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SINGULAR FINITE HORIZON FULL INFORMATION $\mathcal{H}^\infty$ CONTROL VIA REDUCED ORDER RICCATI EQUATIONS

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In this paper we consider the standard finite horizon, full information $\mathcal{H}^\infty$ control problem when the direct feedthrough matrix, which links the control input to the controlled output, is not full column rank. Using a differential game approach, we show that, in this case, the solution of the problem can be obtained solving a *reduced order* Riccati differential equation.

1. INTRODUCTION

In this paper we consider the finite horizon, full information $\mathcal{H}^\infty$ control problem for linear time-varying systems. Full information means that, as often it happens in practical situations (see for example [4]) the *exogenus* inputs, including command signals and disturbances, are available for the feedback (for the definition of full information problem see [5]).

This problem has been solved in the *nonsingular case* (in other words when the direct feedthrough matrix $\boldsymbol{D}$ between the control input and the controlled output is full column rank), see for example [5], [6] and [9].

Our goal is to discuss the $\mathcal{H}^\infty$ problem when the above-mentioned $\boldsymbol{D}$ matrix *is not* full column rank, the so-called *singular* problem. Our main result consists in proving that, in this case, the original $\mathcal{H}^\infty$ problem is *equivalent* to another $\mathcal{H}^\infty$ problem related to a *reduced order* system.

The machinery uses a dynamic games approach ([1], [2]) leading to a singular minmax problem. Using a suitable decomposition of the state space introduced in the literature by Butman [3] (see also [7]) and considering the class of solutions of full information type, we will show that this game is equivalent to another game acting on a reduced order state equation.

This work is a first attempt of generalization to the time-varying setting of the results contained in the paper by Stoorvogel [8], where the singular $\mathcal{H}^\infty$ control problem for time-invariant systems has been solved by means of an elegant decomposition of the state space involving the concept of strongly controllable subspace.

The paper is organized as follows. In Section 2 we state precisely the problem we deal with, showing the connections with the differential game theory. In Section 3 a theorem concerning the equivalence between the original singular minmax problem and a certain *reduced order* minmax problem is proved when $\boldsymbol{D} = \mathbf{0}$. In Section 4 we come back to the $\mathcal{H}^\infty$ setting and state our main result when $\boldsymbol{D} = \mathbf{0}$, while in Section 5 the case $\boldsymbol{D} \neq \mathbf{0}$ is discussed, showing that it can be solved using the same machinery. Finally in Section 6 some concluding remarks and plans for future research are given.

2. PRELIMINARIES AND PROBLEM STATEMENT

Let $\Omega := [t_0, t_f]$ any compact interval on the real line. We denote by $\mathcal{L}^2(\Omega)$ the space of the real vector-valued functions which are square integrable on Ω. The usual norm in $\mathcal{L}^2(\Omega)$ is denoted by $\|\cdot\|_2$. Given a linear time-varying system

$$\mathcal{G} := \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\,\boldsymbol{x}(t) + \boldsymbol{B}(t)\,\boldsymbol{u}(t), & \boldsymbol{x}(t_0) = \mathbf{0} \\ \boldsymbol{y}(t) = \boldsymbol{C}(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t)\,\boldsymbol{u}(t) \end{cases} \quad t \in \Omega, \tag{1}$$

it uniquely defines a linear operator from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)$ denoted by G. $\|G\|$ denotes the operator norm induced by the norm in $\mathcal{L}^2(\Omega)$. Given any matrix $\boldsymbol{F} \in \mathbb{R}^{n\times m}$ (with $n \geq m$), $\boldsymbol{F}^\dagger$ denotes the left pseudoinverse of $\boldsymbol{F}$.

We consider the finite horizon full information $\mathcal{H}^\infty$ control problem for the linear time-varying system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\,\boldsymbol{x}(t) + \boldsymbol{B}(t)\,\boldsymbol{u}(t) + \boldsymbol{H}(t)\,\boldsymbol{w}(t), & \boldsymbol{x}(t_0) = \mathbf{0} \\ \boldsymbol{z}(t) = \boldsymbol{C}(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t)\,\boldsymbol{u}(t) \end{cases} \quad t \in \Omega, \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control input, $\boldsymbol{w}(t) \in \mathbb{R}^l$ is the exogenus input, and $\boldsymbol{z}(t) \in \mathbb{R}^p$ is the controlled output. We shall assume that all the involved matrices are continuously differentiable and, without loss of generality, that the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are full column and row rank respectively.

Since all matrices and vectors in the paper are time-varying, to avoid cumbersome notation, we will omit the time argument, if this is not cause of ambiguity.

The problem we shall consider in this paper is precisely defined as follows.

Problem 1. Given a positive real number γ, find, if existing, a causal linear control $K : \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$, $(\boldsymbol{x}, \boldsymbol{w}) \to \boldsymbol{u}$, such that $\|T_{zw}\| < \gamma$, where T_{zw} denotes the closed loop operator mapping $\boldsymbol{w}$ to $\boldsymbol{z}$.

Problem 1 has been solved for the *full column rank* $\boldsymbol{D}$ case in [5] using a dynamic games approach. The following lemma connects the $\mathcal{H}^\infty$ theory with the dynamic games theory.

Lemma 1. ([9, 5]) Let

$$J(\boldsymbol{u}, \boldsymbol{w}) = \gamma^2 \|\boldsymbol{w}\|_2^2 - \|\boldsymbol{z}\|_2^2 = \int_\Omega (\gamma^2 \boldsymbol{w}^T \boldsymbol{w} - \boldsymbol{z}^T \boldsymbol{z})\,\mathrm{d}t\,.$$

Then, for a given control law $\tilde{\boldsymbol{u}}$, $\|T_{zw}\| < \gamma$ *if and only if* for some $\mu > 0$

$$J(\tilde{\boldsymbol{u}}, \boldsymbol{w}) \geq \mu \|\boldsymbol{w}\|_2^2, \qquad \forall\, \boldsymbol{w} \in \mathcal{L}^2(\Omega). \tag{3}$$

By virtue of Lemma 1 the solution of the $\mathcal{H}^\infty$ problem requires the study of the dynamic game [1]

$$\begin{cases} \min\limits_{\boldsymbol{w}} \max\limits_{\boldsymbol{u}} J(\boldsymbol{u}, \boldsymbol{w}) \\ \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{H}\boldsymbol{w}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{z} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}. \end{cases} \tag{4}$$

Lemma 2. ([2, 5]) The zero-sum dynamic game (4) *with $\boldsymbol{D}$ full column rank* admits a unique feedback saddle point solution *if and only if* there exists a positive semidefinite matrix $\boldsymbol{P}$ which satisfies the Riccati differential equation

$$-\dot{\boldsymbol{P}} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} + \frac{1}{\gamma^2}\boldsymbol{P}\boldsymbol{H}\boldsymbol{H}^T\boldsymbol{P} + \boldsymbol{C}^T\boldsymbol{C} - \left(\boldsymbol{P}\boldsymbol{B} + \boldsymbol{C}^T\boldsymbol{D}\right)(\boldsymbol{D}^T\boldsymbol{D})^{-1}\left(\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{D}^T\boldsymbol{C}\right),$$
$$\boldsymbol{P}(t_f) = \boldsymbol{0}. \tag{5}$$

In this case the solution is given by

$$\boldsymbol{u}^* = -\left(\boldsymbol{D}^\dagger\boldsymbol{C} + \boldsymbol{B}^T(\boldsymbol{D}^T\boldsymbol{D})^{-1}\boldsymbol{P}\right)\boldsymbol{x} \tag{6a}$$

$$\boldsymbol{w}^* = \tfrac{1}{\gamma^2}\boldsymbol{H}^T\boldsymbol{P}\boldsymbol{x}. \tag{6b}$$

For arbitrary $\boldsymbol{u}$, $\boldsymbol{w} \in \mathcal{L}^2(\Omega)$, let

$$\begin{aligned} \boldsymbol{u}_0 &= \boldsymbol{D}\left(\boldsymbol{u} + \left(\boldsymbol{D}^\dagger\boldsymbol{C} + \boldsymbol{B}^T(\boldsymbol{D}^T\boldsymbol{D})^{-1}\boldsymbol{P}\right)\boldsymbol{x}\right) \\ \boldsymbol{w}_0 &= \boldsymbol{w} - \frac{1}{\gamma^2}\boldsymbol{H}^T\boldsymbol{P}\boldsymbol{x}\,, \end{aligned}$$

then

$$J(\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{x}_0^T\boldsymbol{P}(t_0)\,\boldsymbol{x}_0 + \gamma^2\|\boldsymbol{w}_0\|_2^2 - \|\boldsymbol{u}_0\|_2^2,$$

where $J(\boldsymbol{u}, \boldsymbol{w})$ is the same as in Lemma 1.

Now assume there exists a positive semidefinite $\boldsymbol{P}$ satisfying (5). In this case from Lemma 2 we have that the feedback control law $\boldsymbol{u}^*$ defined in (6a) is such that $\boldsymbol{u}_0 = 0$; consequently, letting $\boldsymbol{x}_0 = \boldsymbol{0}$, the corresponding optimal cost becomes $J(\boldsymbol{u}^*, \boldsymbol{w}) = \gamma^2\|\boldsymbol{w}_0\|_2^2$. Now it is possible to prove (see for example [5] and [9]) the existence of a positive scalar k such that, for all $t \in \Omega$, $\|\boldsymbol{w}_0\|_2^2 \geq k\|\boldsymbol{w}\|_2^2$. From this follows that

$$J(\boldsymbol{u}^*, \boldsymbol{w}) \geq \gamma^2 k\|\boldsymbol{w}\|_2^2\,, \quad \forall\, \boldsymbol{w} \in \mathcal{L}^2(\Omega)\,, \tag{7}$$

and, according to Lemma 1, this means that the control law (6a) solves the $\mathcal{H}^\infty$ Problem 1.

[1] The dynamic game requires nonzero initial condition to avoid the trivial solution $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

In this paper we consider the more general situation in which $\boldsymbol{D}$ *is not full column rank*, i. e. $\operatorname{rank}(\boldsymbol{D}) = m_1 < m$. When this happens the minmax problem (4) becomes singular and Lemma 2 does not hold.

We will show that when $\boldsymbol{D}$ is not full column rank and we are under Assumption 1, *solving Problem 1 is equivalent to solve another* $\mathcal{H}^\infty$ *control problem related to a reduced order state equation.*

3. A REDUCED ORDER DIFFERENTIAL GAME

Throughout this and the next section we shall assume that $\boldsymbol{D} = \boldsymbol{0}$; this greatly simplifies the machinery. How to deal with the more general nonzero $\boldsymbol{D}$ case will be detailed in Section 5. When $\boldsymbol{D} = \boldsymbol{0}$, if the number of inputs m equals the number of states n, the solution of Problem 1 is trivial, that is $\boldsymbol{u} = -\boldsymbol{B}^{-1}\boldsymbol{H}\boldsymbol{w}$; therefore we shall assume that $n > m$.

Our goal in this section is to prove that, when $\boldsymbol{D} = \boldsymbol{0}$ and we consider solutions of full information type, problem (4) is equivalent to another minmax problem acting on a *reduced order* state equation.

We use a procedure introduced, in the optimal control setting, by Butman [3]. Let $\boldsymbol{E}$ a time-varying continuously differentiable matrix, $\boldsymbol{E}(t) \in \mathbb{R}^{n\times(n-m)}$, such that for all $t \in \Omega$

$$\boldsymbol{E}^T\boldsymbol{B} = \boldsymbol{0}, \qquad \boldsymbol{E}^T\boldsymbol{E} = \boldsymbol{I}\,. \tag{8}$$

Note that the existence of $\boldsymbol{E}$ is guaranteed from the fact that $n > m$ and that $\boldsymbol{B}$ is full column rank.

Now consider the following decomposition of the state space

$$\boldsymbol{x} = \boldsymbol{E}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{v}\,. \tag{9}$$

Observe that, by virtue of (8) and (9), we can write

$$\boldsymbol{y} = \boldsymbol{E}^T\boldsymbol{x} \tag{10a}$$

$$\boldsymbol{v} = \boldsymbol{B}^\dagger\boldsymbol{x}\,. \tag{10b}$$

Differentiating (10a) we obtain

$$\dot{\boldsymbol{y}} = \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}\boldsymbol{v} + \tilde{\boldsymbol{H}}\boldsymbol{w}, \tag{10}$$

where

$$\tilde{\boldsymbol{A}} = \dot{\boldsymbol{E}}^T\boldsymbol{E} + \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E} \tag{12a}$$

$$\tilde{\boldsymbol{B}} = \dot{\boldsymbol{E}}^T\boldsymbol{B} + \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{B} \tag{12b}$$

$$\tilde{\boldsymbol{H}} = \boldsymbol{E}^T\boldsymbol{H}. \tag{12c}$$

Differentiating (9) we have

$$\dot{\boldsymbol{x}} = \boldsymbol{E}\dot{\boldsymbol{y}} + \dot{\boldsymbol{E}}\boldsymbol{y} + \boldsymbol{B}\dot{\boldsymbol{v}} + \dot{\boldsymbol{B}}\boldsymbol{v}\,. \tag{13}$$

Equaling the expression for $\dot{\boldsymbol{x}}$ in (2) and (13) and premultiplying both sides by $\boldsymbol{B}^\dagger$ we obtain

$$\boldsymbol{B}^\dagger \boldsymbol{A}\boldsymbol{E}\boldsymbol{y} + \boldsymbol{B}^\dagger \boldsymbol{A}\boldsymbol{B}\boldsymbol{v} + \boldsymbol{B}^\dagger \boldsymbol{H}\boldsymbol{w} + \boldsymbol{u} = \dot{\boldsymbol{v}} + \boldsymbol{B}^\dagger \dot{\boldsymbol{E}}\boldsymbol{y} + \boldsymbol{B}^\dagger \dot{\boldsymbol{B}}\boldsymbol{v}, \tag{14}$$

where we have used the fact that $\boldsymbol{B}^\dagger \boldsymbol{E} = \boldsymbol{0}$. From (14) it follows

$$\boldsymbol{u} = \dot{\boldsymbol{v}} + \boldsymbol{B}^\dagger \big(\dot{\boldsymbol{B}} - \boldsymbol{A}\boldsymbol{B}\big)\, \boldsymbol{v} + \boldsymbol{B}^\dagger \big(\dot{\boldsymbol{E}} - \boldsymbol{A}\boldsymbol{E}\big)\, \boldsymbol{y} - \boldsymbol{B}^\dagger \boldsymbol{H}\boldsymbol{w}\,. \tag{15}$$

Replacing in the output equation of system (2) (with $\boldsymbol{D} = \boldsymbol{0}$) equality (9), we obtain

$$\boldsymbol{z} = \tilde{\boldsymbol{C}}\boldsymbol{y} + \tilde{\boldsymbol{D}}\boldsymbol{v}, \tag{16}$$

where

$$\tilde{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{E} \tag{17a}$$

$$\tilde{\boldsymbol{D}} = \boldsymbol{C}\boldsymbol{B}. \tag{17b}$$

Now let us consider the following two systems

$$\mathcal{G} \quad := \quad \left\{ \begin{array}{lcll} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{H}\boldsymbol{w}, & \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{v} & = & \boldsymbol{B}^\dagger \boldsymbol{x}, & \end{array} \right. \tag{18}$$

$$\mathcal{F} \quad := \quad \left\{ \begin{array}{lcll} \dot{\boldsymbol{y}} & = & \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}\boldsymbol{v} + \tilde{\boldsymbol{H}}\boldsymbol{w}, & \qquad \boldsymbol{y}(t_0) = \boldsymbol{E}^T \boldsymbol{x}_0 \\ \boldsymbol{u} & = & \boldsymbol{B}^\dagger \big(\dot{\boldsymbol{E}} - \boldsymbol{A}\boldsymbol{E}\big)\, \boldsymbol{y} + \boldsymbol{B}^\dagger \big(\dot{\boldsymbol{B}} - \boldsymbol{A}\boldsymbol{B}\big)\, \boldsymbol{v} - \boldsymbol{B}^\dagger \boldsymbol{H}\boldsymbol{w} + \dot{\boldsymbol{v}} & \end{array} \right. \tag{19}$$

System (18) defines an operator $G : (\boldsymbol{u}, \boldsymbol{w}) \to \boldsymbol{v}$, while system (19) defines an operator $F : (\boldsymbol{v}, \boldsymbol{w}) \to \boldsymbol{u}$. It is simple to show that, for fixed $\boldsymbol{w}$'s, F is the inverse of G and viceversa, that is $G(F(\boldsymbol{v}, \boldsymbol{w}),\, \boldsymbol{w}) = \boldsymbol{v}$ and $F(G(\boldsymbol{u}, \boldsymbol{w}),\, \boldsymbol{w}) = \boldsymbol{u}$.

We can redefine the cost function in the following way

$$\tilde{J}(\boldsymbol{v}, \boldsymbol{w}) := J(F(\boldsymbol{v}, \boldsymbol{w}),\, \boldsymbol{w}) = J(\boldsymbol{u}, \boldsymbol{w}), \tag{20}$$

and consider the new dynamic game acting on a reduced order state equation (because $\boldsymbol{y}(t) \in \mathbb{R}^{n-m}$)

$$\left\{ \begin{array}{l} \min\limits_{\boldsymbol{w}} \max\limits_{\boldsymbol{v}} \tilde{J}(\boldsymbol{v}, \boldsymbol{w}) \\ \dot{\boldsymbol{y}} = \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}\boldsymbol{v} + \tilde{\boldsymbol{H}}\boldsymbol{w}, \qquad \boldsymbol{y}(t_0) = \boldsymbol{E}^T \boldsymbol{x}_0 \\ \tilde{\boldsymbol{z}} = \tilde{\boldsymbol{C}}\boldsymbol{y} + \tilde{\boldsymbol{D}}\boldsymbol{v}. \end{array} \right. \tag{21}$$

In the next theorem we will show that problem (21) is equivalent to the original problem (4).

Theorem 1. $(\boldsymbol{u}^*, \boldsymbol{w}^*)$ is a feedback solution of problem (4) with $\boldsymbol{D} = \boldsymbol{0}$ *if and only if* $(\boldsymbol{v}^*, \boldsymbol{w}^*)$ is a feedback solution of problem (21), where

$$\boldsymbol{v}^* = \boldsymbol{v}^*(\boldsymbol{w}) = G(\boldsymbol{u}^*, \boldsymbol{w}) \tag{22a}$$

$$\boldsymbol{u}^* = \boldsymbol{u}^*(\boldsymbol{w}) = F(\boldsymbol{v}^*, \boldsymbol{w}). \tag{22b}$$

Proof. If $(\boldsymbol{v}^*, \boldsymbol{w}^*)$ is solution of (21), we have that

$$\tilde{J}(\boldsymbol{v}, \boldsymbol{w}^*) \le \tilde{J}(\boldsymbol{v}^*, \boldsymbol{w}^*) \le \tilde{J}(\boldsymbol{v}^*, \boldsymbol{w}), \quad \forall\, (\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega). \tag{23}$$

In order to prove that $(\boldsymbol{u}^*, \boldsymbol{w}^*)$, with $\boldsymbol{u}^*$ satisfying (22b), is solution of (4), we have to show that

$$J(\boldsymbol{u}, \boldsymbol{w}^*) \le J(\boldsymbol{u}^*(\boldsymbol{w}^*),\, \boldsymbol{w}^*) \le J(\boldsymbol{u}^*(\boldsymbol{w}),\, \boldsymbol{w}), \quad \forall\, (\boldsymbol{u}, \boldsymbol{w}) \in \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega). \tag{24}$$

By contradiction suppose there exists a feedback solution $\hat{\boldsymbol{u}}(\boldsymbol{w}) \neq \boldsymbol{u}^*(\boldsymbol{w})$ such that

$$J(\hat{\boldsymbol{u}}(\boldsymbol{w}^*),\, \boldsymbol{w}^*) > J(\boldsymbol{u}^*(\boldsymbol{w}^*),\, \boldsymbol{w}^*) \tag{25}$$

and let

$$\hat{\boldsymbol{v}}(\boldsymbol{w}) = G(\hat{\boldsymbol{u}}(\boldsymbol{w}),\, \boldsymbol{w})\,. \tag{26}$$

We obtain

$$\tilde{J}(\hat{\boldsymbol{v}}(\boldsymbol{w}^*),\, \boldsymbol{w}^*) = J(\hat{\boldsymbol{u}}(\boldsymbol{w}^*),\, \boldsymbol{w}^*) > J(\boldsymbol{u}^*(\boldsymbol{w}^*),\, \boldsymbol{w}^*) = \tilde{J}(\boldsymbol{v}^*, \boldsymbol{w}^*) \tag{27}$$

which contradicts (23); therefore the left inequality in (24) is proven. The proof of the right inequality follows the same guidelines.

The proof that if $(\boldsymbol{u}^*, \boldsymbol{w}^*)$ is a solution of (4) then $(\boldsymbol{v}^*, \boldsymbol{w}^*)$ is a solution of (21) is analogous. □

From equations (22) follows that the solutions considered in Theorem 1 are of full information type, that is the player "$\boldsymbol{u}$" have to know the move of the player "$\boldsymbol{v}$" and viceversa; therefore Theorem 1 establishes a one-to-one correspondence between the full information solutions of the game (4) and the full information solutions of the game (21). There is no full state feedback counterpart of Theorem 1; this is the reason for which we cannot extend the technique developed in this paper to full state feedback $\mathcal{H}^\infty$ control problems.

4. MAIN RESULT

In this section we come back to the $\mathcal{H}^\infty$ problem; using Theorem 1 we will show the equivalence between the original Problem 1 and a reduced order $\mathcal{H}^\infty$ problem.

Let us consider the time-varying system

$$\left\{ \begin{array}{lcl} \dot{\boldsymbol{y}} & = & \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}\boldsymbol{v} + \tilde{\boldsymbol{H}}\boldsymbol{w}, \qquad \boldsymbol{y}(t_0) = \boldsymbol{0} \\ \tilde{\boldsymbol{z}} & = & \tilde{\boldsymbol{C}}\boldsymbol{y} + \tilde{\boldsymbol{D}}\boldsymbol{v} \end{array} \right. \qquad t \in \Omega. \tag{28}$$

Problem 2. Given a positive real number γ, find, if existing, a causal linear control $K : \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$, $(\boldsymbol{y}, \boldsymbol{w}) \to \boldsymbol{v}$, such that $\|T_{\tilde{z}w}\| < \gamma$, where $T_{\tilde{z}w}$ denotes the closed loop operator mapping $\boldsymbol{w}$ to $\tilde{\boldsymbol{z}}$.

Theorem 2. Assume $\boldsymbol{D} = \boldsymbol{0}$. Then

i) Problem 1 admits a solution if and only if Problem 2 admits a solution.

ii) If Problem 2 is regular, that is $\tilde{\boldsymbol{D}}$ is full column rank, it admits a solution if and only if there exists a unique positive semidefinite solution $\tilde{\boldsymbol{P}}$ of the reduced order Riccati equation

$$-\dot{\tilde{\boldsymbol{P}}} = \tilde{\boldsymbol{P}}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^T\tilde{\boldsymbol{P}} + \frac{1}{\gamma^2}\tilde{\boldsymbol{P}}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{H}}^T\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{C}}^T\tilde{\boldsymbol{C}} - \big(\tilde{\boldsymbol{P}}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}^T\tilde{\boldsymbol{D}}\big)(\tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{D}})^{-1}\big(\tilde{\boldsymbol{B}}^T\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{C}}\big),$$
$$\tilde{\boldsymbol{P}}(t_f) = \boldsymbol{0}\,; \tag{29}$$

in this case the control law

$$\boldsymbol{u} = \boldsymbol{K}_1\boldsymbol{x} + \boldsymbol{K}_2\boldsymbol{w} \tag{30}$$

with

$$\boldsymbol{K}_1 = \dot{\tilde{\boldsymbol{K}}}_1\boldsymbol{E}^T + \tilde{\boldsymbol{K}}_1\left(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{K}}_1\right)\boldsymbol{E}^T + \boldsymbol{B}^\dagger\big(\dot{\boldsymbol{B}} - \boldsymbol{A}\boldsymbol{B}\big)\tilde{\boldsymbol{K}}_1\boldsymbol{E}^T + \boldsymbol{B}^\dagger\big(\dot{\boldsymbol{E}} - \boldsymbol{A}\boldsymbol{E}\big)\boldsymbol{E}^T \tag{31a}$$

$$\boldsymbol{K}_2 = \tilde{\boldsymbol{K}}_1\tilde{\boldsymbol{H}} - \boldsymbol{B}^\dagger\boldsymbol{H} \tag{31b}$$

$$\tilde{\boldsymbol{K}}_1 = -\left(\tilde{\boldsymbol{D}}^\dagger\tilde{\boldsymbol{C}} + \tilde{\boldsymbol{B}}^T(\tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{D}})^{-1}\tilde{\boldsymbol{P}}\right) \tag{31c}$$

is optimal for the original Problem 1, i.e. it is such that $\|T_{zw}\| < \gamma$.

P r o o f. (i) It is a straight consequence of Lemma 1 and of equality (20).

(ii) If $\tilde{\boldsymbol{D}}$ is full column rank problem (21) is regular and, applying Lemma 2, the solution is

$$\boldsymbol{v}^* = \tilde{\boldsymbol{K}}_1\boldsymbol{y} \tag{32a}$$

$$\boldsymbol{w}^* = \tilde{\boldsymbol{K}}_2\boldsymbol{y}, \tag{32b}$$

where $\tilde{\boldsymbol{K}}_1$ has the expression (31c) and

$$\tilde{\boldsymbol{K}}_2 = \frac{1}{\gamma^2}\tilde{\boldsymbol{H}}^T\tilde{\boldsymbol{P}}. \tag{33}$$

Substituting equations (32) into system (19), it is readily seen that the solution of the original problem (4), by virtue of Theorem 1, is given by

$$\begin{aligned} \boldsymbol{u}^*(t) &= F(\boldsymbol{v}^*, \boldsymbol{w})\,(t) \\ &= \boldsymbol{K}_1(t)\,\boldsymbol{x}(t) + \boldsymbol{K}_2(t)\,\boldsymbol{w}(t) + \mathrm{d}\delta(t - t_0), \end{aligned} \tag{34}$$

where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have the expressions (31a) and (31b), $\delta(t)$ is the delta function centered at 0, and

$$\boldsymbol{x} = \boldsymbol{E}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{v}^* \tag{35a}$$

$$\mathrm{d} = (\boldsymbol{v}^*(t_0^+) - \boldsymbol{v}^*(t_0^-)). \tag{35b}$$

Now, elaborating with some algebra the equations in (18) and (19), it is possible to show that the control law (30) assures that $\boldsymbol{v} = \tilde{\boldsymbol{K}}_1\boldsymbol{y}$. Let

$$\boldsymbol{v}_0 = \boldsymbol{v} - \tilde{\boldsymbol{K}}_1\boldsymbol{y} \tag{36a}$$

$$\boldsymbol{w}_0 = \boldsymbol{w} - \tilde{\boldsymbol{K}}_2\boldsymbol{y}. \tag{36b}$$

From Lemma 2 with $\boldsymbol{x}(t_0) = \boldsymbol{0}$

$$\tilde{J}(\boldsymbol{v}, \boldsymbol{w}) = \gamma^2\|\boldsymbol{w}_0\|_2^2 - \|\boldsymbol{v}_0\|_2^2, \tag{37}$$

hence, since $\boldsymbol{v}_0 = \boldsymbol{0}$, we have that

$$\tilde{J}(\boldsymbol{v}, \boldsymbol{w}) = \gamma^2\|\boldsymbol{w}_0\|_2^2\,. \tag{38}$$

Under the control law (30), $\boldsymbol{w}$ and $\boldsymbol{w}_0$ are the input and the output respectively of the system

$$\begin{cases} \dot{\boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)\,\boldsymbol{x} + (\boldsymbol{B}\boldsymbol{K}_2 + \boldsymbol{H})\,\boldsymbol{w}, \quad \boldsymbol{x}(t_0) = \boldsymbol{0} \\ \boldsymbol{w}_0 = -\tilde{\boldsymbol{K}}_2\boldsymbol{E}^T\boldsymbol{x} + \boldsymbol{w}. \end{cases} \tag{39}$$

This system is invertible and the inverse is

$$\begin{cases} \dot{\boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1 + \boldsymbol{B}\boldsymbol{K}_2\tilde{\boldsymbol{K}}_2\boldsymbol{E}^T + \boldsymbol{H}\tilde{\boldsymbol{K}}_2\boldsymbol{E}^T)\,\boldsymbol{x} + (\boldsymbol{B}\boldsymbol{K}_2 + \boldsymbol{H})\,\boldsymbol{w}_0, \quad \boldsymbol{x}(t_0) = \boldsymbol{0} \\ \boldsymbol{w} = \tilde{\boldsymbol{K}}_2\boldsymbol{E}^T\boldsymbol{x} + \boldsymbol{w}_0. \end{cases} \tag{40}$$

Since system (40) cannot have finite escape time, we can find $\mu > 0$ such that

$$\gamma^2\|\boldsymbol{w}_0\|_2^2 \geq \mu\|\boldsymbol{w}\|_2^2 \tag{41}$$

which implies

$$J(\boldsymbol{u}, \boldsymbol{w}) = \tilde{J}(\boldsymbol{v}, \boldsymbol{w}) \geq \mu\|\boldsymbol{w}\|_2^2. \tag{42}$$

Now from Lemma 1 the statement of the theorem readily follows. □

Remark 1. Note that $\boldsymbol{v}^*(t)$ is discontinuous at the point $t = t_0$; indeed the initial condition of the state equation requires that $\boldsymbol{v}^*(t_0^-) = \boldsymbol{B}^\dagger\boldsymbol{x}_0$, while the optimal control law requires that $\boldsymbol{v}^*(t_0^+) = \tilde{\boldsymbol{K}}_1(t_0)\,\boldsymbol{y}_0 = \tilde{\boldsymbol{K}}_1(t_0)\,\boldsymbol{E}^T(t_0)\,\boldsymbol{x}_0$. Hence the solution $\boldsymbol{u}^*$ of problem (4) contains an impulse at t_0; this is not surprising since $\boldsymbol{u}$ is not constrained to be bounded. Conversely, due to the zero initial condition in the statement of the $\mathcal{H}^\infty$ problem, the impulse, appearing in the solution of the associated singular dynamic game, is not present in the control law given in Theorem 2.

If $\tilde{\boldsymbol{D}}$ is not full column rank, one can either apply again the reduction procedure if $n-m>m$ or to replace, as suggested in [9], the output equation in problem (21) with

$$\tilde{\boldsymbol{z}} = \begin{pmatrix} \tilde{\boldsymbol{C}} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} \tilde{\boldsymbol{D}} \\ \beta \boldsymbol{I} \end{pmatrix} \boldsymbol{v}, \tag{43}$$

where β is a sufficiently small positive number. Obviously the same trick could be applied directly to the original problem (4). However now the advantage is that, in any case, we are dealing with a reduced order state equation.

5. EXTENSION TO THE NONZERO D CASE

Now we will show that the case $0 < \operatorname{rank}(\boldsymbol{D}) = m_1 < m$ can be treated using the same machinery introduced in Section 3. Let us denote by $\boldsymbol{V}$ a continuously differentiable matrix, $\boldsymbol{V}(t) \in \mathbb{R}^{m\times m}$, such that

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{I}, \qquad \boldsymbol{D}\boldsymbol{V} = \begin{pmatrix} \boldsymbol{D}_1 & \boldsymbol{0} \end{pmatrix}, \tag{44}$$

where $\boldsymbol{D}_1(t) \in \mathbb{R}^{p\times m_1}$ is full column rank. Letting $\boldsymbol{u} = \boldsymbol{V}\boldsymbol{r}$, $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}_1^T & \boldsymbol{r}_2^T \end{pmatrix}^T$, and $\boldsymbol{B}\boldsymbol{V} = \begin{pmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{pmatrix}$, with $\boldsymbol{B}_1(t) \in \mathbb{R}^{n\times m_1}$ and $\boldsymbol{B}_2(t) \in \mathbb{R}^{n\times(m-m_1)}$ full column rank, system (2) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{r}_1 + \boldsymbol{B}_2\boldsymbol{r}_2 + \boldsymbol{H}\boldsymbol{w}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{0} \\ \boldsymbol{z} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_1\boldsymbol{r}_1 \end{cases} \qquad t \in \Omega. \tag{45}$$

Let

$$\boldsymbol{x} = \boldsymbol{E}\boldsymbol{y} + \boldsymbol{B}_2\boldsymbol{v}_1, \tag{46}$$

where $\boldsymbol{E}$ is continuously differentiable, $\boldsymbol{E}(t) \in \mathbb{R}^{n\times(n-m+m_1)}$, $\boldsymbol{E}^T\boldsymbol{E} = \boldsymbol{I}$ and $\boldsymbol{E}^T\boldsymbol{B}_2 = \boldsymbol{0}$. Differentiating (46) and following the same guidelines of Section 3, we obtain the reduced order system ($\boldsymbol{y}(t) \in \mathbb{R}^{n-m+m_1}$)

$$\begin{cases} \dot{\boldsymbol{y}} &= \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}_1\boldsymbol{r}_1 + \tilde{\boldsymbol{B}}_2\boldsymbol{v}_1 + \tilde{\boldsymbol{H}}\boldsymbol{w}, \quad \boldsymbol{y}(t_0) = \boldsymbol{0} \\ \tilde{\boldsymbol{z}} &= \tilde{\boldsymbol{C}}\boldsymbol{y} + \boldsymbol{D}_1\boldsymbol{r}_1 + \tilde{\boldsymbol{D}}_2\boldsymbol{v}_1, \end{cases} \tag{47}$$

where $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{H}}$, and $\tilde{\boldsymbol{C}}$ are still given from equations (12a), (12c) and (17a) respectively, and

$$\tilde{\boldsymbol{B}}_1 = \boldsymbol{E}^T\boldsymbol{B}_1 \tag{48a}$$

$$\tilde{\boldsymbol{B}}_2 = \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{B}_2 + \dot{\boldsymbol{E}}^T\boldsymbol{B}_2 \tag{48b}$$

$$\tilde{\boldsymbol{D}}_2 = \boldsymbol{C}\boldsymbol{B}_2. \tag{48c}$$

Letting

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{r}_1^T & \boldsymbol{v}_1^T \end{pmatrix}^T \tag{49a}$$

$$\tilde{\boldsymbol{B}} = \begin{pmatrix} \tilde{\boldsymbol{B}}_1 & \tilde{\boldsymbol{B}}_2 \end{pmatrix} \tag{49b}$$

$$\tilde{\boldsymbol{D}} = \begin{pmatrix} \boldsymbol{D}_1 & \tilde{\boldsymbol{D}}_2 \end{pmatrix}, \tag{49c}$$

system (47) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{y}} &= \tilde{\boldsymbol{A}}\boldsymbol{y} + \tilde{\boldsymbol{B}}\boldsymbol{v} + \tilde{\boldsymbol{H}}\boldsymbol{w}, \quad \boldsymbol{y}(t_0) = \boldsymbol{0} \\ \tilde{\boldsymbol{z}} &= \tilde{\boldsymbol{C}}\boldsymbol{y} + \tilde{\boldsymbol{D}}\boldsymbol{v}. \end{cases} \tag{50}$$

Using the same machinery of Sections 3 and 4, it is simple to prove Theorem 2 in the general case considered in the current section. In particular Problem 1 admits a solution if and only if Problem 2, stated for system (50), admits a solution. Moreover, if the matrix $\tilde{\boldsymbol{D}}$ defined in (49c) is full column rank, Problem 2 admits a solution if and only if the reduced order Riccati equation

$$-\dot{\tilde{\boldsymbol{P}}} = \tilde{\boldsymbol{P}}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^T\tilde{\boldsymbol{P}} + \frac{1}{\gamma^2}\tilde{\boldsymbol{P}}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{H}}^T\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{C}}^T\tilde{\boldsymbol{C}} - \big(\tilde{\boldsymbol{P}}\tilde{\boldsymbol{B}} + \tilde{\boldsymbol{C}}^T\tilde{\boldsymbol{D}}\big)(\tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{D}})^{-1}\big(\tilde{\boldsymbol{B}}^T\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{C}}\big),$$

$$\tilde{\boldsymbol{P}}(t_f) = \boldsymbol{0} \tag{51}$$

has a unique positive semidefinite solution $\tilde{\boldsymbol{P}}$.

Now letting, as in (31c),

$$\tilde{\boldsymbol{K}}_1 = -\Big(\tilde{\boldsymbol{D}}^\dagger\tilde{\boldsymbol{C}} + \tilde{\boldsymbol{B}}^T(\tilde{\boldsymbol{D}}^T\tilde{\boldsymbol{D}})^{-1}\tilde{\boldsymbol{P}}\Big) = \begin{pmatrix}\tilde{\boldsymbol{K}}_{11} \\ \tilde{\boldsymbol{K}}_{21}\end{pmatrix}, \tag{52}$$

after some algebra it is possible to show that the solution of Problem 1 is given by

$$\begin{aligned} \begin{pmatrix}\boldsymbol{r}_1 \\ \boldsymbol{r}_2\end{pmatrix} &= \boldsymbol{K}_1\boldsymbol{x} + \boldsymbol{K}_2\boldsymbol{w} \\ &= \begin{pmatrix}\boldsymbol{K}_{11} \\ \boldsymbol{K}_{21}\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{K}_{22}\end{pmatrix}\boldsymbol{w}, \end{aligned} \tag{53}$$

where

$$\boldsymbol{K}_{11} = \tilde{\boldsymbol{K}}_{11}\boldsymbol{E}^T \tag{54a}$$

$$\boldsymbol{K}_{21} = \dot{\tilde{\boldsymbol{K}}}_{21}\boldsymbol{E}^T + \tilde{\boldsymbol{K}}_{21}(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{K}}_1)\,\boldsymbol{E}^T + \boldsymbol{B}_2^\dagger(\dot{\boldsymbol{B}}_2 - \boldsymbol{A}\boldsymbol{B}_2)\tilde{\boldsymbol{K}}_{21}\boldsymbol{E}^T + \boldsymbol{B}_2^\dagger(\dot{\boldsymbol{E}} - \boldsymbol{A}\boldsymbol{E})\,\boldsymbol{E}^T \tag{54b}$$

$$\boldsymbol{K}_{22} = \tilde{\boldsymbol{K}}_{21}\tilde{\boldsymbol{H}} - \boldsymbol{B}_2^\dagger\boldsymbol{H}\,. \tag{54c}$$

In terms of the variable $\boldsymbol{u}$, the optimal control law (53) is given by

$$\boldsymbol{u} = \boldsymbol{V}\begin{pmatrix}\boldsymbol{K}_{11} \\ \boldsymbol{K}_{21}\end{pmatrix}\boldsymbol{x} + \boldsymbol{V}\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{K}_{22}\end{pmatrix}\boldsymbol{w}\,. \tag{55}$$

6. CONCLUSIONS

In this paper the singular finite horizon full information $\mathcal{H}^\infty$ control problem has been considered. Using the dynamic games theory and a suitable state space decomposition, we have shown that the original problem is equivalent to a reduced order one. If a certain assumption is satisfied, this new problem is regular and can be solved via standard methods. Future research will be devoted to investigate two open problems: full state feedback and the extension to the output feedback case.

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