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NUMERICAL ALGORITHM FOR NONSMOOTH STABILIZATION OF TRIANGULAR FORM SYSTEMS[1]

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The aim of this contribution is to present a simple method for finding nonsmooth stabilizers in cases when the smooth ones are not available. More precisely, we adress the stabilization of a certain class of single-input nonlinear systems; namely, the class of systems that are state equivalent to the so-called singular triangular form. It is based on the formal application of the exact linearization scheme to the systems with linear part having noncontrollable unstable mode. Such a formal approach leads to the stabilizer possesing singularities and a regularization process is suggested to remove them. This approach is realized and tested by computer simulations for various nonlinear systems.

1. INTRODUCTION: NONSMOOTH STABILIZATION

We are interested in the static state feedback stabilization of the smooth nonlinear controlled dynamical system without outputs:

$$\dot{x} = f(x) + \sum_{k=1}^{m} g_k(x)\, u_k, \quad u = (u_1, \ldots, u_m)' \in \mathbb{R}^m, \ \ x \in \mathbb{R}^n. \tag{1}$$

Namely, let $x_0 \in \mathbb{R}^n$ be an equilibrium of (1), i.e. $f(x_0) = 0$, then this system is called globally (locally) asymptotically stabilizable at x_0 if there exists feedback law (asymptotic stabilizer of (1))

$$\alpha(x) = (\alpha_1(x), \ldots, \alpha_m(x))', \quad \alpha : \mathbb{R}^n \to \mathbb{R}^m, \tag{2}$$

such that the corresponding closed loop system (i.e. system (1) with $u = \alpha(x)$, α given by (2)) has x_0 as its globally (locally) asymptotically stable equilibrium point. System is called stabilizable if there exists feedback that makes the corresponding closed loop system stable. System is called nonasymptotically stabilizable if it is stabilizable, but it is not asymptotically stabilizable. Stabilizability will be called

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Table 1. Stabilization of the nonlinear system (NLS) and modification of eigenvalues of its aproximate linearization by linear feedback $u = Kx$.

No.	Best real parts of $eig(F + GK)$	Relation	Stabilizability type of NLS (1)
1.	negative	$\Rightarrow$	local smooth asymptotic
2.	nonpositive	$\Leftarrow$	local smooth
3.	nonpositive	? $\Rightarrow$?	local smooth or nonsmooth
4.	exist(s) positive	?$\Rightarrow$?	local nonsmooth

linear (smooth, nonsmooth, continuous, etc.) dependingly on the *best available* character of the stabilizer α. For the survey of the stabilization topic see e. g. [19].

The usual and most straightforward way to study local asymptotic stabilization is to consider the approximate linearization of (1) at x_0, namely, the linear system:

$$\dot{y} = Fy + Gu, \tag{3}$$

related with the original nonlinear system as follows:

$$y = x - x_0, \quad F = \frac{\partial f}{\partial x}(x_0), \quad G = [g_1(x_0)|...|g_m(x_0)].$$

Relations between the best available real parts of eigenvalues of $F + GK$, where K is arbitrary $m \times n$ matrix defining a linear feedback $u = Kx$, and various types of the local stabilizability of (1) at x_0 are collected in Table 1.

Question marks in Table 1 mean that the problem is open and should be solved by higher-order, intristically nonlinear, methods. Problem 3 of this table is usually referred as the so-called critical case of the stabilization and may be tackled e. g. by means of Lyapunov function method or center manifold approach (see e. g. [2, 17]).

Problem 4 is exactly the area to which we aim contribute here: how to find stabilizer that is nonsmooth when no smooth is available? Due to the infinetesimal character of the relation between nonlinear system and its approximate linearization (i. e. system (3) is completely determined by (1) considered in an arbitrarily small neighbourhood of x_0) the only thing that is sure regarding Problem 4 is that the stabilizer (if any) should be nonsmooth just at x_0. Actually, if a smooth feedback (2) stabilizes system (1), then obviously $F + G\alpha_x(x_0)$ has eigenvalues with nonpositive real parts. It is therefore reasonable to search a stabilizer that is everywhere except x_0 smooth (and hopefully continuous at x_0). Such a stabilizer particularly guarantees that all solutions of the corresponding closed loop system are well defined with the only possible nonuniqueness at x_0.

The above task is a challenging truely nonlinear problem, important both theoretically and practically (e. g. to justify robustness of stabilizers in critical cases: small perturbation of the Problem 3 leads either to the Problem 1 or to the Problem 4). Nevertheless, it is also very dificult and as the consequence only rare results on this topic with a rather limited applicability are available (see [16, 11]). For the discrete time version of this problem see also [18]. For a recent state of the art in the area of the nonsmooth stabilization see [4] and references in there.

2. STABILIZATION VIA EXACT LINEARIZATION

Exact linearization is probably the main breakthrough made by the differential-geometric approach in the nonlinear control theory. Contrary to the approximate linearization (3) it aims to find reasonable *exact* transformations, (e. g. nonlinear change of coordinates in the state space, feedback of various levels of complexity, etc.) taking the original nonlinear system (1) to a controllable linear system (CLS).

Let us recall that $GL(m,\mathbb{R})$ stands for the group of all $m \times m$ regular matrices.

Definition 1. System (1) is called smoothly locally feedback linearizable at x_0 if there exist a neighbourhood U_{x_0} of x_0 and a neighbourhood V_0 of $0 \in \mathbb{R}^n$, feedback[3] of the form

$$u = \alpha(x) + \beta(x)v, \quad \alpha(x) \in C^\infty(U_{x_0}, \mathbb{R}^m), \quad \beta(x) \in C^\infty(U_{x_0}, GL(m,\mathbb{R})),$$

and a diffeomorphism

$$\mathcal{D} : V_0 \to U_{x_0}, \quad x = \mathcal{D}(y),$$

transforming the system (1) into a controllable linear system

$$\dot{y} = Fy + Gv, \quad y \in \mathbb{R}^n,\ v \in \mathbb{R}^m, \tag{4}$$

where F, G are $(n \times n)$ and $(n \times m)$ matrices, respectively.

The system is called smoothly state linearizable if it is feedback linearizable with $\beta(x) \equiv I_m$ and $\alpha(x) \equiv 0$. Where no confusion arises, various adjectives may be ommited to shorten the exposition.

Remark 1. Linearization is a particular case of the system equivalence: two nonlinear systems are called mutually feedback (state) equivalent, if they can be transformed into one another using appropriate transformations. Linearizability then means equivalence to a linear controllable system and both terminology will be used in the sequel. Where appropriatte, we often call this linearization as the exact one to stress the difference with respect to the aproximate linearization (3). The whole remark applies also to various kinds of global linearization and equivalence that will be later introduced.

Definition 2. System (1) is called *globally feedback (state) linearizable at x_0 to a linear system on $\mathbb{R}^n$* if it is at this point locally linearizable and $V_0 = \mathbb{R}^n$. It is called *globally linearizable on $\mathbb{R}^n$* if $U_{x_0} = M$. System that is linearizable globally on M to a linear system on $\mathbb{R}^n$ is called *globally linearizable*.

For further details see surveys [6, 20, 10] or books [13, 17].

Positive solution of the smooth exact linearization task gives immediately the solution to the stabilization problem, nevertheless this concerns only Problem No. 1

[3] To avoid confusion, let us remind that the term ‘feedback’ is used in the control theory in different senses: first, as the closing of the open loop system or, secondly, as the transformation leading to the new open loop system with a new input variable.

and 2 of Table 1. Actually, the only additional contribution of the smooth exact linearization in comparision with the approximate one is that the former one enables also to find (or at least to estimate) basin of attraction of the asymptotically stabilized equilibrium. Particularly, in case of the global smooth exact linearization this approach leads to the global stabilization. The reason is that the exact smooth linearization and approximate one are always mutually linearly equivalent (i. e. via linear change of coordinates and a linear feedback). As a consequence, a feedback linearizable system has a controllable approximate linearization (3) and therefore may be stabilized using the linear feedback.

In this paper we aim to adapt the described exact linearization approach to be applicable to the nonsmooth stabilization. First, let us describe the smooth case in some detail.

During the rest of the paper we concentrate ourselves on the single-input case, i. e. $m = 1$ (the multi-input case is analogous, though more technical).

General, coordinate free conditions for the feedback linearization can be given using a quite abstract differential geometric language — see previously mentioned references. To simplify the exposition, let us consider the so-called systems in triangular form (TF). Actually, as we illustrate in the sequel, this is an intermediate step to linearize the system using transformation of the state and feedback provided certain regularity condition is satisfied. Abstract geometrical conditions for transforming the system into TF may be found in [9].

To put it more explicitely, let us consider locally around $x_0 \in I\!\!R^n$ the following single-input system

$$\dot{x} = f(x) + g(x)\,u, \quad x = (x_1, \ldots, x_n)' \in I\!\!R^n, \quad f(x_0) = 0,$$

$$g = (g_1(x), 0, \ldots, 0)', \quad f = (f_1(x), f_2(x), f_3(x_2, \ldots, x_n), \ldots, f_n(x_{n-1}, x_n))', \quad (5)$$

such a system is said to be in TF (or TF-system). The TF (5) is called as a regular one (RTF) if

$$g(x_0) \neq 0, \quad \frac{\partial f_i}{\partial x_{i-1}}(x_0) \neq 0, \quad i = 2, \ldots, n. \quad (6)$$

otherwise it is called as a singular one (STF). The TF (5) is called as a locally bijective one (BTF) if the following mapping from $I\!\!R^{n+1}$ to $I\!\!R^{n+1}$

$$(x_1, x_2, \ldots, x_n, u)' \to (f_2(x), \ldots, f_n(x), x_n, g_1(x)\,u)' \quad (7)$$

is bijective locally around the origin. Obviously, a RTF is always a BTF, the converse is not true, see e. g. $\dot{x}_1 = u,\ \dot{x}_2 = x_1^3$.

Let us remind (see e. g. [12]) that a single-input system (1) is smoothly locally at x_0 feedback exact linearizable if and only if it takes in some smooth coordinates RTF.

Moreover, the following straightforward algorithm for the local (for the global aspects consult ([6])) smooth exact feedback linearization is applicable:

Algorithm 1.

1. Consider smooth system (5 – 6) such that for some $p \geq 1$ it holds $f_k(x) = x_{k-1},\ k > p$. We denote this type of system as Σ_p. Notice, that Σ_n is the general triangular form (5). Then, applying the smooth local change of the coordinates (thanks to (6)) of the form $\tilde{x} = (x_1, \ldots, x_{p-2},\ f_p(x_{p-1}, \ldots$ $\ldots, x_n),\ x_p, \ldots, x_n)'$ we obtain after straightforward computations system of type Σ_{p-1}.
2. Starting with Σ_n and repeating the Step 1 $n-1$ times we will finally obtain system of the type Σ_1:

$$\dot{\tilde{x}}_1 = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u, \quad \dot{\tilde{x}}_k = \tilde{x}_{k-1}, \quad k = 2, \ldots, n,$$

   where $\tilde{f}$, $\tilde{g}$ are scalar functions with $\tilde{g} \neq 0$. Now, introducing a new input variable v via the following feedback

$$v = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\, u, \tag{8}$$

   we have CLS in the Brunovsky canonical form (see [3]).

So, using the above algorithm one can compute (at least locally at $x_0 \in I\!R^n$ — see [6] for global aspects) for any system of type (5 – 6) in a straightforward way the smooth change of coordinates $y = \mathcal{D}(x)$, $\det(\mathcal{D}_x(x_0)) \neq 0$ and the feedback $v = \beta_1(x) + \beta_2(x)\, u$, $\beta_2(x_0) \neq 0$ taking (5) to the linear system in Brunovsky canonical form:

$$\dot{y}_1 = v, \quad \dot{y}_2 = y_1, \ldots, \dot{y}_{n-1} = y_{n-2}, \quad \dot{y}_n = y_{n-1}, \tag{9}$$

where $y = (y_1, \ldots, y_n)'$. System (9) can be easily stabilized using linear feedback

$$v = linst(y) = \sum_{i=1}^{n} y_i c_i,\ c = (c_1, \ldots, c_n)' \in I\!R^n,$$

(see [15, 22] for details) and therefore the system of type (5,6) is stabilized (at least locally) by the smooth feedback

$$u = \alpha(x) = \left(linst(\mathcal{D}(x)) - \beta_1(x)\right)/\beta_2(x). \tag{10}$$

Previous approach has been adopted by the control community for a long time, see e. g. [14] for the toolbox trying explore it. Nevertheless, in spite of the generality of the system form (5), this approach often fails since conditions (6) need not be valid. Namely, it is easy to observe that $\beta_2(x_0) \neq 0$ *if and only if* (6) is valid, i. e. (6) is the crucial regularity condition. Moreover, it is intuitively understable to the people dealing with numerical computations that practically the approach should fail even when the derivatives in (6) are nonzero, but too close to zero. In other words, one cannot get rid of the violation of (6) by claiming it 'nongeneric'. Adaptation of the described algorithm to be able to work with cases when (6) is violated (or 'nearly' violated) is therefore of a great interest.

The cases, when some of equalities (6) are violated, are just the cases when the approximate linearization of (5) is not controllable. Actually, simple computations show that the approximate linearization (3) of (5) has the form

$$F=\begin{bmatrix} * & * & * & * & \cdots & * & * \\ \frac{\partial f_2}{\partial x_1} & * & * & * & \cdots & * & * \\ 0 & \frac{\partial f_3}{\partial x_2} & * & * & \cdots & * & * \\ 0 & 0 & \frac{\partial f_4}{\partial x_3} & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial f_{n-1}}{\partial x_{n-2}} & * & * \\ 0 & 0 & \cdots & 0 & 0 & \frac{\partial f_n}{\partial x_{n-1}} & * \end{bmatrix}(x_0), \quad G=\begin{bmatrix} g_1(x_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If some of the corresponding uncontrollable modes are unstable, then (5) is not stabilizable using smooth feedback. In other words, successful regularization of singularities of (10) may give nonsmooth stabilization of smoothly nonstabilizable system.

3. REGULARIZED ALGORITHM AND NONSMOOTH STABILIZATION

We present and develop here a simple heuristic idea to regularize the stabilization Algorithm 1. This idea is based on the rather straightforward observation that the violation of (6) do not prevent from the formal applying of the described Algorithm 1 and the only trouble is that for the resulting stabilizer given by (10) one has $\alpha(x)\to\infty$ when x approaches some singular subset of $I\!R^n$. This singular subset contains at least the stabilized equilibrium x_0 and is 'negligible' (more exactly, it is a submanifold of the dimension $< n$). Moreover, it can be easily seen (using geometric approach in the spirit of e.g. [13]) that for any nonzero input trajectories starting near this singular subset cross it transversaly (i.e. with a nonzero angle) in isolated points. All these observation immediately suggest the following heuristic adaptation of Algorithm 1:

Algorithm 2.

1. Apply Algorithm 1 and compute feedback $\alpha(x)$ according to (10).
2. Find singular subset $\mathcal{S}\subset I\!R^n$ on which (10) is not defined.
3. Construct sequence of functions $\alpha_k : I\!R^n\to I\!R,\ k=1,2,\dots$ in such a way that each α_k is everywhere continuous, it is everywhere except x_0 smooth and $\alpha_k\to\alpha,\ k\to\infty$ (we skip mathematical technical details regarding the proper definition of this convergence[4]).
4. Select a particular, sufficiently large integer k and consider α_k as the everywhere continuous and everywhere except x_0 smooth stabilizer.

[4] The idea is the following: consider a sequence of open sets $\mathcal{S}_1,\mathcal{S}_2,\dots$ with $\mathcal{S}_1\cap\mathcal{S}_2\dots=\mathcal{S}\setminus\{x_0\}$. Then, use the partition unity technique to construct for each $k=1,2,\dots$ smooth on $I\!R^n\setminus\{x_0\}$ and continuous on $I\!R^n$ function α_k that equals to α on $\mathcal{S}_k$.

Basic heuristic idea of the above algorithm was independently introduced and tested in [5] and [14], nevertheless, both these papers used a very primitive sequence $\{\alpha_k\}$ (particularly α_k were discontinuous) and they considered only one single-input two-dimensional system.

We concentrate ourselves here both on the selection of the sequence $\{\alpha_k\}$ fulfilling all requirements of Algorithm 2 as well as on the three-dimensional simulations in order to test experimentally viability of this approach.

4. NUMERICAL SIMULATIONS

We describe here shortly the application of Algorithm 2 on the two typical examples.

Example 1. Consider two dimensional system (cf. [2, 5, 7, 11, 14, 16])

$$\dot{x_1} = x_1 + x_2^3, \quad \dot{x_2} = u.$$

New coordinates $y_1 = x_1,\ y_2 = x_1 + x_2^3$, and feedback $v = y_2 + 3x_2^2 u$ leads to

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = v.$$

Observe, that coordinate transformation has the nonsmooth inverse and the feedback mapping cannot be inverted for $x_2 = 0$. This is exactly due to the fact that (6) is violated here. Applying (10) we therefore obtain discontinuous, unbounded, 'stabilizer'

$$u = \alpha_{unb} = (-ax_1 - (b+1)\,(x_1 + x_2^3))/(3x_2^2)$$

where $a, b \in I\!R$ are such that the matrix

$$\begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

has eigenvalues in the open left complex halfplane. Now, let us proceed along the lines suggested by Algorithm 2. We introduce a regularizing parameter *bound* > 0 and define for each its value a regularized feedback α_{bound} in such a way that $\alpha_\infty = \alpha_{unb}$ (then the sequence of Algorithm 2.3 may be taken as $\alpha_{bound_k},\ k = 1, 2, \ldots,$ where $bound_1, bound_2, \ldots$ is arbitrary positive monotounous unbounded sequence of reals). The most trivial idea how to regularize the above singular stabilizer α_{unb} gives discontinuous at the origin, but bounded stabilizer (first introduced in [5])

$$\begin{aligned} & u = \alpha_{bound}(x) \\ = \quad & \min\{|\alpha_{unb}(x)|, bound\}\mathrm{sign}(\alpha_{unb}(x)), \quad bound > 0, \quad \alpha_{bound}(0) = ? \end{aligned} \tag{11}$$

We suggest here more sophisticated idea giving continuous and a. e. smooth stabilizer

$$\begin{aligned} & u = \alpha_{cont}(x) \\ = \quad & \min\{|\alpha_{unb}(x)|, bound|x_1|^{1/3}\}\mathrm{sign}(\alpha_{unb}(x)), \quad bound > 0, \quad \alpha_{cont}(0) = 0. \end{aligned} \tag{12}$$

Notice, that infinitesimaly for $x \to 0$ we have that the inequality $|x_1| \le K|x_2|^{1/3}$, $K = K(bound, a, b) > 0$, defines the set where $\alpha_{unb}(x) = \alpha_{cont}(x)$. For K sufficiently large it contains (near the origin) the curved sectors $x_1(x_1 + x_2^3) < 0$ (the maximal set where $|x_1(t)|$ strictly decreases along the trajectories). Together with the topological linearization arguments of [5, 7] this fact gives even an oportunity to prove theoretically the stability of this closed loop system. We skip the detailed proof, nevertheless, let us underline that for the case of real eigenvalues of the linearized system this proof is straightforward since any trajectory crosses the singularity $x_2 = 0$ only one time.

After numerous simulations made for both types of regularizations, the continuous one appears as more suitable also from the numerical point of view. The reason is that for nonreal eigenvalues of the linearized system the origin approaching trajectory intersects the singularity in its arbitrarily small neighbourhood. For the stabilizer (11) this means that $\dot{x}_2(t) = \pm bound$ very close to the origin and causes failure of the numerical integration procedure. As expected, continuous regularization (12) completely avoids these problems.

Example 2. This example tests the previous approach for the increased dimensionality. Namely, we consider the previous example with added integrator:

$$\dot{x_1} = x_1 + x_2^3, \quad \dot{x_2} = x_3, \quad \dot{x}_3 = u.$$

Transformations $y_1 = x_1,\ y_2 = x_2^3 + x_1,\ y_3 = 3x_2^2x_3 + x_2^3 + x_1,\ v = 3x_2^2u + 6x_2x_3^2 + 3x_2^2x_3 + x_2^3 + x_1$, takes it into the linear form

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = v.$$

Stabilizing linear feedback is $v = -ay_1 - by_2 - cy_3$, where matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{bmatrix},$$

has eigenvalues λ_1, λ_2, λ_3 belonging to the open left complex halplane (particularly, $a = -\lambda_1\lambda_2\lambda_3,\ b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\ c = -(\lambda_1 + \lambda_2 + \lambda_3)$) and after appropriatte computations we obtain unbounded stabilizer

$$u = \alpha_{unb}(x) = -(1+c)x_3 - (1+b+c)x_2/3 - (6x_2x_3^2 + (a+b+c+1)x_1)/(3x_2^2).$$

The analogous idea as in Example 1 will be exploited here, the difference is that we use *two* regularizing parameters $bound_{1,2} > 0$. Having in mind observations made during simulations of Example 1, we consider here only the continuous regularization α_{cont}, defined for each pair $bound_{1,2} > 0$ as follows (observe, that contrary to Example 1 we have $\alpha_{cont} = \alpha_{unb}$ for $bound_{1,2} = 0$ — this is a matter of notation only)

$$\begin{aligned} & u = \alpha_{cont} \\ = \quad & -(1+c)x_3 - (1+b+c)x_2/3 - \frac{(6x_2x_3^2 + (a+b+c+1)x_1)}{\max\{3x_2^2, 3bound_1x_1^{2/3} + 3bound_2x_3^2\}} \to 0 \\ & \text{for } x \to 0. \end{aligned}$$

A similar justification of the above regularization as in the case $n = 2$ is possible: the set of $x \in I\!R^3$ where $\alpha_{unb}(x) = \alpha_{cont}(x)$ is given by $bound_1 x_1^{2/3} + bound_2 x_3^2 \leq x_2^2,\ x \neq 0$, its closure contains the origin, projection of its boundary to $x_{1,2}$-plane is given by $bound_1^{1/2} x_1^{1/3} = \pm x_2$ and for $bound_{1,2} \to 0$ this set tends to $I\!R^3 \setminus \{x_2 = 0\}$.

The following observations are based on numerous computer simulations of the above regularized stabilizer:

- after selecting $\lambda_{1,2,3}$ in the open left complex halfplane one can easily adjust parameters $bound_{1,2} > 0$ within a wide intervals of values such that trajectories starting from numerous initial conditions converge to the origin with the selected precision — see Figs. 1−4 for the illustration
- for $bound_{1,2} > 0$ *too large* α_{unb} and α_{cont} mutually differs too much and as the result closed loop system may be unstable,
- for $bound_{1,2} > 0$ *too small* interuption of the numerical integration procedure occurs (remind that if $bound_{1,2} = 0$ then $\alpha_{unb} = \alpha_{cont}$ everywhere) since numerical values of α_{cont} may be too large.
- effect of $bound_{1,2}$ is illustrated in Figs. 1−3
- algorithm is robust with respect to all parameters and initial states
- good numerical properties of the above continuous regularization are illustrated on Fig. 4: even fast oscillations in the linearized system did not prevent from approaching the origin with extremely high precision
- $x_1(t)$ is converging to zero faster than $x_{2,3}(t)$ — this fact complies with the discrete-time case (see [18]).

5. CONNECTION WITH TOPOLOGICAL LINEARIZATION

Finally, we shortly discuss the possibility of a more rigorous justification of the Algorithm 2. The good basis for this seems to be recently developed notion of the topological (nonsmooth) linearization (see [5, 7] and compare them with a recent [18] dealing with the discrete-time case). Without going into the details, that are out of scope for this contribution (being focused on numerical simulations), the relation of Algorithm 2 with the topological linearization may be characterized as follows.

Topological linearization concept introduces generalizations of state transformation and feedback that are only continuous and moreover they are understood in the functional spaces sense. Particularly, let Ω be a suitable normed functional space of all admissible input signals, then the 'generalized' feedback is understood as a continuous map from $\Omega \times I\!R^n$ into Ω satisfying certain compatibility conditions (see [7] for details). It was also showed there that Example 1 is topologically linearizable. This particularly means that for the sequence α_k constructed according to Algorithm 2 we have that along each trajectory $\alpha_k(x(t)) \to \alpha(x(t))$ in the sense of Ω-norm and this fact justifies the expectation that for a sufficiently great k feedback $\alpha_k(x)$ stabilizes the system.

Fig. 1. $\lambda_{1,2,3} = -10,\ bound_{1,2} = 0.1.$

Fig. 2. $\lambda_{1,2,3} = -10,\ bound_{1,2} = 0.01.$

Fig. 3. $\lambda_{1,2,3} = -10,\ bound_{1,2} = 0.001.$

Fig. 4. $\lambda_1 = -2,\ \lambda_{2,3} = -2 \pm 5\mathrm{i},\ bound_{1,2} = 0.05.$

In other words, singular unbounded stabilizing feedback is well defined in a certain generalized sense and may be arbitrarily approximated by more regular feedbacks in the sense of a reasonable functional space norm. In this respect, topological linearization serves as the theoretical explanation of the practically observed 'negligibility' of the previously investigated singularities.

The shortcommings of the topological linearization consist in their complicated definition, difficulties in proof techniques, especially for higher dimensions. As a consequence, the only proved result is planar, namely, it was proved in [7] that a planar single-input system is topologically linearizable if and only if it is state equivalent to BTF (7–5). The necessary and sufficient conditions for arbitrary single-input system (1) to be state equivalent to BTF were obtained in [9]). In the same paper, the following characterization of BTF-systems was obtained.

Proposition 1. Consider the smooth nonlinear system (1) that is state equivalent to the BTF in a neighbourhood of the origin $\mathcal{N}_0$. Then, there exist

1. an open set $\mathcal{N}$, $\overline{\mathcal{N}} = \mathcal{N}_0$,
2. an open set $\mathcal{P}$, $\overline{\mathcal{P}} = \mathcal{P}_0$, $\mathcal{P}_0$ being a neighbourhood of the origin in $I\!R^n$,
3. an open set $\mathcal{R}$, $\overline{\mathcal{R}} = \mathcal{R}_0$ with $\mathcal{R}_0$, $\mathcal{S}_0$ — neighbourhoods of the origin in $I\!R$,
4. $\mathcal{D} \in C^\infty(\mathcal{N}_0, \mathcal{P}_0) \cap DIFF(\mathcal{N}, \mathcal{P})$[5], $\alpha \in C^\infty(\mathcal{R}_0 \times \mathcal{N}_0, \mathcal{S}_0)$, $\forall\, x \in \mathcal{N}$ $\alpha(x, \cdot) \in DIFF(\mathcal{R}_0, \mathcal{S}_0)$,

such that for any piecewise continuous input $u(t) \in \mathcal{R}_0\ \forall\, t \in [t_0, t_1] \subset I\!R$ and the corresponding trajectory of $x(t) \in \mathcal{N}_0\ \forall\, t \in [t_0, t_1] \subset I\!R$ it holds

$$\dot{y}_1 = v, \quad \dot{y}_2 = y_1, \quad \dot{y}_{n-1} = y_{n-2}, \quad \dots \quad \dot{y}_n = y_{n-1}, \tag{13}$$

where $y = (y_1, \dots, y_n)^T \in I\!R^n$, $\;v \in I\!R$ and

$$y = \mathcal{D}(x), \quad v = \alpha(x, u), \quad x \in \mathcal{N}_0,\ u \in \mathcal{R}_0.$$

Moreover, let x be the corresponding triangular the coordinates, then the set $\mathcal{N}_0 \setminus \mathcal{N} = \cup_{j=2}^{n} \mathcal{N}_j$ and the sets $\mathcal{N}_j = \{x \in \mathcal{N}_0 \mid \frac{\partial f_j}{\partial x_{j-1}}(x) = 0\}$, $j = 2, \dots n$, are not invariant with respect to the original nonlinear system.

This proposition immediatelly supports the idea of applicability of Algorithm 2 for general single-input BTF-systems: apart from singularities stabilizers are-well defined and singularities are not invariant with respect to system trajectories. The last property ensures that the trajectory of the closed-loop system corresponding to the regularized stabilizer always leaves the proximity of a singularity and stays within the set where regularized stabilizer equals to the singular one (10). If $linst(y)$ is chosen in such a way that (9) with $v = linst(y)$ has *real* negative eigenvalues, one may easily prove using Proposition 1 that regularized stabilizer (10) stabilizes nonlinear system in question.

[5]the set of all diffeomorphisms between $\mathcal{N}$ and $\mathcal{R}$.

6. CONCLUDING REMARKS

Simple idea for the adaptation of the stabilization method via exact linearization to the nonsmooth stabilization was developed and studied. It was justified both by numerical simulations and partially theoreticaly using rather abstract and recently introduced concept of the topological linearization of nonlinear systems.

It is appropriate to note that nonsmooth stabilization and feedback linearization approach was also studied (much more succesfully) for the discrete time systems — see [18]. Relation between nonsmooth stabilization of discrete time systems and topological linearization of continuous time systems is studied in [8].

Presented algorithm is realizable in a rather straightforward way. Moreover, other quoted results on nonsmooth stabilization are not constructive, they usually have the character of pure existence results. Exception is [11], nevertheless its approach is questionable for higher-dimensional cases.

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