(s)\,D^{-1}(s)$$
$$T_M(s) = N_M(s)\,E^{-1}(s)$$

and using (26), we observe that

$$K^{-1}(s)\,D(s) = E(s)\;N_M^{-1}(s)\;N_P(s)$$

is a polynomial matrix if and only if N_M divides N_P on the left. This means that the equation

$$N_M(s)\,X(s) = N_P(s)$$

must be solvable for a polynomial matrix X. A necessary and sufficient condition is that the matrices $[T_P \;\; T_M]$ and T_M have identical finite zeros structures [8].

Having achieved the match desired, we can check for stability of the closed-loop system. In the case of static state feedback, $D^{-1}K$ is required to be stable, which means that the equation

$$N_P(s)\,Y(s) = N_M(s)$$

is to have a stable rational solution Y. Thus a stable match can be achieved if and only if the matrices $[T_P \;\; T_M]$ and T_P have identical finite unstable zeros structures [8]. In the scalar case, this amounts to the requirement that all non-minimum-phase zeros of T_P must be included in T_M.

In case the match is to be achieved via output feedback, additional conditions must be satisfied, viz. Theorem 4(b) and Theorem 5(c). These conditons, however, involve deeper properties of T_P and T_M than just their finite or infinite zeros. An example is included to illustrate the application of transfer function equivalence to model matching.

Example 1. Consider a plant (1) given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [\,-1 \quad -1\,]$$

with the input-state transfer function

$$T(s) = \frac{1}{s^2+s+1}\begin{bmatrix} 1 \\ s \end{bmatrix}$$

and the input-output transfer function

$$T_P(s) = \frac{s-1}{s^2+s\;+1}.$$

Which models $T_M(s)$ of McMillan degree less than or equal to 2 can be matched with this plant using dynamic/static state feedback and dynamic/static output feedback?

For dynamic state feedback (5), the relative degree of T_M should equal that of T_P, hence 1. This gives the model class

$$T_{M5}(s) = c\frac{s+b}{s^2+a_1 s+a_0}$$

where a_0, a_1, b, and $c \neq 0$ vary over real numbers.

For static state feedback (4), T_{M5} should have in addition either one zero at 1 or no finite zero at all. This yield the model class

$$T_{M4}(s) = c\frac{s-1}{s^2+a_1s+a_0}$$

where a_0, a_1, and $c \neq 0$ are any real numbers. The case of no finite zero occurs when $1 + a_1 + a_0 = 0$.

For dynamic output feedback (10), the model class T_{M5} is further constrained by the condition (b) of Theorem 4. However, our particular T_P has relative degree 1 and so has

$$K_{SP}^{-1}(s) = \frac{1}{c}\frac{(a_1-b-2)s^2+(a_0-a_1-b-1)\,s-(a_0+b)}{(s^2+s\ +1)(s+b)}. \tag{27}$$

Thus no further constraint applies and the achievable model class is

$$T_{M10}(s) = T_{M5}(s).$$

For static output feedback (9), the model class T_{M4} is further constrained by the condition (c) of Theorem 5. We calculate

$$K_{SP}^{-1}(s) = \frac{1}{c}\frac{(a_1-1)s\ -(1-a_0)}{s^2+s\ +1} \tag{28}$$

and align its numerator with that of T_P. This results in $a_1 - 1 = 1 - a_0$ and the achievable class is given by

$$T_{M9}(s) = c\frac{s-1}{s^2+(2-a_0)\,s+a_0}$$

where a_0 is any real number.

Let us now check for the ability of the above compensation schemes to stabilize the system. The dynamic state feedbacks (5) that achieve T_{M5} are given by (15) as

$$\begin{aligned} F(s) &= -\frac{1}{s+b}[\,\tau s\ -(a_0+b)(a_1-b-2)\,s+(a_0-a_1-b-1-\tau)\,] \\ G &= c \end{aligned} \tag{29}$$

where τ is any real parameter. Thus

$$D^{-1}KP^{-1} = \frac{c}{(s+1)(s^2+a_1s+a_0)}$$

and (29) can never stabilize (1) unless $F(s)$ is constant.

The static state feedback (4) that achievers T_{M4} is given by (18) as

$$F = [\,1-a_0 \quad 1-a_1\,], \quad G = c$$

and

$$D^{-1}K = \frac{c}{s^2+a_1s+a_0}$$

stable implies the constraint $a_0 > 0$, $a_1 > 0$.

The dynamic output feedback (10) that achieves T_{M10} is given by (20) as

$$\begin{aligned} F'(s) &= -\frac{(a_1 - b - 2)s^2 + (a_0 - a_1 - b - 1)s - (a_0 + b)}{(s-1)(s+b)} \\ G' &= c \end{aligned} \tag{30}$$

and again (30) cannot stabilize (1) unless F' is constant.

The static output feedback (9) that achieves T_{M9} is given by (23) as

$$F' = a_0 - 1, \quad G' = c$$

and

$$D^{-1}K = \frac{c}{s^2 + (s - a_0)\, s + a_0}$$

stable implies the constraint $0 < a_0 < 2$.

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