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SYNTHESIS OF CHAOTIC SYSTEMS[1]

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Scenario for the chaos synthesis was developed and tested. Namely, choose:

(i) A SISO (single-input single-output) dissipative (i. e. sum of its poles is negative) linear system of the third or higher order having hyperbolic (i. e. with nonzero real parts only), semistable (i. e. both positive and negative real parts are always present) poles.

(ii) Nonlinear static output feedback being odd and strictly monotonous function. The corresponding closed-loop system should have two additional nontrivial equilibria such that the appropriate approximate linearizations has again poles with properties of (i).

(iii) Zeros of the linear systems that are attracting and parametrized by the feedback gain according to the Root Locus Method.

It will be demonstrated that the nonlinear system synthetized according (i) – (iii) exibits chaotic behaviour (i. e. bounded nonstationary motion with sensitive dependence on initial data) for a wide range of its parameters.

1. CONTROL SYSTEMS CLASSIFICATION

We introduce control systems both (linear and nonlinear) as dynamical systems with the parameters. A natural and clear geometrical interpretation of these systems will be suggested.

1.1. Linear control system is the following linear dynamical system with parameters:

$$\mathrm{d}x/\mathrm{d}t = F_p x, \quad x \in \mathbb{R}^n,$$

where F_p is $n \times n$ matrix depending linearly or additively on parameter $p \in \mathbb{R}^m$. Typically, $F_K = F + GK$ or $F_L = F + LH$ where K, L, G, K are appropriate matrices from linear systems theory (see e. g. [14, 10]). Fixing a parameter $p \in \mathbb{R}^m$ and limiting ourselves to a semisimple hyperbolic $(n \times n)$ matrices F_p ([14]), we may describe the systems geometrically by the decomposition of the state space $\mathbb{R}^n$ into the collection of one-dimensional subspace(s) corresponding to the real eigenvalues

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and/or of two-dimensional subspace(s) corresponding to the pairs of complex conjugate eigenvalues. Changing parameter p the previous structure may vary, but the only equilibrium of the system remains fixed.

1.2. Weakly nonlinear control system is the following nonlinear dynamical system with parameters:

$$\mathrm{d}x/\mathrm{d}t = f_p(x), \quad x \in \mathbb{R}^n, \;\; p \in \mathbb{R}^m$$

e. g. $f_K(x) = f_0(x) + g(Kx)$ or $f_L(x) = f_0(x) + Lh(x)$, where f_0, g, $h : \mathbb{R}^n \to \mathbb{R}^n$ and K, L are appropriate matrices of parameters. We suppose that there always exists (at least locally) unique solution passing through the given initial state. We consider this weakly nonlinear system around an equilibrium point that is hyperbolic, then it is topologically equivalent (at least locally) to a linear system. Nevertheless, dependence on parameter $p \in \mathbb{R}^m$ remains nonlinear. Weakly nonlinear systems behaviour is qualitatively the same as in the linear case. Instead of stable and ustable subspaces we have in weakly nonlinear case curved stable and unstable manifolds tangent to them. Topological transformation of a given weakly nonlinear system into the linear one may be intuitively viewed as "deformation" of curved manifolds into their tangents. Weakly nonlinear systems analysis and synthesis may be regarded as very similiar to the linear case.

1.3. Strongly nonlinear control systems is the following nonlinear dynamical systems with parameters

$$\mathrm{d}x/\mathrm{d}t = f_p(x), \quad x \in \mathbb{R}^n, \;\; p \in \mathbb{R}^m,$$

that is globally defined. We suppose that there always exists globally defined unique solution passing through the given initial state. Strongly nonlinear systems are not globally topologically equivalent to a linear system. They may have several isolated equilibria. The behaviour of these systems is locally everywhere unstable, globally it is chaotic, i. e. bounded nonstationary behaviour with the sensitive dependence on initial data, topological transitivity and mixing property.

This paper is devoted to the synthesis of the strongly nonlinear systems.

2. CHAOTIC SYSTEMS DESIGN

With the change of paradigm, now evaluating chaos in some cases positively, [4, 6, 8, 12] – mainly due to its mixing properties, we have postulated a *new control goal: synthesis of the chaotic system*. The usual way is to search global bifurcations – homoclinic or heteroclinic orbits, [17]. This way enables a rigorous (but very complicated) proof of the chaos presence. Nevertheless, it is mainly related with the analysis of systems rather then with their synthesis. See [5] for an application of this approach to a particular class of systems.

We suggest here rather different approach. Namely, we provide a generalizations of Chua's circuit, [3, 4, 11]. Contrary to the analysis of this Chua's circuit (see [3, 11]), we concentrate ourselves on the synthesis. Our generalizations of Chua's circuit are based on the observation that the Chua's circuit may be viewed as Lur'e nonlinear system with a single static nonlinearity. The explicit use of this Lur'e structure for chaotic system was initiated by Genesio and Tesi in [7]. We will design chaotic behaviour searching suitable parameters representing the scalar nonlinearity $NL(y)$ with the odd symmetry and the linear subspace(s) of real and complex conjugate poles and zeros.

The chaotic system will consists from the linear system with the scalar input $U(s) \in \mathbb{C}$ and the scalar output $Y(s) \in \mathbb{C}$ with the transfer function

$$Y(s) = \frac{g(s-z_1)(s-z_2)\dots(s-z_{n-1})}{(s-s_1)(s-s_2)\dots(s-s_n)}\,U(s),$$

and nonlinear, static, odd, strictly monotonous output feedback

$$U(s) = NL(Y(s)) = -NL(-Y(s)).$$

The most simple (up to the weight) symmetric, nontrivial nonlinearity is

$$NL(y(t)) = -y^3(t)/3 + qy^5(t)/5 = -NL(-y(t)).$$

A state space realization of the nonlinear circuit in the canonical Frobenius observable form with the output $y = x_n$ gives the nonlinear system

$$\mathrm{d}x/\mathrm{d}t = Fx + G\,(-x_n^3/3 + qx_n^5/5),$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \quad F = \begin{bmatrix} 0 & \dots & 0 & F_1 \\ 1 & & & F_2 \\ & \ddots & & \dots \\ & & 1 & F_n \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ \dots \\ G_n \end{bmatrix}.$$

For $q \neq 0$, the solutions of $F_1x_n + G_1(-x_n^3/3 + qx_n^5/5) = 0$ are the output coordinates of the equilibria:

$$y_{eq0} = x_{n,eq0} = 0,\ y_{eq1} = x_{n,eq1} = \sqrt{(\frac{1/3-\mathrm{SQRT}}{2q/5})},\ y_{eq2} = x_{n,eq2} = \sqrt{(\frac{1/3+\mathrm{SQRT}}{2q/5})},$$

$$\mathrm{SQRT} = \sqrt{(\frac{1}{9} - \frac{qF_1}{5G_1})},\ y_{eq3} = -y_{eq1} = x_{n,eq3},\ y_{eq4} = -y_{eq2} = x_{n,eq4}.$$

Than for $i = 1, 2, \dots, n$: $x_{i,eqk} = -F_ix_{n,eqk} - G_i(-x^3_{n,eqk}/3 + qx^5_{n,eqk}/5)$ and eq_k, $k = 1, 2$ have the coordinates $x_{j,eqk}$ $(j = 1, \dots, n)$, respectivelly. By the symmetry: $eq_3 = -eq_1$, $eq_4 = -eq_2$. Of course, $eq_0 = 0 \in \mathbb{R}^n$.

For $q = 0$ we have instead of 5 equilibria only 3 equilibria instead of 5. The solution of $F_1x_n + G_1(-x_n^3/3) = 0$, gives the output coordinates of the equilibria $y_{eq0} = x_{n,eq0} = 0$, $y_{eq1} = x_{n,eq1} = \sqrt{(3F_1/G_1)}$, $y_{eq3} = x_{n,eq2} = -y_{eq2}$. Than for

$i = 1, 2, \dots, n$: $x_{i,eq1} = -F_i x_{n,eq1} - G_i(-x^3_{n,eq1}/3)$, The eq_1 has the coordinates $x_{j,eq1}$ $(j = 1, \dots, n)$. From the symmetry: $eq_2 = -eq_1$. Again, $eq_0 = 0 \in \mathbb{R}^n$.

For both $q \neq 0$ and $q = 0$ the equilibria lie on the same line and they are symmetric with respect to the origin $0 \in \mathbb{R}^n$. Moreover due the invariance $\mathrm{d}x/\mathrm{d}t - Fx - G(-x^3/3 + qx^5/5) = \mathrm{d}(-x)/\mathrm{d}t - F(-x) - G(-(-x^3)/3 + q(-x^5)/5) = 0$ each solution $x(t)$ has a symmetric counterpart $-x(t)$. Consult [3, 17] to understand the importance of the *symmetry*: one symmetrical equation reduces the dimensionality of the variety on which the solutions evolve by one; the space of symmetric functions is smaller than the space of all functions and so the number of parameters needed to be investigated on chaotic behavior is smaller.

According to the idea of the Chua's circuit, we will choose the linear system as hyperbolic, dissipative, nonpotential system. For the prescription of characteristic polynomial

$$s^n - F_n s^{n-1} \dots - F_2 s - F_1 = (s - s_1)(s - s_2) \dots (s - s_n)$$

with the poles $s_1, s_2, \dots, s_n$, we shall use the Vietà's formulas

$$F_1 = (-1)^{n-1} s_1 s_2 \dots s_n, \ \dots, \ F_n = s_1 + s_2 + \dots + s_n.$$

The condition of *dissipativity* (negativity of the divergence of the right-hand side vector field gives:

$$s_1 + s_2 + \dots + s_n + g\left(-x_n^2 + qx_n^4\right) < 0.$$

According to the theorem of Liouville

$$\mathrm{d}V/\mathrm{d}t = \textstyle\int_{D(t)} \mathrm{div} f(x)\, \mathrm{d}x,$$

where $V(t)$ is the volume of the variety $D(t)$ (see [2]). The dissipativity guarantees that any initial volume converges to a zero volume which may be an attractor or that all trajectories tend to some set with zero volume, [9].

Similarly the G_i from the gain g and the zeros z_i

$$G_1 = (-1)^{n-1} g z_1 \dots z_{n-1}, \ \dots, \ G_{n-1} = -g(z_1 + \dots. + z_{n-1}), \ G_n = g.$$

For $q \neq 0$, near the equilibria states $eq_0 = 0$, eq_1, $eq_3 = -eq_1, eq_2, eq_4 = -eq_2$ the nonlinear system, with the output nonlinearity $-y^3/3 + qy^5/5$ is behaving locally as the linear systems with the state matrices F, $F + G_{\mathrm{mid}}$, $F + G_{\mathrm{off}}$, where $G_{\mathrm{mid}} = G(-x^2_{n,eq1} + qx^4_{n,eq1})$, $G_{\mathrm{off}} = G(-x^2_{n,eq2} + qx^4_{n,eq2})$ and where $-x^2 + qx^4 = (-x^3/3 + qx^5/5)'$. The state matrices have the eigenvalues s_j, s_{j1}, $s_{j3} = s_{j1}$, s_{j2}, $s_{j4} = s_{j2}$, and the eigenvectors v_{j0}, v_{j1}, $v_{j3} = v_{j1}$, v_{j2}, $v_{j4} = v_{j2}$ $(j = 1, \dots, n)$ which are either the real lines – for the real eigenvalues or the real planes – for pairs of the complexely conjugate eigenvalues, going through the equilibria eq_0, eq_1, eq_2, eq_3, eq_4, eq_5. The Root Locus Method of Bode and Evans originated from the stability analysis using the poles with the change of the gain with the change of the working point. For us, the working points are both the three equilibria and the middle points – parameterized by $x_n = y$. The equation of the Root Locus, parameterized by the output x_n is:

$$(s - s_1)(s - s_2) \dots (s - s_n) + (x_n^2 - qx_n^4)\, g\,(z - z_1)(z - z_2) \dots (z - z_{n-1}) = 0.$$

For $x_n = 0$, the roots are $s_1, s_2, \ldots, s_n$; for x_n inproper, the roots are $z_1, z_2, \ldots, z_{n-1}$ and the inproper root.

The condition of *hyperbolicity* guarantees that the solution is contracting in some directions and expanding in some other directions. This condition together with synergistic effect of several equilibrium hyperbolic points leads to the solutions that expand from the vicinity of an equilibrium and is for some time catched at the vicinity of the other equilibrium, etc. [11].

The condition of *nonpotencionality* or the existence of either stable or unstable planes corresponding to complex conjugate poles guarantees the rotation which is, together with hyperbolicity, part of the mechanism of the Smale's horseshoe [3, 17]. The nonpotentionality guarantees that during the stay at the vicinity of equilibrium there are several turns near that equilibrium.

3. AN EXAMPLE OF SYNTHESIZED CHAOTIC SYSTEM

Fig. 1. Upper left: The slope of feedback quintic nonlinearity parameterizes the Root Locus giving rise of the chaos near the inner three of the five equilibria. On upper left there is given the output symmetric quintic nonlinearity $-y^3/3 + qy^5/5$ with the slope $-y^2 + qy^4$. Upper right: The Root Locus parameterized by output y starting from the poles $\times$ of the center equilibrium, going through the poles $\star$ of the middle equilibria, than turning back to poles $\times$ of the center equilibria, going through the poles $+$ of the far-off-center equilibria and then reaching the zeros $\circ$. In all the equilibria, the divergence is negative. On the bottom: The integrations start near the center equilibrium $eq0$, the right middle off-center equilibrium $eq1$, and right far-off-center equilibrium $eq2$ on the eigenvectors with the unstable eigenvalues. The first solution starting near $eq0$ is expanded to the vicinity of right middle equilibrium $eq2$, rotates there and expands to the vicinity of the center equilibrium $eq0$, rotates there and then expands to the left middle equilibrium $eq3$, etc. keeping the symmetry. Similarly for the two other solutions starting near $eq1$ and $eq2$. After some transient time, all three solutions get mixed in the proximity of the chaotic attractor of the designed system.

We present the most advanced example: system of sixth order and quintic nonlinearity. Nevertheless, initially we started with the third order and cubic nonlinearity and then we investigated the orders $4, 5, 6$. Then we used quintic nonlinearity for the third order and continued up to order 6. To demonstrate the chaotic behaviour, we know from our previous research (see [15]), that the good choice is to start the integration of chaotic systems in the following way. We integrated our strongly nonlinear system starting near equilibria on unstable directions − which corresponds to an unstable eigenvalues of appropriate approximate linearizations. For the specific dimension of the state space $\mathbb{R}^6$ the specific poles $s_1, \ldots, s_6$, the zeros $z_1, \ldots, z_5$, the gain g and the weight of the quintic term q were designed; the results are shown on Figure 1. It demonstrates even the basic mixing properties of the chaos: the three solutions get mixed. The search in the space $\mathbb{R}^{13}$ of 6 real components of the 6 poles, 5 real components of the 5 zeros, 1 real gain and 1 real weight was successful due to the knowledge of the classical Root Locus Method.

The computations were done in PC–MatLab (later in AT–MatLab) and implemented on diskettes CanonChaos II ($q = 0$), III ($q \neq 0$) − in $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$, $\mathbb{R}^6$.

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