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GENERALIZED BEZOUTIAN FOR A PERIODIC COLLECTION OF RATIONAL MATRICES[1]

Carmen Coll, Rafael Bru and Vicente Hernández

We introduce the concept of periodic generalized Bezoutians associated with discrete-time linear periodic systems or periodic rational matrices. Given a periodic collection of rational matrices, we characterize the dimension of minimal periodic realizations by means of the rank of the associated periodic generalized Bezoutian matrix.

1. INTRODUCTION

The usual Bezoutian matrix involves a pair of scalar polynomials and provides information concerning coprimeness and greatest common divisor of polynomials. Helmke and Fuhrmann [7] surveyed this theory and gave some connections with realization problems. Anderson and Jury [1] extended the concept of Bezoutian to a quadruple of polynomial matrices and proved that the rank of the generalized Bezoutian matrix is equal to the McMillan degree of the transfer matrix. Other authors, Bakri [2], Wimmer [10], studied the relationships between generalized Bezoutians and Hankel matrices. Lerer and Tismenetsky [8], gave a natural generalization of the classical Bezoutian matrix of two polynomials for a family of several matrix polynomials. Gohberg and Shalom [5] defined the H-Bezoutian and T-Bezoutian, where only two matrix polynomials are needed. The H-Bezoutian and T-Bezoutian are connected with Hankel and Toeplitz matrices, respectively. Our goal is to extend some of these results to discrete-time linear periodic systems.

Section 2 has a preliminary character. In Section 3 we give the notion of periodic generalized Bezoutian of a periodic collection of rational matrices and we construct a left (right) matrix fraction description of the rational matrix at time s from a left (right) matrix fraction description of the rational matrix at time 0. Section 4 contains relations between the periodic Hankel and periodic Bezoutian matrices associated with a periodic collection of rational matrices and we characterize the dimension of minimal periodic realizations by means of the rank of the associated periodic generalized Bezoutian matrix.

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2. REALIZATIONS OF PERIODIC RATIONAL MATRICES

Consider the discrete-time linear N-periodic system, $N \in Z^+$,

$$\left.\begin{aligned} x(k+1) &= A(k)\,x(k) + B(k)\,u(k) \\ y(k) &= C(k)\,x(k), \end{aligned}\right\} \tag{1}$$

with $A(k+N) = A(k) \in \mathbb{R}^{n\times n}$, $B(k+N) = B(k) \in \mathbb{R}^{n\times m}$ and $C(k+N) = C(k) \in \mathbb{R}^{p\times n}$, $k \in Z$. This system is denoted by $(C(\cdot),\, A(\cdot),\, B(\cdot))_N$. The invariant system associated with the periodic system (1) at time s, $s \in Z$, is defined by (see [9]),

$$\left.\begin{aligned} x_s(k+1) &= A_s\,x_s(k) + B_s\,u_s(k) \\ y_s(k) &= C_s\,x_s(k) + E_s\,u_s(k), \end{aligned}\right\} \tag{2}$$

with

$$\begin{aligned} x_s(k) &= x(s+kN), \\ u_s(k) &= \operatorname{col}\left[u(s+kN),\, u(s+kN+1), \ldots, u(s+kN+N-1)\right], \\ y_s(k) &= \operatorname{col}\left[y(s+kN),\, y(s+kN+1), \ldots, y(s+kN+N-1)\right], \end{aligned}$$

where

$$\begin{aligned} A_s &= \phi_A(s+N,\, s) \in \mathbb{R}^{n\times n}, \\ B_s &= [\phi_A(s{+}N,\, s{+}1)\,B(s),\, \phi_A(s{+}N,\, s{+}2)\,B(s{+}1), \ldots, B(s{+}N{-}1)] \in \mathbb{R}^{n\times mN}, \\ C_s &= \operatorname{col}\left[C(s),\, C(s{+}1)\,\phi_A(s{+}1,\, s), \ldots, C(s{+}N{-}1)\,\phi_A(s{+}N{-}1,\, s)\right] \in \mathbb{R}^{pN\times n}, \\ E_s &= \left[E^s_{ij}\right], \quad E^s_{ij} \in \mathbb{R}^{p\times m}, \quad i,\, j = 1, \ldots, N, \\ E^s_{ij} &= 0, \quad i \le j, \\ E^s_{ij} &= C(s+i-1)\,\phi_A(s+i-1,\, s+j)\,B(s+j-1), \quad i > j, \end{aligned}$$

and $\Phi_A(k, k_0)$, $k \ge k_0$, is the transition matrix of (1). This system will be denoted by (C_s, A_s, B_s, E_s).

The transfer matrix of the periodic system (1) at time s, $s \in Z$, was defined by Grasselli and Longhi [6] as

$$G_s(z) = C_s(zI - A_s)^{-1}\,B_s + E_s \in \mathbb{R}^{pN\times mN}(z). \tag{3}$$

The rational matrix $G_s(z)$ is proper with a strictly lower block-triangular polynomial part. All such $G_s(z)$ matrices constitute a ring with identity element I.

The same authors proved the following relation between the transfer matrix of (1), at consecutive times s and $s+1$

$$G_{s+1}(z) = S_{1,p}(z)\,G_s(z)\,S^{-1}_{1,m}(z), \tag{4}$$

where the matrices $S_{1,p}(z)$, $S_{1,m}(z)$ are

$$S_{1,t}(z) = \begin{bmatrix} 0 & I_{(N-1)t} \\ zI_t & 0 \end{bmatrix}.$$

Note that the product of two transfer matrices $G_s(z)\, H_s(z)$ and the inverse $G_s^{-1}(z)$ get transformed by expression (4) in the same way as $G_s(z)$.

From the periodicity of the system (1) the invariant systems (2) satisfy $(C_{s+N},\, A_{s+N},\, B_{s+N},\, E_{s+N}) = (C_s,\, A_s,\, B_s,\, E_s)$, and therefore the transfer matrices satisfy $G_{s+N}(z) = G_s(z)$.

Now, we give the definition of a periodic realization of a periodic collection of rational matrices.

Definition 1. Consider the periodic collection of rational matrices

$$\{H_s(z),\, s \in Z\}\,, \quad H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN\times mN}(z). \tag{5}$$

The periodic system $(C(\cdot),\, A(\cdot),\, B(\cdot))_N$ is a *periodic realization* of (5) if $H_s(z) = C_s(zI - A_s)^{-1}\, B_s + E_s$. The size of $A(\cdot)$ is called the dimension of the realization. A periodic realization is called a minimal periodic realization if there exists no other periodic realization having a lower dimensional state vector.

In [3] we proved the following result.

Theorem 1. The periodic collection of rational matrices given by (5) has a periodic realization, if and only if,

$$H_{s+1}(z) = S_{1,p}(z)\, H_s(z)\, S_{1,m}^{-1}(z), \tag{6}$$

and $H_0(z)$ is proper with strictly lower block-triangular polynomial part.

Remark 1. From (6) if $H_0(z)$ satisfies the above condition then $H_s(z),\ s \in Z$ are also proper rational matrices with strictly lower block-triangular polynomial parts.

3. PERIODIC GENERALIZED BEZOUTIAN

Consider the transfer matrix of the periodic system (1) at time s, given by (3). Let

$$G_s(z) = D_s^{-1}(z)\, N_s(z) = R_s(z)\, P_s^{-1}(z), \tag{7}$$

be a left and right polynomial matrix fraction description of $G_s(z)$, where $D_s(z)$, $N_s(z)$, $R_s(z)$ and $P_s(z)$ are real polynomial matrices with respectively sizes $pN \times pN$, $pN \times mN$, $pN \times mN$ and $mN \times mN$, with $D_s(z)$ and $P_s(z)$ nonsingular. By (7) it readily follows that $D_s(z)\, R_s(z) - N_s(z)\, R_s(z) = 0$. To define the generalized Bezoutian, we construct the following nonsquare polynomial matrices

$$M_s(z) \;=\; [\, D_s(z),\ -N_s(z)\,] \in \mathbb{R}^{pN\times(p+m)\,N}[z], \tag{8}$$

$$L_s(z) \;=\; \begin{bmatrix} R_s(z) \\ P_s(z) \end{bmatrix} \in \mathbb{R}^{(p+m)\,N\times mN}[z]. \tag{9}$$

Let k_s and h_s be the degrees of $M_s(z)$ and $L_s(z)$, respectively. Note that $M_s(z)\, L_s(z) = 0$ then the matrix $M_s(\lambda)\, L_s(\mu)\, /\, (\lambda - \mu)$ is a polynomial matrix in two different variables, λ and μ. Now, the definition of generalized Bezoutian given by Gohberg and Shalom [5], allows us to establish the following definition for a periodic linear system.

Definition 2. (i) The polynomial matrix given by

$$\beta(M_s,\, L_s,\, \lambda,\, \mu) = \frac{M_s(\lambda)\, L_s(\mu)}{\lambda - \mu} = \sum_{i=0}^{k_s-1} \sum_{j=0}^{h_s-1} \beta_{ij}^s\, \lambda^i\, \mu^j, \tag{10}$$

with $\beta_{ij}^s \in \mathbb{R}^{pN\times mN}$ is called the *generalized Bezoutian* associated with the periodic system (1) at time s.

(ii) The *generalized Bezoutian matrix* associated with the periodic system (1) at time s, is the block matrix

$$B(M_s,\, L_s) = \begin{bmatrix} \beta_{00}^s & \beta_{01}^s & \cdots & \beta_{0,h_s-1}^s \\ \beta_{10}^s & \beta_{11}^s & \cdots & \beta_{1,h_s-1}^s \\ \vdots & \vdots & & \vdots \\ \beta_{k_s-1,0}^s & \beta_{k_s-1,1}^s & \cdots & \beta_{k_s-1,h_s-1}^s \end{bmatrix}, \tag{11}$$

whose entries are the coefficients of (10).

Remark 2. (i) The generalized Bezoutian of the periodic system (1) at time s is, by definition, the generalized Bezoutian of the invariant system $(C_s,\, A_s,\, B_s,\, E_s)$.

(ii) By the periodicity of the system (1) it follows that

$$\beta\,(M_{s+N},\, L_{s+N},\, \lambda,\, \mu) = \beta\,(M_s,\, L_s,\, \lambda,\, \mu)\,,$$
$$B\,(M_{s+N},\, L_{s+N}) = B(M_s,\, L_s).$$

(iii) The generalized Bezoutian and the generalized Bezoutian matrix at time s of the periodic collection of rational matrices (5), can be defined by analogous form. It suffices to consider a left and right polynomial matrix fraction description of $H_s(z)$

$$H_s(z) = D_s^{-1}(z)\, N_s(z) = R_s(z)\, P_s^{-1}(z)$$

and use the expressions (8) – (11).

In the following result we prove some properties of the polynomial matrices $S_{s,t}(z)$ defined for any $s \in Z$ and $t = p,\, m$ as follows

$$S_{s,t}(z) = \begin{bmatrix} 0 & z^j\, I_{(N-i)\,t} \\ z^{j+1}\, I_{it} & 0 \end{bmatrix} \tag{12}$$

where $s = jN + i,\ i = 0, 1, \ldots, N-1$. Note that $S_{s,t}(z)$ is an extension of the polynomial matrix $S_{1,t}(z)$ introduced in expression (4).

Lemma 1. The polynomial matrices $S_{s,t}(z)$ given by (12) satisfy the following relations:

i) $S_{s+N,t}(z) = z\, S_{s,t}(z),\ S_{s-N,t}(z) = z^{-1}\, S_{s,t}(z),\ s \in Z,$

ii) $S_{s_1+s_2,t}(z) = S_{s_1,t}(z),\ S_{s_2,t}(z),\ s_1,\, s_2 \in Z.$

P r o o f. (i)

$$S_{s+N,t}(z) = \begin{bmatrix} 0 & z^{j+1}\, I_{(N-i)\,t} \\ z^{j+2}\, I_{it} & 0 \end{bmatrix} = z \begin{bmatrix} 0 & z^{j}\, I_{(N-i)\,t} \\ z^{j+1}\, I_{it} & 0 \end{bmatrix} = z\, S_{s,t}(z),$$

$$S_{s-N,t}(z)= \begin{bmatrix} 0 & z^{j-1}\, I_{(N-i)\,t} \\ z^{j}\, I_{it} & 0 \end{bmatrix} = z^{-1} \begin{bmatrix} 0 & z^{j}\, I_{(N-i)\,t} \\ z^{j+1}\, I_{it} & 0 \end{bmatrix} = z^{-1}\, S_{s,t}(z).$$

(ii) Suppose $s_1 = j_1\, N + i_1,\ s_2 = j_2\, N + i_2,\ i_1,\, i_2 = 0, 1, \ldots, N-1$, and $i_1 + i_2 = j_3\, N + i_3,\ j_3 = 0, 1,\ i_3 = 0, 1, \ldots, N-1$. Then $s_1 + s_2 = (j_1 + j_2 + j_3)\, N + i_3$. By relation (i)

$$\begin{aligned} & S_{s_1,t}(z)\, S_{s_2,t}(z) = z^{j_1+j_2}\, S_{i_1,t}(z)\, S_{i_2,t}(z) = \\ = \ & z^{j_1+j_2} \begin{bmatrix} 0 & I_{(N-i_1)\,t} \\ z\, I_{i_1,t} & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{(N-i_2)\,t} \\ z\, I_{i_2,t} & 0 \end{bmatrix} = \\ = \ & z^{j_1+j_2+j_3}\, S_{i_3,t}(z) = S_{s_1+s_2,t}(z). \end{aligned}$$

□

Using the relation (4), in the following result we construct a left (right) matrix fraction description of the transfer matrix at time s from a left(right) matrix fraction description of the transfer matrix at time 0.

Proposition 1. If $G_0(z) = D_0^{-1}(z)\, N_0(z) = R_0(z)\, P_0^{-1}(z)$ is a left and right polynomial matrix fraction description of the transfer matrix of (1) at time 0, then

$$\begin{aligned} G_s(z) &= \{D_0(z)\, S_{N-s,p}(z)\}^{-1}\ N_0(z)\, S_{N-s,m}(z) \\ &= S_{s,p}(z)\, R_0(z)\ \{S_{s,m}(z)\, P_0(z)\}^{-1} \end{aligned}$$

is a left and right polynomial matrix fraction description of the transfer matrix of (1) at time s, $s \in Z$ where $S_{s,t}(z)$ is the polynomial matrix defined in (12).

P r o o f. Since the transfer matrices of the periodic system (1) satisfies, $G_{s+1}(z) = S_{1,p}(z)\, G_s(z)\, S_{1,m}^{-1}(z),\ s \in Z$, and by property (ii) of Lemma 1 we obtain

$$\begin{aligned} G_s(z) &= S_{1,p}^{s}(z)\, G_0(z)\, S_{1,m}^{-s}(z) \\ &= S_{s,p}(z)\, G_0(z)\, S_{s,m}^{-1}(z), \quad s \in Z. \end{aligned}$$

From the left and right matrix fraction description of $G_0(z)$, we deduce

$$\begin{aligned} G_s(z) &= S_{s,p}(z)\, D_0^{-1}(z)\, N_0(z)\, S_{s,m}^{-1}(z) \\ &= S_{s,p}(z)\, R_0(z)\, P_0^{-1}(z)\, S_{s,m}^{-1}(z). \end{aligned}$$

By Lemma 1, $S_{s,p}(z)\, S_{N-s,p}(z) = S_{N,p}(z) = z\, S_{0,p}(z) = z\, I_{pN}$, then $S_{s,p}(z) = z\, S_{N-s,p}^{-1}(z)$. Now we obtain

$$\begin{aligned} G_s(z) &= z\, S_{N-s,p}^{-1}(z)\, D_0^{-1}(z)\, N_0(z)\, z^{-1}\, S_{N-s,m}(z) \\ &= \{D_0(z)\, S_{N-s,p}(z)\}^{-1}\ N_0(z)\, S_{N-s,m}(z). \end{aligned}$$

Analogously, we have

$$\begin{aligned} G_s(z) &= S_{s,p}(z)\, R_0(z)\, P_0^{-1}(z)\, S_{s,m}^{-1}(z) \\ &= S_{s,p}(z)\, R_0(z)\, \{S_{s,m}(z)\, P_0(z)\}^{-1} . \end{aligned}$$

Remark 3. From these matrix fraction descriptions, we construct the following nonsquare polynomial matrices:

$$\left.\begin{array}{l} M_0(z) = [\, D_0(z),\; -N_0(z)\,] , \\ M_s(z) = [\, D_0(z)\, S_{N-s,p}(z),\; -N_0(z)\, S_{N-s,m}(z)\,] ,\;\; s \in Z \end{array}\right\} \tag{13}$$

$$\left.\begin{array}{l} L_0(z) = \begin{bmatrix} R_0(z) \\ P_0(z) \end{bmatrix} , \\ L_s(z) = \begin{bmatrix} S_{s,p}(z)\, R_0(z) \\ S_{s,m}(z)\, P_0(z) \end{bmatrix} ,\;\; s \in Z \end{array}\right\} \tag{14}$$

Note that

$$\left.\begin{array}{l} M_s(z) = M_0(z)\, \mathrm{diag}\,(S_{N-s,p}(z),\; S_{N-s,m}(z)) , \\ L_s(z) = \mathrm{diag}\,(S_{s,p}(z),\; S_{s,m}(z))\;\, L_0(z). \end{array}\right\} \tag{15}$$

Thus, from a matrix fraction description of the transfer matrix of (1) at time 0, we can, by Definition 2, determine the generalized Bezoutian of the periodic system (1) at time s, $s \in Z$.

Remark 4. All the above results remain valid for the periodic collection of rational matrices (5), provided that the expression (6) holds.

4. MINIMAL REALIZATIONS. PERIODIC BEZOUTIAN AND HANKEL MATRICES

Consider the periodic collection of rational matrices given by (5), $H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN\times mN}(z)$, $s \in Z$. In the following, we assume that this collection satisfies the conditions of Theorem 1.

To characterize the dimension of the minimal periodic realizations of (5) by means of the rank of the corresponding generalized periodic Bezoutian matrix, we need the concept of Hankel matrix associated with a sequence of Markov parameters. The (α, β)th block Hankel matrix associated with the rational matrix $H_s(z)$ is defined by

$$H^s_{\alpha,\beta} = \begin{pmatrix} H^s_0 & H^s_1 & H^s_2 & \cdots & H^s_{\beta-1} \\ H^s_1 & H^s_2 & H^s_3 & \cdots & H^s_{\beta} \\ \vdots & \vdots & \vdots & & \vdots \\ H^s_{\alpha-1} & H^s_{\alpha} & H^s_{\alpha+1} & \cdots & H^s_{\alpha+\beta-2} \end{pmatrix} \in \mathbb{R}^{\alpha pN\times\beta mN}$$

where H^s_j are the Markov parameters of $H_s(z)$ that is

$$H_s(z) = \sum_{j=0}^{\infty} H^s_j \, z^{-j}.$$

By the periodicity of (5) we have that $H^{s+N}_{\alpha,\beta} = H^s_{\alpha,\beta}$.

It was proved that (see [3]), under the conditions of Theorem 1, the periodic collection of rational matrices given by (5) admits the following polynomial matrix fraction descriptions

$$H_s(z) = D_s^{-1}(z)\, N_s(z) = R_s(z)\, P_s^{-1}(z), \quad s \in Z \tag{16}$$

where

$$\left.\begin{aligned} &D_s(z) = z\, d_0(z)\, I_{pN}, \quad P_s(z) = z\, d_0(z)\, I_{mN} \\ &N_s(z) = D_s(z)\, H_s(z), \\ &R_s(z) = H_S(z)\, P_s(z), \end{aligned}\right\}$$

and $d_0(z) = a_r\, z^r + a_{r-1}\, z^{r-1} + \cdots + a_0,\ a_r = 1$, is the monic least common denominator of the elements of the proper rational matrix $H_0(z)$.

Using the relation between the Hankel and the Bezoutian matrices, associated with a rational matrix, we establish the following result for a periodic collection of rational matrices provided that the conditions of Theorem 1 hold.

Proposition 2. The periodic generalized Bezoutian matrix and the periodic $(r+1,\, r+1)$th block Hankel matrix associated with the periodic collection of rational matrices (5) satisfy the following relation

$$B\,(M_s,\, L_s) = T\,(a_0\, I_{pN}, \ldots, a_r\, I_{pN})\ H^s_{r+1,\, r+1}\, T\,(a_0\, I_{mN}, \ldots, a_r\, I_{mN})\,,$$

where $T(a_0\, I_{tN}, \ldots, a_r\, I_{tN})$ is the matrix defined by

$$T\,(a_0\, I_{tN}, \ldots, a_r\, I_{tN}) = \begin{pmatrix} a_0\, I_{tN} & a_1\, I_{tN} & \cdots & a_r\, I_{tN} \\ a_1\, I_{tN} & a_2\, I_{tN} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{r-1}\, I_{tN} & a_r\, I_{tN} & \cdots & 0 \\ a_r\, I_{tN} & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. Given the rational matrix $H_s(z)$ of the collection (5), we know from (16) that $D_s(z) = a_r\, I_{pN}\, z^{r+1} + a_{r-1}\, I_{pN}\, z^r + \cdots + a_0\, I_{pN}\, z$ and $P_s(z) = a_r\, I_{mN}\, z^{r+1} + a_{r-1}\, I_{mN}\, z^r + \cdots + a_0\, I_{mN}\, z$. The result is obtained by using the relation between the Hankel and the Bezoutian matrices associated with a rational matrix, given by Anderson and Jury [1]. □

From this Proposition and using the characterization of minimal periodic realizations by the block Hankel matrix associated with an input-output periodic application, given in [4], we establish the following result.

Theorem 2. The rank of the periodic generalized Bezoutian matrix of (5) at time s is the dimension of the minimal periodic realizations of (5).

Proof. By Proposition 2, the $(r+1,\, r+1)$th block Hankel matrix $H^{s}_{r+1,\, r+1}$ and the periodic Bezoutian matrix at time s, are related by antidiagonal block-triangular matrices with nonsingular antidiagonal blocks. Futher, the rank of the block Hankel matrix is the dimension of the minimal periodic realization of the input-output periodic application defined by the Markov parameters. Then this property is translated to the periodic generalized Bezoutian matrix. □

Using the relation given in [9] for the periodic block Hankel matrix at times s and $s+1$

$$H^{s+1}_{r+1,\, r+1}\, A^{\mathrm{T}}_{r+1}(mN) = A_{r+1}(pN)\, H^{s}_{r+1,\, r+1}, \quad s \in Z \tag{17}$$

with

$$A_{r+1}(tN) = \begin{pmatrix} S_{1,t}(0) & T_{1,t}(0) & 0_{tN} & \cdots & 0_{tN} & 0_{tN} \\ 0_{tN} & S_{1,t}(0) & T_{1,t}(0) & \cdots & 0_{tN} & 0_{tN} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0_{tN} & 0_{tN} & 0_{tN} & \cdots & S_{1,t}(0) & T_{1,t}(0) \\ 0_{tN} & -a_0\, T_{1,t}(0) & -a_1\, T_{1,t}(0) & \cdots & -a_{r-2}\, T_{1,t}(0) & S_{1,t}(-a_{r-1}) \end{pmatrix}$$

and

$$S_{1,t}(z) = \begin{bmatrix} 0 & I_{(n-1)\,t} \\ z\, I_t & 0 \end{bmatrix}, \quad T_{1,t}(z) = \begin{bmatrix} 0 & z\, I_{(N-1)\,t} \\ I_t & 0 \end{bmatrix},$$

where the a_i's are the coefficients of the monic least common denominator of the elements of the proper rational matrix $H_0(z)$, $d_0(z) = a_r\, z^r + a_{r-1}\, z^{r-1} + ..a_0$, $a_r = 1$, we obtain the following result.

Proposition 3. There exists a left and a right polynomial matrix fraction description of (5) such that the associated periodic generalized Bezoutian matrix at consecutive times s and $s+1$ satisfies the following relation

$$A^{B}_{r+1}(pN)\, B(M_s,\, L_s) = B\,(M_{s+1},\, L_{s+1})\, \left\{A^{B}_{r+1}(mN)\right\}^{\mathrm{T}}, \tag{18}$$

where $A^{B}_{r+1}(tN)$ is the matrix whose column (row) blocks are the row (column) blocks of $A_{r+1}(tN)$.

Proof. Consider the decompositions (16) and let $H^{s}_{\alpha,\beta}$ be the (α,β)th block Hankel matrix and $B(M_s, L_s)$ be the generalized Bezoutian matrix associated with the rational matrix $H_s(z)$. By Proposition 2, we have

$$B(M_s, L_s) = T\,(a_0\, I_{pN}, \ldots, a_r\, I_{pN})\;\, H^{s}_{r+1,\, r+1} T\,(a_0\, I_{mN}, \ldots, a_r\, I_{mN})\,,$$

and

$$B\,(M_{s+1},\, L_{s+1}) = T\,(a_0\, I_{pN}, \ldots, a_r\, I_{pN})\;\, H^{s+1}_{r+1,\, r+1}\, T\,(a_0,\, I_{mN}, \ldots, a_r\, I_{mN})\,.$$

From (17) we obtain

$$\begin{aligned} & T\left(a_0\, I_{pN}, \ldots, a_r\, I_{pN}\right)\, A_{r+1}(pN)\, T^{-1}\left(a_0\, I_{pN}, \ldots, a_r\, I_{pN}\right)\, B\left(M_s,\, L_s\right) = \\ = \quad & B\left(M_{s+1},\, L_{s+1}\right)\, T^{-1}\left(a_0\, I_{mN}, \ldots, a_r\, I_{mN}\right)\, A^{\mathrm{T}}_{r+1}(mN)\, T\left(a_0\, I_{mN}, \ldots, a_r\, I_{mN}\right). \end{aligned}$$

To obtain the relation (18) it remains to show that

$$T\left(a_0,\, I_{pN}, \ldots, a_r\, I_{pN}\right)\, A_{r+1}(pN) = A^{B}_{r+1}(pN)\, T\left(a_0\, I_{pN}, \ldots, a_r\, I_{pN}\right), \tag{19}$$

and

$$A^{\mathrm{T}}_{r+1}(mN)\, T\left(a_0\, I_{mN}, \ldots, a_r\, I_{mN}\right) = T\left(a_0\, I_{mN}, \ldots, a_r\, I_{mN}\right)\, \left\{A^{B}_{r+1}(mN)\right\}^{\mathrm{T}}. \tag{20}$$

Note that

$$\begin{aligned} & T\left(a_0\, I_{tN}, \ldots, a_r\, I_{tN}\right)\, A_{r+1}(tN) = \\ = & \begin{bmatrix} a_0 S_{1,t}(0) & a_1 S_{1,t}(0) & \cdots & a_{r-1}T_{1,t}(0) + a_r S_{1,t}(-a_{r-1}) \\ a_1, S_{1,t}(0) & a_1 T_{1,t}(0) + a_2 S_{1,t}(0) & \cdots & a_r T_{1,t}(0) \\ \vdots & \vdots & & \vdots \\ a_{r-1} S_{1,t}(0) & a_{r-1} T_{1,t}(0) + a_r S_{1,t}(0) & \cdots & 0_{tN} \\ a_r S_{1,t}(0) & a_r T_{1,t}(0) & \cdots & 0_{tN} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} & A^{B}_{r+1}(tN)\, T\left(a_0\, I_{tN}, \ldots, a_r\, I_{tN}\right) = \\ = & \begin{bmatrix} a_0 S_{1,t}(0) & a_1 S_{1,t}(0) & \cdots & a_r S_{1,t}(0) \\ a_1 S_{1,t}(0) & a_1 T_{1,t}(0) + a_2 S_{1,t}(0) & \cdots & a_r T_{1,t}(0) \\ \vdots & \vdots & & \vdots \\ a_{r-1} S_{1,t}(0) & a_{r-1} T_{1,t}(0) + a_r S_{1,t}(0) & \cdots & 0_{tN} \\ a_r S_{1,t}(0) & a_r T_{1,t}(0) & \cdots & 0_{tN} \end{bmatrix}. \end{aligned}$$

But $a_{r-1}\, T_{1,t}(0) + a_r S_{1,t}(-a_{r-1}) = a_r\, S_{1,t}(0)$. Then expression (20)

$$T\left(a_0,\, I_{pN}, \ldots, a_r\, I_{tN}\right)\, A_{r+1}(tN) = A^{B}_{r+1}(tN)\, T\left(a_0\, I_{tN}, \ldots, a_r\, I_{tN}\right),$$

holds.

From this expression we obtain

$$A^{\mathrm{T}}_{r+1}(tN)\, \left\{T\left(a_0\, I_{tN}, \ldots, a_r\, I_{tN}\right)\right\}^{\mathrm{T}} = \left\{T\left(a_0\, I_{tN}, \ldots, a_r\, I_{tN}\right)\right\}^{\mathrm{T}} \left\{A^{B}_{r+1}(tN)\right\}^{\mathrm{T}}$$

and by the symmetry of the matrix $T(a_0\, I_{tN}, \ldots, a_r\, I_{tN})$ we deduce expression (21). □

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Prof. Dr. Carmen Coll and Raphael Bru, Dpto. de Matemtica Aplicada, Universidad Politécnica de Valencia, Apartado de Correos 22 012, 46 071 Valencia. Spain.

Dr. Vicente Hernández, Dpto. de Sistemas Informáticos y Computación, Universidad Politécnica de Valencia, Apartado de Correos 22 012, 46 071 Valencia. Spain.