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A CLASSIFICATION OF GENERALISED STATE SPACE REDUCTION METHODS FOR LINEAR MULTIVARIABLE SYSTEMS

N. P. Karampetakis,[1] A. I. Vardulakis and A. C. Pugh

Two algorithms which reduce a general polynomial matrix model of a linear multivariable system Σ to an equivalent model in generalised state space (g.s.s.) form are proposed. The nature of this equivalence is established.

1. INTRODUCTION

Consider a linear multivariable system Σ described by a polynomial matrix model:

$$(\Sigma): \quad A(\rho)\beta(t) = B(\rho)\,u(t) \tag{1.1a}$$

$$y(t) = C(\rho)\beta(t) + D(\rho)\,u(t) \tag{1.1b}$$

where $\rho := \mathrm{d}/\mathrm{d}t$, $A(\rho) \in \mathbb{R}[\rho]^{r\times r}$ with $\mathrm{rank}_{\mathbb{R}} A(\rho) = r$, $B(\rho) \in \mathbb{R}[\rho]^{r\times m}$, $C(\rho) \in \mathbb{R}[\rho]^{p\times r}$ and $D(\rho) \in \mathbb{R}[\rho]^{p\times m}$, $\beta(t)$ the *pseudostate* of Σ, $u(t)$ the input vector and $y(t)$ the output vector. The normalised form $\Sigma^{(N)}$ of Σ [9] is

$$(\Sigma^{(N)}) : \mathcal{T}(\rho)\,\xi(t) = \mathcal{U}\,u(t) \tag{1.2a}$$

$$y(t) = \mathcal{V}\,\xi(t), \tag{1.2b}$$

where

$$\mathcal{T}(\rho) = \begin{pmatrix} A(\rho) & B(\rho) & 0 \\ -C(\rho) & D(\rho) & I_p \\ 0 & -I_m & 0 \end{pmatrix} \in \mathbb{R}[\rho]^{\tilde{r}\times\tilde{r}},\ \mathcal{U} = \begin{pmatrix} 0 \\ 0 \\ I_m \end{pmatrix} \in \mathbb{R}^{\tilde{r}\times m};\ \xi(t) = \begin{pmatrix} \beta(t) \\ -u(t) \\ y(t) \end{pmatrix}$$

$$\mathcal{V} = (0\ 0\ I_p) \in \mathbb{R}^{p\times\tilde{r}} \quad \tilde{r} = r + p + m \tag{1.3}$$

Σ, $\Sigma^{(N)}$ may equally be represented by the polynomial matrices

$$P(s) = \begin{bmatrix} A(s) & B(s) \\ -C(s) & D(s) \end{bmatrix}; \quad \mathcal{P}(s) = \begin{bmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{bmatrix}. \tag{1.4}$$

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The problem to be studied is, given the polynomial matrix description (PMD) $[A(\rho), B(\rho), C(\rho), D(\rho)]$ of Σ determine a system

$$(\Sigma_R): \quad E\dot{x}(t) = Ax(t) + Bu(t) \tag{1.5a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.5b}$$

"equivalent" to Σ in the sense that they exhibit identical system properties.

There are essentially two different ways to solve the above problem. The first and the most common method is to produce an algorithm, which reduces a general PMD of a linear multivariable system Σ to Σ_R, and to show step by step that all the required properties of Σ remain invariant [2, 8, 9]. The second and more direct way, [1], is to produce a reduction algorithm and to show it is achieved via a *system equivalence transformation*, which has the property of preserving the desired properties. The system equivalence tools available are *strong system equivalence* [1] and *full system equivalence* [3]. Now, strong system equivalence is composed of two separate system transformations whereas full system equivalence is composed of only one. For this reason we use *full system equivalence* in the sequel.

2. PRELIMINARY RESULTS

Consider the set $P(p,m)$ of $(r+p)\times(r+m)$ polynomial matrices where the integer $r \geq \max\{-p,-m\}$. A matrix transformation important in systems theory is

Definition 1. [3] $T_1(s), T_2(s) \in P(p,m)$ are said to be *fully equivalent* (f.e.) in case there exist polynomial matrices $M(s)$, $N(s)$ such that:

$$[M(s) \quad T_2(s)] \begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{2.3}$$

where the compound matrices in (2.1) are such that

(i) they have full normal rank, (2.2a)

(ii) they have no finite nor infinite zeros, (2.2b)

(iii) the following McMillan degree conditions hold

$$\delta_M([M(s) \quad T_2(s)]) = \delta_M(T_2(s)); \quad \delta_M\left(\begin{bmatrix} T_1(s) \\ -N(s) \end{bmatrix}\right) = \delta_M(T_1(s)). \tag{2.2c}$$

Let $\mathcal{P}(p,m)$ be the set of $(r+p)\times(r+m)$ Rosenbrock system matrices (1.4), then

Definition 2. [3] $P_1(s), P_2(s) \in \mathcal{P}(p,m)$ are said to be *full system equivalent* (f.s.e.) if $\exists$ polynomial matrices $M(s)$, $N(s)$, $X(s)$, $Y(s)$ such that

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ -C_1(s) & D_1(s) \end{bmatrix} = \begin{bmatrix} A_2(s) & B_2(s) \\ -C_2(s) & D_2(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} \tag{2.3}$$

where (2.3) is a (f.e.) transformation.

Some interesting results concerning (f.s.e.) are [3, 5]:

Theorem 1. (i) (f.s.e.) is an equivalence relation on $P(p, m)$.

(ii) Under (f.s.e.) the following are invariant

(a) the generalized order $\boldsymbol{f}$, the order $\boldsymbol{n}$ and the Rosenbrock degree $\boldsymbol{d_R}$,

(b) the transfer function and thus the finite and infinite transmission poles and zeros,

(c) the finite and infinite system poles and zeros,

(d) the finite and infinite invariant zeros,

(e) the sets of finite and infinite input (output) decoupling zeros.

(f) the set of input (output) dynamical indices.

(iii) Every system matrix $P(s)$ is (f.s.e.) with its normalized form $\mathcal{P}(s)$.

Theorem 2. [4] Let $P_1(s)$, $P_2(s) \in \mathcal{P}(p, m)$ with transfer function matrices $G_1(s), G_2(s)$ be strongly irreducible system matrices i. e. possessing no finite nor infinite decoupling zeros, then

$$P_1(s) \overset{\text{f.s.e.}}{\sim} P_2(s) \Longleftrightarrow G_1(s) = G_2(s).$$

3. GENERALISED STATE SPACE REALIZATIONS FOR LINEAR MULTIVARIABLE SYSTEMS

The problem of reducing a linear multivariable system to an "equivalent" (g.s.s.) system has been considered by many authors [1, 2, 8, 9]. The solutions can actually be classified under two different theoretical algorithms which are proposed in this section.

Algorithm 1

Step 1. Given $[A(s), B(s), C(s), D(s)]$ be the PMD of (1.1) of Σ form $\mathcal{T}(s) \in \boldsymbol{R}[s]^{\bar{r}\times\bar{r}}$ where $\bar{r} = r + p + m$.

Step 2. Compute a strongly irreducible realisation $[A_0(s), B_0(s), C_0(s), D_0(s)]$ of $\mathcal{T}(s)$ in the sense of Verghese [9] i. e. $\mathcal{T}(s) = C_0(s)A_0(s)^{-1}B_0(s) + D_0(s)$ with

$$\begin{pmatrix} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & I \end{pmatrix}; \quad \begin{pmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \\ 0 & -I \end{pmatrix}$$

having no finite nor infinite zeros, and the polynomial matrices $A_0(s)$, $B_0(s)$, $C_0(s)$, $D_0(s)$ are matrix pencils.

Step 3. The system matrix of the "equivalent" (g.s.s.) realization $\Sigma_{\mathcal{T}}$ of Σ is then

$$P_{\mathcal{T}}(s) = \left(\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & \mathcal{U} \\ \hline 0 & -\mathcal{V} & 0 \end{array}\right) \tag{3.1}$$

In the case where $C_0(s)$, $B_0(s)$ are constant matrices and $D_0(s) = 0$ then the algorithm is that proposed in [1, 9]. Similarly the reduction algorithm of [2, 5] also arises from Algorithm 1 by effecting a specific strongly irreducible realization of $\mathcal{T}(s)$. We now wish to determine the nature of the equivalence between Σ and $\Sigma_{\mathcal{T}}$.

Theorem 3. The linear multivariable systems Σ of (1.1) and $\Sigma_{\mathcal{T}}$ of (3.1) are (f.s.e.).

Proof. Clearly the following holds

$$\begin{gathered}\left(\begin{array}{c|c} 0 & 0 \\ I_{\tilde r} & 0 \\ \hline 0 & I_p \end{array}\right)\left(\begin{array}{c|c} \mathcal{T}(s) & \mathcal{U} \\ \hline -\mathcal{V} & 0 \end{array}\right) \\ = \left(\begin{array}{cc|c} A_0(s) & B_0(s) & 0 \\ -C_0(s) & D_0(s) & \mathcal{U} \\ \hline 0 & -\mathcal{V} & 0 \end{array}\right)\left(\begin{array}{c|c} -A_0(s)^{-1}B_0(s) & 0 \\ I_{\tilde r} & 0 \\ \hline 0 & I_m \end{array}\right)\end{gathered} \tag{3.2}$$

where $[A_0(s), B_0(s), C_0(s), D_0(s)]$ is a strongly irreducible realisation of $\mathcal{T}(s)$ and the polynomial matrices $A_0(s), B_0(s), C_0(s), D_0(s)$ are pencils. We need to show that (3.2) is a (f.s.e.) transformation or specifically that $A_0(s)^{-1}B_0(s)$ is a polynomial matrix and that the compound matrices formed from (3.2) satisfy the (f.e.) conditions (2.2).

It is obvious that the Rosenbrock system matrices

$$P_1(s) = \begin{pmatrix} I & 0 \\ 0 & \mathcal{T}(s) \end{pmatrix} \quad \text{and} \quad P_2(s) = \begin{pmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \end{pmatrix} \tag{3.3}$$

are strongly irreducible and have the same transfer function matrix $\mathcal{T}(s)$. Thus from Theorem 2 $P_1(s)$, $P_2(s)$ are (f.s.e.) and therefore there exist polynomial matrices $M(s), N(s)$, $X(s), Y(s)$ such that

$$\begin{pmatrix} M(s) & 0 \\ X(s) & I_{\tilde r} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \mathcal{T}(s) \end{pmatrix} = \begin{pmatrix} A_0(s) & B_0(s) \\ -C_0(s) & D_0(s) \end{pmatrix}\begin{pmatrix} N(s) & Y(s) \\ 0 & I_{\tilde r} \end{pmatrix} \tag{3.4}$$

where (3.4) is a (f.e.) transformation. From the (1,2) equation of (3.4) we have that $Y(s) = -A_0(s)^{-1}B_0(s)$ and thus $A_0(s)^{-1}B_0(s)$ is a polynomial matrix.

The compound matrix formed by the left matrices of the relation (3.2) satisfies the McMillan degree conditions of (2.2) and further it can be easily transformed under constant, nonsingular column operations to

$$\begin{pmatrix} 0 & | & 0 & A_0(s) & B_0(s) & 0 \\ 0 & | & I_{\tilde r} & -C_0(s) & D_0(s) & 0 \\ \hline I_p & | & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

which obviously satisfies (2.2b) because $[A_0(s), B_0(s), C_0(s), D_0(s)]$ is strongly irreducible.

From the McMillan degree condition which holds for the right compound matrix of the (f.e.) transformation (3.4) we have that since $Y(s) = -A_0(s)^{-1}B_0(s)$

$$\delta_M \begin{pmatrix} I & 0 \\ 0 & \mathcal{T}(s) \\ \hline -N(s) & A_0(s)^{-1}B_0(s) \\ 0 & -I_{\tilde r} \end{pmatrix} = \delta_M \begin{pmatrix} I & 0 \\ 0 & \mathcal{T}(s) \end{pmatrix} \tag{3.6}$$

(3.6) implies [3] that $N(s)$ is constant which plays no role in the McMillan degree conditions and so we conclude that

$$\delta_M \begin{pmatrix} \mathcal{T}(s) \\ A_0(s)^{-1}B_0(s) \end{pmatrix} = \delta_M(\mathcal{T}(s)) \tag{3.7}$$

Thus the compound matrix formed by the right matrices of the relation (3.2) satisfies the McMillan degree conditions of (2.2) because of (3.7), and since it can be easily transformed under constant and nonsingular row operations to the following form

$$\begin{pmatrix} 0 & | & 0 \\ \mathcal{T}(s) & | & 0 \\ A_0(s)^{-1}B_0(s) & | & 0 \\ -I_{\tilde r} & | & 0 \\ \hline 0 & | & I_m \end{pmatrix} \tag{3.8}$$

it follows that it has no finite nor infinite zeros. Thus (3.2) is an (f.e.) transformation.

It then follows from Theorem 1 (iii) and the transitivity of (f.s.e.) that the system matrix $P(s)$ corresponding to Σ of (1.1) is (f.s.e.) to the system matrix (3.1) of $\Sigma_{\mathcal{T}}$. □

While the first algorithm is based on the realization of $\mathcal{T}(s)$ defined in (1.3), a second reduction may be based on a realization of $\mathcal{T}(s)^{-1}$.

Algorithm 2

Step 1. Given Σ of (1.1), form $\mathcal{T}(s) \in \mathbb{R}[s]^{\tilde r \times \tilde r}$ of (1.3) where $\tilde r = r + p + m$.

Step 2. Compute a strongly irreducible realization $[E, A \in \boldsymbol{R}^{\lambda\times\lambda}, B \in \boldsymbol{R}^{\lambda\times\tilde{r}}, C \in \boldsymbol{R}^{\tilde{r}\times\lambda}, D \in \boldsymbol{R}^{\tilde{r}\times\tilde{r}}]$ of $\mathcal{T}(s)^{-1}$ in the sense of Verghese i. e. $\mathcal{T}(s)^{-1} = C(sE-A)^{-1}B + D$ where the compound matrices

$$\begin{pmatrix} sE-A & B \end{pmatrix}; \quad \begin{pmatrix} sE-A \\ -C \end{pmatrix}$$

have no finite nor infinite zeros.

Step 3. The "equivalent" generalized state space system $\Sigma_{\mathcal{T}^{-1}}$ of Σ will be the following

$$(\Sigma_{\mathcal{T}^{-1}}) : E\dot{x}(t) = Ax(t) + B\mathcal{U}u(t) \tag{3.9a}$$

$$y(t) = \mathcal{V}Cx(t) + \mathcal{V}D\mathcal{U}u(t). \tag{3.9b}$$

Theorem 4. The linear multivariable systems Σ of (1.1) and $\Sigma_{\mathcal{T}^{-1}}$ of (3.9) are (f.s.e.).

Proof. This follows in a similar way to the result in Theorem 3. The (f.s.e.) transformations which relates the systems (1.2) and (3.9) are the following

$$\begin{pmatrix} \mathcal{T}(s)\, C(sE-A)^{-1} & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} sE-A & B\mathcal{U} \\ -\mathcal{V}C & \mathcal{V}D\mathcal{U} \end{pmatrix} = \begin{pmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix} \begin{pmatrix} C & -D\mathcal{U} \\ 0 & I_m \end{pmatrix} \tag{3.10}$$

$$\begin{pmatrix} B & 0 \\ \mathcal{V}D & I_p \end{pmatrix} \begin{pmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix} = \begin{pmatrix} sE-A & B\mathcal{U} \\ -\mathcal{V}C & \mathcal{V}D\mathcal{U} \end{pmatrix} \begin{pmatrix} (sE-A)^{-1}B\mathcal{T}(s) & 0 \\ 0 & I_m \end{pmatrix} \tag{3.11}$$

□

As we can see from Algorithms 1 and 2 the construction of an "equivalent" generalized state space realization of the system Σ in (1.1) centres on the computation of a strongly irreducible realization either of $\mathcal{T}(s)$ or $\mathcal{T}(s)^{-1}$. [1] and [9] ([8]) gave a solution to this problem with the construction of a strongly irreducible realization of $\mathcal{T}(s)$ $(\mathcal{T}(s)^{-1})$ in terms of finite and infinite Jordan pairs of $\mathcal{T}(s)$ $(\mathcal{T}(s)^{-1})$. A consequence of this kind of solution is, that these algorithms are not easily implemented. In contrast [2] proposed a more practical solution to the above problem.

4. AN EXTENSION OF TAN AND VAN DEWALL'S MODEL

A different approach to the implementation of the above algorithms is given by an extension of the known generalized state space realization method for MFDs (matrix fraction descriptions) presented by [6]. We first present some useful lemmas.

Lemma 1. [7] Let $\mathcal{T}(s) \in \mathbb{R}(s)^{p\times m}$ with $\operatorname{rank}_{\mathbb{R}(s)}\mathcal{T}(s) = m$ (resp. $= p$). Then $\mathcal{T}(s)$ is column (row) reduced at $s = \infty$ iff the pole-zero structure at $s = \infty$ of $\mathcal{T}(s)$ is given by the pole-zero structure of its columns (rows) taken separately.

Lemma 2. [7] Let $\mathcal{T}(s)\in\mathbb{R}(s)^{p\times m}$ and $\mathcal{T}(s)=Q_1(s)^{-1}R_1(s)$ $(=R_2(s)\,Q_2(s)^{-1})$, be a left (right) MFD of $\mathcal{T}(s)$. If the compound matrix $[Q_1(s)\,R_1(s)]$, $([Q_2(s)^T\,R_2(s)^T]^T)$ possesses no zeros in $\mathbb{C}U\{\infty\}$ then

$$\delta_M(Q_1(s)^{-1}R_1(s)) = \sum_{i=1}^{k} q_i[Q_1\ R_1] = \delta_M(Q_1(s)\ R_1(s))$$

$$\left[\delta_M(R_2(s)Q_2(s)^{-1}) = \sum_{i=1}^{k} q_i[Q_2^T\ R_2^T]^T = \delta_M(Q_2(s)^T\ R_2(s)^T)^T\right] \tag{4.1}$$

where $q_i[Q_1\ R_1] \geq 0$ $(q_i[Q_2^T\ R_2^T]^T \geq 0)$ are the degrees of the infinite poles of $[Q_1(s)\ R_1(s)]$ $([Q_2(s)^T\ R_2(s)^T]^T)$.

If (1.2) is the normalised form of Σ then let

$$S^{\infty}_{\mathcal{T}(s)}(s) := \text{diag}\left[s^{q_1}, s^{q_2}, \dots, s^{q_k}, \frac{1}{s^{q_{k+1}}}, \dots, \frac{1}{s^{q_{\tilde{r}}}}\right] \tag{4.2}$$

be the Smith McMillan form at $s=\infty$ of $\mathcal{T}(s)$. The following readily implementable algorithm for the construction of an "equivalent" (g.s.s.) reduction of Σ is proposed. The algorithm represents a generalisation of the proposed in [6, 10].

Algorithm 3

Step 1. Compute a unimodular matrix $U(s)\in\mathbb{R}[s]^{\tilde{r}\times\tilde{r}}$ such that

$$\mathcal{T}(s)^{-1} = [U(s)] \times [\mathcal{T}(s)U(s)]^{-1} \tag{4.3}$$

where the following compound matrix is column reduced

$$\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix} := \begin{pmatrix} \mathcal{T}(s)U(s) \\ U(s) \end{pmatrix} \tag{4.4}$$

Step 2. Define,

$$S^{\infty}_{[Q(s)^T\ R(s)^T]^T}(s) = \begin{pmatrix} \text{diag}\,[s^{\bar{q}_1}, s^{\bar{q}_2}, \dots, s^{\bar{q}_{\tilde{r}}}] \\ 0_{\tilde{r}\times\tilde{r}} \end{pmatrix}. \tag{4.5}$$

Define the matrix

$$S^T(s) = \begin{pmatrix} s^{\bar{q}_1} & s^{\bar{q}_1-1} & \dots & 1 & \dots & & & & 0 \\ & & 0 & & \dots & & & & 0 \\ & & \vdots & & \ddots & & & & 0 \\ & & 0 & & \dots & s^{\bar{q}_{\tilde{r}}} & s^{\bar{q}_{\tilde{r}}-1} & \dots & 1 \end{pmatrix} \tag{4.6}$$

and write the polynomial matrices $Q(s)$ and $R(s)$ as follows

$$Q(s) = Q_c S(s); \quad R(s) = R_c S(s) \tag{4.7}$$

where Q_c and R_c are constant matrices.

Step 3. Construct the core realization

$$E_c s - A_c = \text{block diag}\{E_{c1}s - A_{c1}, \ldots, E_{c\tilde{r}}s - A_{c\tilde{r}}\} \tag{4.8}$$

$$E_{ci}s - A_{ci} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ -1 & s & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \ldots & -1 & s \end{bmatrix} \in \mathbb{R}[s]^{(\bar{q}_i+1)\times(\bar{q}_i+1)} \tag{4.9}$$

$$B_c^T = \text{blockdiag}[B_1, B_2, \ldots, B_{\tilde{r}}]; \quad B_i = [1\ 0\ \ldots\ 0] \in \mathbb{R}^{1\times(\bar{q}_i+1)} \tag{4.10}$$

$$C_c = I_n, \quad n = \sum_{i=1}^{\tilde{r}} \bar{q}_i + \tilde{r}. \tag{4.11}$$

The "equivalent" (g.s.s.) model of the system Σ in (1.1) is then

$$(\Sigma_R) : E\dot{x}(t) = Ax(t) + Bu(t) \tag{4.12a}$$

$$y(t) = Cx(t) + Du(t), \tag{4.12b}$$

where

$$E = E_c, \quad A = A_c - B_c Q_c, \quad B = B_c\mathcal{U}, \quad C = \mathcal{V}R_cC_c = \mathcal{V}R_c. \tag{4.13}$$

Theorem 5. The linear multivariable systems Σ_R of (4.12) and Σ of (1.1) are (f.s.e.).

P r o o f. It is easily checked ([6]) that the PMD $[sE_c - A_c + B_cQ_c, B_c, R_c]$ is strongly irreducible and realizes $\mathcal{T}(s)^{-1}$ i.e. $\mathcal{T}(s)^{-1} = R_c(sE_c - A_c + B_cQ_c)^{-1}B_c$. Thus from Theorem 4 the (g.s.s.) system (4.12) will be (f.s.e.) with the system $\Sigma^{(N)}$ in (1.2), and thus with the system Σ in (1.1). □

Remark 1. The dimension of $x(t)$ of the (g.s.s.) system Σ_R in (4.12) is

$$\lambda = \tilde{r} + \delta_M(\mathcal{T}(s)). \tag{4.14}$$

To see this note that from (4.8), (4.9) and (4.13)

$$\lambda = \sum_{i=1}^{\tilde{r}} (\bar{q}_i + 1). \tag{4.15}$$

The compound matrix $[Q(s)^T\ R(s)^T]^T$ has been constructed from $[I_{\tilde{r}}\ \mathcal{T}(s)^T]^T$ with unimodular operations which have no finite zeros. Thus $[Q(s)^T\ R(s)^T]^T$ possess no finite zeros. $[Q(s)^T\ R(s)^T]^T$ has also no infinite zeros by Lemma 1 because it is column reduced. Hence from Lemma 2

$$\delta_M(\mathcal{T}(s)) \overset{[7]}{=} \delta_M(\mathcal{T}(s)^{-1}) = \delta_M(R(s)\,Q(s)^{-1}) = \delta_M([R(s)^T\ Q(s)^T]^T) = \sum_{i=1}^{\tilde{r}} \bar{q}_i. \tag{4.16}$$

The combination of relations (4.15) and (4.16) gives the result.

A similar more practical version of Algorithm 1 can also be given.

5. ILLUSTRATIVE EXAMPLE FOR ALGORITHM 3

Consider the linear system Σ described by the following system equations

$$\begin{aligned} (\Sigma):\ (\rho^2+5\rho+6)\,\beta(t) &= (p+1)\,u(t) \\ y(t) &= (-2\rho+5)\,\beta(t)+(3\rho+2)\,u(t). \end{aligned} \tag{E.1}$$

The normalized system matrix $\mathcal{P}(s)$ of the above system will be following

$$\mathcal{P}(s)=\begin{pmatrix} \mathcal{T}(s) & \mathcal{U} \\ -\mathcal{V} & 0 \end{pmatrix}=\left(\begin{array}{ccc|c} s^2+5s+6 & s+1 & 0 & 0 \\ 2s-5 & 3s+2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 \end{array}\right). \tag{E.2}$$

Step 1. The polynomial matrix $\mathcal{T}(s)^{-1}$ may be written as

$$\mathcal{T}(s)^{-1}=I_3\mathcal{T}(s)^{-1}=:R(s)\,Q(s)^{-1}. \tag{E.3}$$

Note that the matrix

$$\begin{pmatrix} Q(s) \\ R(s) \end{pmatrix}=\begin{pmatrix} \mathcal{T}(s) \\ I_3 \end{pmatrix} \tag{E.4}$$

is column reduced with column degrees $\{2,1,0\}$ and so there is no need to determine the unimodular matrix $U(s)$.

Step 2. Let now

$$S(s)^T=\left(\begin{array}{ccc|cc|c} s^2 & s & 1 & & 0 & \\ \hline & & & s & 1 & \\ \cline{4-6} & 0 & & & & 1 \end{array}\right). \tag{E.5}$$

Then

$$Q(s)=(=\mathcal{T}(s))=Q_cS(s)=\begin{pmatrix} 1 & 5 & 6 & 1 & 1 & 0 \\ 0 & 2 & -5 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}S(s) \tag{E.6}$$

$$R(s)=(=I_3)=R_cS(s)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}S(s). \tag{E.7}$$

The core realization is then

$$
sE_c - A_c = \left(\begin{array}{ccc|cc|c}
0 & 0 & 0 & & & \\
-1 & s & 0 & & & \\
0 & -1 & s & & & \\
\hline
 & & & 0 & 0 & \\
 & & & -1 & s & \\
\hline
 & 0 & & & & 0
\end{array}\right);
$$

$$
B_c = \left(\begin{array}{c|c|c}
1 & & \\
0 & 0 & \\
0 & & \\
\hline
 & 1 & \\
 & 0 & \\
\hline
0 & & 1
\end{array}\right) \quad \text{and} \quad C_c = I_6 .
$$

(E.8)

The (f.s.e.) g.s.s. system of (E.1) will be the following

$$
P_R(s) = \begin{pmatrix} sE_c - A_c + B_cQ_c & B_c\mathcal{U} \\ -\mathcal{V}R_c & 0 \end{pmatrix} = \left(\begin{array}{cccccc|c}
1 & 5 & 6 & 1 & 1 & 0 & 0 \\
-1 & s & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & s & 0 & 0 & 0 & 0 \\
0 & 2 & -5 & 3 & 2 & 1 & 0 \\
0 & 0 & 0 & -1 & s & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right). \tag{E.9}
$$

CONCLUSIONS

The problem of the reduction of a linear multivariable system to an "equivalent" (g.s.s.) system has been studied. All the known reduction algorithms proposed by [1, 2, 8, 9] can be classified by two different theoretical reduction algorithms. In either case the problem is reduced to the construction of a strongly irreducible realisation of a square rational matrix. It was shown that these two general reduction algorithms gives rise to g.s.s. models, which are (f.s.e.) to the original. Thus the original system and any g.s.s. reduction of it share the same finite and infinite system properties. A more computationally attractive form of Algorithm 2 has been presented via Algorithm 3 and a similarly attractive form of Algorithm 1 can be developed, although this has not been given here.

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Prof. Dr. N. P. Karampetakis and Prof. Dr. A. C. Pugh, Department of Mathematical Sciences, Loughborough University of Technology, Loughborough, Leics LE11 3TU. U. K.

Prof. Dr. A. I. Vardulakis, Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54006. Greece.