KYBERNETIKA — VOLUME *32* (1996), NUMBER 3, PAGES 307 – 322

KALMAN FILTER SENSITIVITY WITH RESPECT TO PARAMETRIC NOISES UNCERTAINTY

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The influence of the noises uncertainty on the Kalman filter performance is characterized by sensitivity functions. Relationships for computing these functions are derived and used both for synthesizing a Kalman filter with reduced sensitivity (KFRS) and a self-tuning Kalman filter (SKF). The results are illustrated by examples.

1. INTRODUCTION

The influence of the noises uncertainty on the Kalman filter behaviour is widely discussed in literature [7, 8, 12, 13, 14, 15, 17, 20, 22, 24, 25, 26, 31]. Considerable research has been performed, for example, on the estimation of errors bounds under modeling uncertainty [24, 31], the indirect sensitivity functions that characterize the effect of each parameter variation on the error variance [5, 6], the direct sensitivity functions defined by the state vector derivative with respect to the varying parameter [13, 14], etc. The sensitivity functions can be used both as a quantitative characteristics of the filter sensitivity and in a generalized performance index for synthesizing a robust filter. The degradation of the filter performance depending on the data uncertainty is, obviously, closely related to the investigation of the Riccati equation sensitivity to the data uncertainty [1, 4, 10].

Different ideas for synthesizing a robust Kalman filter are suggested in [2, 11, 16, 21, 34]. Methods for synthesizing discrete Kalman filters, robust with respect to data outliers are proposed [2, 11]. The problem of continuous robust Kalman filter design is considered for linear systems with parameter uncertainty in both the state and measurement matrices [34]. The covariance of the filter estimation error is guaranteed to be within certain bounds for all admissible uncertainties. Stratton and Stengel [30] develop a robust filter for predictive wind shear detection with application in aircrafts.

Another group of methods for synthesizing robust Kalman filters is related to presetting the unknown variables within the domains of their possible values. The operations over the unknown values are reduced to operations over the respective domains [23].

An alternative to the approach for synthesizing robust filters is the approach for synthesizing adaptive filters. The estimation of the state vector in the presence of an a priori uncertainty can be done by means of different adaptive algorithms [7, 20]. The numerical characteristics of the filter innovation process are studied and minimized in most of the algorithms and "whitening" of the innovations is made. The various adaptive filtering methods for unknown noise statistics are divided into four categories [19]: Baysian, maximum likelihood, correlation and covariance matching. Indirect methods for tuning are used in a number of papers [3, 7, 20].

In the present paper the influence of the inaccurate noise covariances on the Kalman filter performance is characterized by means of sensitivity functions. A Kalman filter with reduced sensitivity is synthesized through augmentation of the estimation error vector by the sensitivity functions. It is shown that the KFRS possesses robust properties in a wide range of variations of the noise covariances. A self-tuning Kalman filter is described using sensitivity functions and the stochastic approximation method. The filter gain is tuned directly. The obtained results are illustrated by examples.

2. PROBLEM STATEMENT

The state vector $x_k \in \Re^n$ of a linear discrete-time system

$$x_{k+1} = Fx_k + Gv_k \tag{1}$$

is estimated on the observation of the output $y_k \in \Re^r$

$$y_k = Cx_k + w_k, \tag{2}$$

where the system noise $v_k \in \Re^m$ and the measurement noise $w_k \in \Re^r$ are mutually uncorrelated white noises, with covariances V_v and V_w, respectively. Sources of parameter variations in the models (1), (2) are the noise covariances.

The inaccurate values of the covariances that are the data available for the Kalman filter synthesis will be denoted by $\overline{V}_v$ and $\overline{V}_w$. Single-input, single-output (SISO) stationary systems are considered because the generalization for multi-input, multi-output (MIMO) and nonstationary systems constitutes no major difficulty. The normalization $v_k = \sqrt{V_v} v_k^0$ and $w_k = \sqrt{V_w} w_k^0$ is used where v_k^0 and w_k^0 are single covariance white noises ($V_{v^0} = V_{w^0} = 1$). The discrete-time stochastic observer is a linear filter, having the form [9]

$$\hat{x}_{k+1} = F\hat{x}_k + K_{k+1}\left(y_{k+1} - CF\hat{x}_k\right), \tag{3}$$

where $\hat{x}_k$ is the estimate of the state x_k. The estimation error $e_k = x_k - \hat{x}_k$ is described by the linear equation

$$e_{k+1} = Ae_k + B\left[v_k^0, w_{k+1}^0\right]^T, \tag{4}$$

where

$$A = \left(I - K_{k+1}C\right)F, \quad B = \left[\left(I - K_{k+1}C\right)G\sqrt{V_v}, -K_{k+1}\sqrt{V_w}\right],$$

I is the identity matrix, $n^0_{k+1} = \left[v^0_k, w^0_{k+1}\right]^T$ is a generalized white vector noise with single covariance. The variance $D_{e,k}$ of the error e_k is the Lyapunov equation solution

$$D_{e,k+1} = (I - K_{k+1}C)\, Q_k\, (I - K_{k+1}C)^T + K_{k+1} V_w K^T_{k+1}, \tag{5}$$

where

$$Q_k = F D_{e,k} F^T + G V_v G^T,$$

K_{k+1} is the Kalman filter gain. If the gain K_{k+1} is determined from the condition

$$\frac{\partial tr\, D_{e,k+1}}{\partial K_{k+1}} = 0, \tag{6}$$

where tr denotes the trace of the matrix, the following relationship holds

$$K_{k+1} = Q_k C^T \left(C Q_k C^T + V_w\right)^{-1}. \tag{7}$$

When the initial noise covariances are inaccurate, i.e. $\overline{V}_v$ and $\overline{V}_w$, the filter coefficient $\overline{K}_{k+1}$ is determined from the "algorithmic" error variance $\overline{D}_{e,k}$

$$\overline{D}_{e,k+1} = \left(I - \overline{K}_{k+1}C\right) \overline{Q}_k \left(I - \overline{K}_{k+1}C\right)^T + \overline{K}_{k+1}\overline{V}_w \overline{K}^T_{k+1}, \tag{8}$$

where

$$\overline{Q}_k = F\overline{D}_{e,k}F^T + G\overline{V}_v G^T,$$

$$\overline{K}_{k+1} = \overline{Q}_k C^T \left(C\overline{Q}_k C^T + \overline{V}_w\right)^{-1}. \tag{9}$$

The algorithmic variance differs from the actual variance $\tilde{D}_{e,k}$ that is the solution to the equation

$$\tilde{D}_{e,k+1} = \left(I - \overline{K}_{k+1}C\right) \tilde{Q}_k \left(I - \overline{K}_{k+1}C\right)^T + \overline{K}_{k+1} V_w \overline{K}^T_{k+1}, \tag{10}$$

$$\tilde{Q}_k = F\tilde{D}_{e,k}F^T + G V_v G^T.$$

The Kalman filter sensitivity with respect to variations in the covariances V_v and V_w is estimated by means of the direct sensitivity functions

$$s^v_k = \frac{\partial e_k}{\partial V_v}, \tag{11}$$

$$s^w_k = \frac{\partial e_k}{\partial V_w}. \tag{12}$$

The following stochastic equations are obtained after the differentiation of (4) with respect to V_v and V_w

$$s^v_{k+1} = A s^v_k + \frac{B}{2\sqrt{V_v}} \left[v^0_k, 0\right]^T, \tag{13}$$

$$s^w_{k+1} = A s^w_k + \frac{B}{2\sqrt{V_w}} \left[0, w^0_{k+1}\right]^T. \tag{14}$$

The variances of s_k^v and s_k^w are solutions to the Lyapunov equations

$$D_{s,k+1}^v = (I - K_{k+1}C)\left(FD_{s,k}^vF^T + \frac{GG^T}{4V_v}\right)(I - K_{k+1}C)^T, \tag{15}$$

$$D_{s,k+1}^w = (I - K_{k+1}C)\,FD_{s,k}^wF^T\,(I - K_{k+1}C)^T + \frac{K_{k+1}K_{k+1}^T}{4V_w}. \tag{16}$$

Example 1. The influence of the inaccurate initial information upon the Kalman filter performance is illustrated by a simple example for the steady-state mode of the filter ($D_{k+1} = D_k = D$ and $K_{k+1} = K$). The system is described by the equations

$$\begin{array}{rclcrcl} x_{k+1} & = & x_k + v_k, & & V_v & = & 2, \\ y_k & = & x_k + w_k, & & V_w & = & 4. \end{array}$$

It is obtained from (5) and (7)

$$D_e = 0.5V_v\left(\sqrt{1 + 4V_wV_v^{-1}} - 1\right) = 2,$$

$$K = 2\left(1 + \sqrt{1 + 4V_wV_v^{-1}}\right)^{-1} = 0.5.$$

For an inaccurately known noise covariance $\overline{V}_w$, from (8) and (9) it follows that

$$\overline{D}_e = \sqrt{1 + 2\overline{V}_w} - 1, \quad \overline{K} = 2\left(1 + \sqrt{1 + 2\overline{V}_w}\right)^{-1}.$$

The actual error variance $\tilde{D}_e$ is determined from (10) and (9) (for V_v and $\overline{V}_w$)

$$\tilde{D}_e = \left(6\overline{K}^2 - 4\overline{K} + 2\right)\left[\overline{K}\left(2 - \overline{K}\right)\right]^{-1}, \quad \overline{K} = 2\left(1 + \sqrt{1 + 2\overline{V}_w}\right)^{-1}.$$

Fig. 1.

The variance D_s^w of the sensitivity function s^w is computed from (16)

$$D_s^w = \left(4\overline{V}_w\sqrt{1+2\overline{V}_w}\right)^{-1}.$$

The graphics of the corresponding variances depending on $\overline{V}_w$ are shown in Figure 1. For $\overline{V}_w = V_w = 4$ all the error variances coincide, i. e. $D_e = \tilde{D}_e = \overline{D}_e$. Considerable deviation of the algorithmic error variance from the actual $\tilde{D}_e$ is observed for an inaccurate noise covariance $\overline{V}_w$, as expected (an effect of the inner filter divergence). As $\overline{V}_w$ increases, the variance D_s^w of the sensitivity function s^w decreases which conforms to the theory.

3. SYNTHESIS OF A KALMAN FILTER WITH REDUCED SENSITIVITY

A standard approach for synthesizing systems with reduced sensitivity is to include the sensitivity functions in the performance index. The generalized Kalman filter error is further used

$$\epsilon_k^* = \begin{bmatrix} e_k \\ \alpha s_k^v \\ \beta s_k^w \end{bmatrix}, \tag{1}$$

where α and β are weighting coefficients as it is assumed that all components of the vectors s_k^v and s_k^w are taken with some weights. The quantities referring to the synthesis of the KFRS will be denoted by an asterisk. Putting together (4), (13) and (14), yields that the generalized error (1) satisfies the equation

$$\epsilon_{k+1}^* = A^*\epsilon_k^* + B^* n_{k+1}^0, \tag{2}$$

where

$$A^* = \begin{bmatrix} (I-K_{k+1}C)\,F & 0 & 0 \\ 0 & (I-K_{k+1}C)\,F & 0 \\ 0 & 0 & (I-K_{k+1}C)\,F \end{bmatrix},$$

$$B^* = \begin{bmatrix} (I-K_{k+1}C)\,G\sqrt{V_v} & -K_{k+1}\sqrt{V_w} \\ \alpha\,(I-K_{k+1}C)\,\frac{G}{2\sqrt{V_v}} & 0 \\ 0 & -\beta\frac{K_{k+1}}{2\sqrt{V_w}} \end{bmatrix}.$$

In its structure (2) is analogous to (4). Therefore the following Lyapunov equation holds

$$D_{\epsilon,k+1}^* = A^* D_{\epsilon,k}^* A^{*T} + B^* B^{*T} \tag{3}$$

which is analogous to (5) after a respective substitution of the matrices A and B with the matrices A^* and B^*. It is shown in Appendix 1 that the KFRS has the form

$$\hat{x}_{k+1} = F\hat{x}_k + K_{k+1}^*\,(y_{k+1} - CF\hat{x}_k)\,, \tag{4}$$

$$K_{k+1}^* = Q_k^* C^T \left(C Q_k^* C^T + V_w^* \right)^{-1}, \qquad (5)$$
$$Q_k^* = F D_{e,k}^* F^T + G V_v^* G^T,$$

$$V_v^* = V_v + \frac{\alpha^2}{4V_v}, \quad V_w^* = V_w + \frac{\beta^2}{4V_w} \qquad (6)$$

and the variance D_k^* is the solution to (5), i. e.

$$D_{e,k+1}^* = \left(I - K_{k+1}^* C\right) \left(F D_{e,k}^* F^T + G V_v G^T\right) \left(I - K_{k+1}^* C\right)^T + K_{k+1}^* V_w K_{k+1}^{*T}. \qquad (7)$$

The actual variance $\tilde{D}_k^*$ of the error e_k is computed from

$$\tilde{D}_{e,k+1}^* = \left(I - \overline{K}_{k+1}^* C\right) \left(F \tilde{D}_{e,k}^* F^T + G V_v G^T\right) \left(I - \overline{K}_{k+1}^* C\right)^T + \overline{K}_{k+1}^* V_w \overline{K}_{k+1}^{*T}. \qquad (8)$$

A compromise between the filter accuracy and robustness can be achieved by an appropriate choosing of the weights α and β (see Appendix 1).

Example 2. The method for synthesizing a KFRS is illustrated for the system in Example 1. It is known that $V_v = 2$ and the accurate noise covariance V_w belongs to the interval $[1, 10]$. The synthesis is performed for two values of $\overline{V}_w : \overline{V}_w = 1.5$ (near to $V_{\min}$) and 5 (close to the middle of the interval).

Fig. 2.

When $\overline{V}_w = 1.5$ it is computed for:
– a standard Kalman filter that

$$\overline{K} = 2 \left(1 + \sqrt{1 + 2\overline{V}_w} \right)^{-1} = 0.6667,$$

$$\tilde{D}_e = \left[\overline{K}^2 \left(2 + V_w\right) - 4\overline{K} + 2 \right] \left[\overline{K} \left(2 - \overline{K}\right) \right]^{-1};$$

– a Kalman filter with reduced sensitivity $\left(\beta = 2\overline{V}_w = 3,\ \text{i.e. } \overline{V}_w^* = 2\overline{V}_w = 3\right)$ that

$$\overline{K}^* = 2\left(1 + \sqrt{1 + 2\overline{V}_w^*}\right)^{-1} = 0.5486,$$

$$\tilde{D}_e^* = \left[\overline{K}^{*2}\left(2 + V_w\right) - 4\overline{K}^* + 2\right]\left[\overline{K}^*\left(2 - \overline{K}^*\right)\right]^{-1},$$

$\tilde{D}_e^*$ being computed from (8).

When $\overline{V}_w = 5$ $\left(\beta = 2\overline{V}_w = 10,\ \text{i.e. } \overline{V}_w^* = 2\overline{V}_w = 10\right)$ the expressions for $\tilde{D}_e$, $\tilde{D}_e^*$ are the same as in the case above, the new filter gains $\overline{K} = 0.4633$, $\overline{K}^* = 0.3583$ being replaced in them. For the standard Kalman filter when the initial information is accurate

$K = 2\left(1 + \sqrt{1 + 2V_w}\right)^{-1}$, $D_e = \left[K^2\left(2 + V_w\right) - 4K + 2\right]\left[K\left(2 - K\right)\right]^{-1}$.

The plots of the variances $\tilde{D}_e$, $\tilde{D}_e^*$, D_e depending on V_w are shown in Figure 2 and Figure 3. It is seen for this example that the KFRS possesses better performance when $\overline{V}_w$ is chosen near to $V_{\min}$ (Figure 2).

Fig. 3.

The generalization of the KFRS synthesis for MIMO systems is performed in the same way. In this case s_k^v and s_k^w are matrices. The generalized error ϵ_k^* comprises the sensitivity functions of every element in the matrices V_v and V_w. If the matrices are diagonal (for mutually uncorrelated system noises, resp. mutually uncorrelated measurement noises)

$$\epsilon_k^* = \begin{bmatrix} e_k \\ \alpha_1 s_k^{v_1} \\ \vdots \\ \alpha_m s_k^{v_m} \\ \beta_1 s_k^{w_1} \\ \vdots \\ \beta_r s_k^{w_r} \end{bmatrix}, \tag{9}$$

where $s_k^{v_i} = \frac{\partial e_{k,i}}{\partial V_{v_i}}$, $i = 1, 2, \ldots, m$, $s_k^{w_j} = \frac{\partial e_{k,j}}{\partial V_{w_j}}$, $j = 1, 2, \ldots, r$. It is shown in Appendix 1 that $V_v^* = V_v + \frac{\alpha^2}{4} V_v^{-1}$, $V_w^* = V_w + \frac{\beta^2}{4} V_w^{-1}$, where

$$\alpha = \begin{bmatrix} \alpha_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \alpha_m \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \beta_r \end{bmatrix}$$

and the Kalman filter equations (4)−(8) hold. The weighting matrices α and β can be computed according to the same conditions as in the scalar case. When any noise covariance element is accurately known, then the respective element of α or β has to be set to zero.

Example 3. The KFRS is designed for the MIMO system from inertial navigation described in [3] for which

$$F = \begin{bmatrix} 0.75 & -1.74 & -0.3 & 0 & -0.15 \\ 0.09 & 0.91 & -0.0015 & 0 & -0.008 \\ 0 & 0 & 0.95 & 0 & 0 \\ 0 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 0 & 0 & 0.905 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 24.64 & 0 & 0 \\ 0 & 0.835 & 0 \\ 0 & 0 & 1.83 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad V_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, V_w = \begin{bmatrix} V_{w_{11}} & 0 \\ 0 & 1 \end{bmatrix},$$

where $V_{w_{11}}$ is supposed to be in the interval $[0.2, 10]$. It is assumed that $\overline{V}_{w_{11}} = 0.4$ and

$$\beta = \begin{bmatrix} 2\overline{V}_{w_{11}} & 0 \\ 0 & 0 \end{bmatrix}.$$

The dependence of $tr\, D_e$, $tr\, \tilde{D}_e^*$, $tr\, \tilde{D}_e$ on $V_{w_{11}}$ is plotted in Figure 4. The same properties as in Example 2 are observed.

Fig. 4.

The case is also considered when it is supposed that $V_{v_{33}} \in [0.2, 3]$ and V_w is accurate, i. e.

$$V_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & V_{v_{33}} \end{bmatrix}, \quad V_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is assumed that $V_{v_{33}} = 0.3$. A KFRS is synthesized for different matrices α:

$$\text{a)} \quad \alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \text{b)} \quad \alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad \text{c)} \quad \alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The plots for these α are shown in Figures 5, 6, 7 respectively. The KFRS in the case b) Figure 6 is more accurate than the filter in the case a) Figure 5 but for shorter interval.

Fig. 5.

Fig. 6.

Fig. 7.

4. A SELF–TUNING KALMAN FILTER

The Kalman filter gain K_k cannot be analytically computed for unknown noise covariances. It is possible to organize a procedure for direct filter gain self-tuning by the stochastic approximation method [32]

$$\overline{K}_k = \overline{K}_{k-1} - \gamma_k \nabla J\left[\nu_k\left(\overline{K}_{k-1}\right)\right], \quad k = 1, 2, \ldots, \tag{1}$$

where γ_k is a step,

$$\nu_k = y_k - CF\hat{x}_{k-1} \tag{2}$$

is the innovation process of the filter, $J\left[\nu_k\left(\overline{K}_{k-1}\right)\right] = \frac{1}{2}\nu_k^T\nu_k$ is the performance index that has to be minimized,

$$\nabla J\left[\nu_k\left(\overline{K}_{k-1}\right)\right] = \frac{1}{2}\frac{\partial\left(\nu_k^T\nu_k\right)}{\partial\overline{K}_{k-1}}$$

– the stochastic gradient. It is assumed that γ_k satisfies the conditions ensuring convergence of the recursive algorithm (1) [32], e. g. $\gamma_k = I/k$. It is also assumed that the system (1)–(2) is completely controllable and completely observable.

For MIMO systems the algorithm for self-tuning has the following form

$$\hat{x}_k = F\hat{x}_{k-1} + \overline{K}_k\nu_k, \tag{3}$$

$$\nabla J(\nu_k) = -\frac{1}{2}\left[\left(I * \left(\nu_k^T CF\right)\right) N_{\hat{x},k-1} + T_{\hat{x},k-1}\left(I * \left(F^T C^T \nu_k\right)\right)\right], \tag{4}$$

$$\overline{K}_k = \overline{K}_{k-1} - \gamma_k \nabla J \left[\nu_k \left(\overline{K}_{k-1} \right) \right], \tag{5}$$

where $\gamma_k = \frac{I}{k}$ and

$$N_{\hat{x},k-1} = \frac{\partial \hat{x}_{k-1}}{\partial \overline{K}_{k-1}} = E_{n\times r}^{n\times r} \left(I * \nu_{k-1} \right), \tag{6}$$

$$T_{\hat{x},k-1} = \frac{\partial \hat{x}_{k-1}^T}{\partial \overline{K}_{k-1}} = \left(I * \nu_{k-1}^T \right) E_{n\times r}^{r\times n} \tag{7}$$

are direct sensitivity functions, $(A * B)$-Kronecker product of the matrices A and B, $E_{n\times r}^{n\times r}$ and $E_{n\times r}^{r\times n}$-permutation matrices [33]. It is supposed that the initial conditions $\hat{x}_0$, $N_{\hat{x},0}$, $T_{\hat{x},0}$ are preset. The relationships (6) and (7) for computing $N_{\hat{x},k-1}$ and $T_{\hat{x},k-1}$ are obtained after differentiation of the filter equation (3) with respect to the gain $\overline{K}_{k-1}$.

The stochastic approximation convergence is slow and can be considerably improved by well selected initial conditions.

For SISO first order systems the equations (4)−(7) have the following form

$$N_{\hat{x},k-1} = T_{\hat{x},k-1} = \nu_{k-1},$$

$$\nabla J \left(\nu_k \right) = -CFN_{\hat{x},k-1}\nu_k = -CF\nu_{k-1}\nu_k.$$

Example 4. The algorithm for self-tuning (3)−(7) is verified for the system (1)−(2) with $F = 0.4$, $G = 1$, $C = 1$, $V_v = 1$, $\overline{V}_w = 0.1$. The accurate value of the covariance $V_w = 1$ is unknown in advance. The work of the algorithm is evaluated by comparing the optimal values $K = 0.52$, $D_e = 0.52$ with the values of $\overline{K}$ and $\tilde{D}_e$ at each step of the self-tuning.

The initial conditions are assumed to be $\hat{x}_0 = N_{\hat{x},0} = T_{\hat{x},0} = 0$. The plots $\overline{K}(k)$ and $\tilde{D}_e(k)$ are shown in Figures 8 and 9. The initial filter gain $\overline{K}_0$ is computed on the basis of the inaccurate noise covariance $\overline{V}_w$. It is possible to find an initial gain through some of the existing algorithms [29] and thus to improve the convergence of the proposed algorithm.

Fig. 8.

5. CONCLUSIONS

The influence of the inaccurate noise covariances on the Kalman filter performance is characterized by the direct sensitivity functions. They can be used directly, as stochastic processes, or indirectly by their variances. A Kalman filter with reduced sensitivity to parametric noises uncertainty is synthesized through augmentation of the estimation error vector by the sensitivity functions taken with some weights. It is shown that the KFRS possesses robust properties in a wide range of variation of the noise covariances. A self-tuning Kalman filter algorithm is presented using sensitivity functions and based on the stochastic approximation. The results obtained are illustrated by examples.

Fig. 9.

APPENDIX 1

The variance matrix $D^*_{\epsilon,k}$ of the generalized error (1) has the form

$$D^*_{\epsilon,k} = \begin{bmatrix} D_{e,k} & \alpha D^v_{es,k} & \beta D^w_{es,k} \\ \star & \alpha^2 D^v_{s,k} & \alpha\beta D^{vw}_{s,k} \\ \star & \star & \beta^2 D^w_{s,k} \end{bmatrix}, \tag{A.1}$$

where $D^v_{es,k}$, $D^w_{es,k}$, $D^{vw}_{s,k}$ are the cross-variances of e_k and s^v_k, e_k and s^w_k, s^v_k and s^w_k. The asterisks replace the respective elements of the symmetric matrix. It is denoted

$$D^*_{k+1} = D_{e,k+1} + \alpha^2 D^v_{s,k+1} + \beta^2 D^w_{s,k+1}. \tag{A.2}$$

Taking into account (3), (5), (15), (16), (A.1) and after transformations, the following is obtained

$$tr\, D^*_{\epsilon,k+1} = tr \left\{ \begin{bmatrix} F_0 D_{e,k} F_0^T & \alpha F_0 D^v_{es,k} F^T & \beta F_0 D^w_{es,k} F^T \\ \star & \alpha^2 F_0 D^v_{s,k} F_0^T & \alpha\beta F_0 D^{vw}_{s,k} F_0^T \\ \star & \star & \beta^2 F_0 D^w_{s,k} F_0^T \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} G_0 V_v G_0^T + K_{k+1} V_w K_{k+1}^T & G_0 \frac{\alpha}{2} G_0^T & K_{k+1} \frac{\beta}{2} K_{k+1}^T \\ \star & G_0 \frac{\alpha^2}{4V_v} G_0^T & 0 \\ \star & \star & K_{k+1} \frac{\beta^2}{4V_w} K_{k+1}^T \end{bmatrix} \right\}, \tag{A.3}$$

where $F_0 = (I - K_{k+1}C)\, F$ and $G_0 = (I - K_{k+1}C)\, G$. On the other hand, from (A.2), (5), (15), (16) and after transformations, the following is derived

$$tr\, D^*_{k+1} = tr \left[F_0 D^*_k F_0^T + G_0 \left(V_v + \frac{\alpha^2}{4V_v} \right) G_0^T + K_{k+1} \left(V_w + \frac{\beta^2}{4V_w} \right) K_{k+1}^T \right]. \tag{A.4}$$

Comparing (A.2), (A.3) and (A.4) yields that

$$tr\, D^*_{\epsilon,k+1} = tr\, D^*_{k+1}.$$

Hence, if the KFRS coefficient K^*_{k+1} is determined from the condition

$$\frac{\partial tr\, D^*_{\epsilon,k+1}}{\partial K^*_{k+1}} = 0, \tag{A.5}$$

the relation (7) remains valid, if the covariances

$$V_v^* = V_v + \frac{\alpha^2}{4V_v}, \quad V_w^* = V_w + \frac{\beta^2}{4V_w} \tag{A.6}$$

are used instead of the covariances V_v and V_w.

For the MIMO case (9), it is established in the same way that the covariances V_v^* and V_w^* are formed in the following manner

$$\begin{aligned} V_v^* &= V_v + \begin{bmatrix} \frac{\alpha_1^2}{4V_{v_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & 0 \\ 0 & \cdots & \frac{\alpha_m^2}{4V_{v_m}} \end{bmatrix} \\ &= V_v + \frac{1}{4} \begin{bmatrix} \alpha_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_m^2 \end{bmatrix} \begin{bmatrix} \frac{1}{V_{v_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{V_{v_m}} \end{bmatrix} = V_v + \frac{\alpha^2}{4} V_v^{-1}, \end{aligned} \tag{A.7}$$

where

$$\alpha = \begin{bmatrix} \alpha_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_m \end{bmatrix}.$$

The diagonal matrix property

$$\begin{bmatrix} V_{v_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_{v_m} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{V_{v_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{V_{v_m}} \end{bmatrix}$$

is taken into account. Analogously

$$V_w^* = V_w + \frac{\beta^2}{4} V_w^{-1}, \tag{A.8}$$

where

$$\beta = \begin{bmatrix} \beta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \beta_r \end{bmatrix}.$$

The inverse matrices V_v^{-1} and V_w^{-1} participate in (A.7) and (A.8) which means that only inputs and outputs with noises participate in the generalized error (9).

The choosing of the weighting coefficients (matrices) α and β is a matter of a compromise between the KFRS accuracy and robustness. From (4), (13) and (14) it is seen that the quantities e_k, s_k^v and s_k^w, elements of the generalized error (1), are solutions to the same type equations, differing from each other by the input variables. If the error e_k and the common sensitivity $s_k = \begin{bmatrix} s_k^v \\ s_k^w \end{bmatrix}$ have to be of equal worth in the error (1), the weights α and β have to be chosen according to the condition

$$B \begin{bmatrix} v_k^0 \\ w_{k+1}^0 \end{bmatrix} = B \frac{\alpha}{2V_v} \begin{bmatrix} v_k^0 \\ 0 \end{bmatrix} + B \frac{\beta}{2V_w} \begin{bmatrix} 0 \\ w_{k+1}^0 \end{bmatrix}$$

that is satisfied if

$$\alpha = 2V_v, \ \beta = 2V_w. \tag{A.9}$$

However, the simultaneous accomplishment of conditions (A.9) results in a proportional augmentation (doubling) of the covariances V_v and V_w, because

$$V_v^* = V_v + \frac{\alpha^2}{4V_v} = 2V_v,$$
$$V_w^* = V_w + \frac{\beta^2}{4V_w} = 2V_w.$$

This does not improve the KFRS accuracy. Really, the Riccati equation solution $\overline{D}_{e,k}$ for $V_v = \overline{V_v}$ and $V_w = \overline{V_w}$ and the solution $\overline{D}_{e,k}^*$ when $V_v^* = 2\overline{V_v}$ and $V_w^* = 2\overline{V_w}$ are related to the condition

$$\overline{D}_{e,k}^* = 2\overline{D}_{e,k},$$

whereas from (7) and (5) it follows that

$$\overline{K}_{k+1}^* = Q_k^* C^T \left(C Q_k^* C^T + V_w^*\right)^{-1} = \quad 2Q_k^* C^T \left(2C Q_k C^T + 2V_w\right)^{-1} = \overline{K}_{k+1},$$

i. e. the KFRS gain and the standard filter gain coincide. That is why the condition (A.9) has to be used only as a point of reference for the weights choosing. The same considerations are valid for MIMO systems.

ACKNOWLEDGMENT

The present work was supported by the Bulgarian National Fund for Scientific Research under grant TH–478/1994.

(Received October 27, 1994.)

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