On a fluid model of neutron star.

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SUMMARY

We consider an initial boundary value problem for the equations of a fluid spherical model of neutron star considered by Lattimer et al. We estimate the asymptotic decay of the solution, which may serve as a crude estimate of a "thermalization time" for the system.

1 Introduction

In a recent paper, C. Monrozeau et al. [27] analyze the influence of neutron superfluidity on the cooling time of inner crust matter in neutron stars, in the case of a rapid cooling of the core of the star. The model used to describe the evolution of temperature [23] in the star follows Lattimer et al. [23]. It supposes a linear dependence of the specific heat as a function of temperature and assumes that a mechanical equilibrium is reached, so the problem reduces to the study of large time asymptotics for a Fast Diffusion Equation satisfied by the temperature.

In a more general setting, it is interesting to consider the complete problem where temperature is coupled to density and velocity fluctuations through a thermo-mechanical system. The simplest description of such a model is achieved [21] through the compressible Navier-Stokes system.

Concerning the fully 3D compressible case with heat conductivity the basic references are the works of Lions [26], Feireisl [11, 12] and Bresch-Desjardins [3], in which global existence of a weak solution is proved. Concerning asymptotics we can mention results done by Feireisl and his collaborator on the problem of the long-time behavior of solutions to the complete system with a time - dependent driving force [13].

However spherical symmetry is considered in the major part of astrophysical literature [4, 5, 14, 21] as a quite reliable approximation (at least when rotation and magnetic aspects are neglected) and in this quasi-monodimensional situation, global existence and uniqueness of a classical solution and its large-time behavior have been obtained in some spherically symmetric cases (see [19, 16, 15]). Our purpose is then to prove well-posedness and large time asymptotics for this model (the compressible Navier-Stokes system for a spherical symmetric flow with specific temperature-dependent specific heat and thermal conductivity) leading to a simple estimate of a "cooling time", in the spirit of Lattimer et al. [23].

The general formulation of the system reads [15]

$$\left\{\begin{array}{c} \rho_t + (\rho v)_r + \dfrac{2\rho v}{r} = 0, \\ \rho(v_t + v v_r) = \left(-p + \mu\left(v_r + \dfrac{2v}{r}\right)\right)_r - 4\nu_r \dfrac{v}{r} + \rho F(r,t), \\ \rho(e_t + v e_r) = q_r + \dfrac{2q}{r} - p\left(v_r + \dfrac{2v}{r}\right) + \mu\left(v_r + \dfrac{2v}{r}\right)^2 - \dfrac{8\nu v v_r}{r} - \dfrac{4\nu v^2}{r^2}, \end{array}\right. \tag{1}$$

in the domain $Q := \omega \times \mathbf{R}^+$ with $\omega := (R_0, R_1)$, where R_0 is the radius of the internal rigid core of the star and R_1 is the exterior boundary, for the density $\rho(r,t)$, the velocity $v(r,t)$ and the temperature $\theta(r,t)$.

The state functions of our model are recovered by using standard thermodynamical arguments [33] from the expression of the specific heat at constant volume $c_V = c_V^0 + \frac{A}{\beta-1}\frac{\theta}{\eta^{1-\beta}}$. The pressure is then $p(\eta,\theta) = \frac{A}{2}\ \frac{\theta^2}{\eta^{2-\beta}}$, the internal energy $e(\eta,\theta) = c_V^0\theta + \frac{A}{2(\beta-1)}\frac{\theta^2}{\eta^{1-\beta}}$, where $c_V^0 > 0$, $A > 0$ and $\beta \geqslant 2$, the specific entropy is $s(\eta,\theta) := c_V^0 \log\theta + \frac{A}{\beta-1}\frac{\theta}{\eta^{1-\beta}}$ and the heat flux q is given by the Fourier law $q(\eta,\theta) := \kappa(\eta,\theta)\theta_r$, with the following constraints on the thermal conductivity

$$\underline{\kappa}(1+\theta^q) \leqslant \kappa(\eta,\theta) \leqslant \overline{\kappa}(1+\theta^q), \tag{2}$$

$$|\kappa_\eta(\eta,\theta)| + |\kappa_{\eta\eta}(\eta,\theta)| \leqslant \overline{K}_1\ (1+\theta^q), \tag{3}$$

$$|\kappa_\theta(\eta,\theta)| \leqslant \overline{K}_2\ (1+\theta^{q-1}), \tag{4}$$

for any $\theta \geqslant 0$, with positive constants $\underline{\kappa}$, $\overline{\kappa}$, $\overline{K}_1$, $\overline{K}_2$ and $q \geqslant 4$. As a simple model, we suppose in the following that the first viscosity coefficient μ is a positive constant and that the second viscosity coefficient ν is zero.

In the original model of [23], the following choice is made: $c_V^0 = 0$ and $\kappa(\eta,\theta) = \frac{A_m}{\theta^m\eta^\alpha}$ for $A_m > 0$, with the two possibilities $(m,\alpha) = (1,1)$ or $(m,\alpha) = (0,2/3)$, corresponding to $q = 0$ and $q = 1$ in (2), (3) and (4). Unfortunately, our results do not cover this situation, but a precise study of the "purely thermal" Lattimer's model considering these values is actually possible and will be the object of a future work [9].

The viscosity of the medium is considered here as a simple regularization, and we consider the simplest model $\mu = Cte > 0$ and $\nu = 0$. Finally $F(r,t)$ is a given external field force (gravitation).

Writing the system in lagrangian (mass) coordinates (x,t), with

$$r(x,t) := r_0(x) + \int_0^t v(x,s)\ ds, \tag{5}$$

where $r_0(x) := \left[R_0^3 + 3\int_0^x \eta^0(y)\ dy\right]^{1/3}$ for $x \in \Omega$, we get

$$\left\{\begin{array}{c} \eta_t = (r^2v)_x, \\ v_t = r^2\left(-p + \dfrac{\mu}{\eta}(r^2v)_x\right)_x + f, \\ e_t = q_x + \left(-p + \dfrac{\mu}{\eta}(r^2v)_x\right)(r^2v)_x, \\ r_t = v, \end{array}\right. \tag{6}$$

in the fixed domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0,M)$, where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radius r depend on the lagrangian mass coordinates.

The lagrangian heat flux is now $q(\eta,\theta) = \kappa(\eta,\theta)\ \frac{r^4}{\eta}\ \theta_x$, and the external field-force is given by the Newton's law $f(x) := -G\frac{M_0}{r^2}$, where G and M_0 are positive constants (we neglect the selfgravitation of the gas), and we denote the stress σ by

$$\sigma(\eta,\theta) := -p(\eta,\theta) + \frac{\mu}{\eta}(r^2v)_x.$$

We consider the boundary conditions

$$\left\{\begin{array}{c} v|_{x=0,M} = 0, \\ q|_{x=0} = 0,\ \ \theta|_{x=M} = \theta_\Gamma, \end{array}\right. \tag{7}$$

for $t > 0$, with $\theta_\Gamma > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad r|_{t=0} = r^0(x), \quad \theta|_{t=0} = \theta^0(x) \qquad \text{on } \Omega. \tag{8}$$

We study weak solutions for the above problem with properties

$$\left\{\begin{array}{c} \eta \in L^\infty(Q_T), \quad \eta_t \in L^\infty([0,T],L^2(\Omega)), \quad \sqrt{\rho}\ (r^2v)_x \in L^\infty([0,T],L^2(\Omega)), \\ v \in L^\infty([0,T],L^4(\Omega)), \quad v_t \in L^\infty([0,T],L^2(\Omega)),\ \sigma_x \in L^\infty([0,T],L^2(\Omega)), \\ \theta \in L^\infty([0,T],L^2(\Omega)), \quad \sqrt{\rho}\ \theta_x \in L^\infty([0,T],L^2(\Omega)). \end{array}\right. \tag{9}$$

and

$$r \in C(Q_T) \ \text{ and for all } t \in [0,T], x \to r(x,t) \text{ is strictly increasing on } \Omega, \tag{10}$$

where $Q_T := \Omega \times (0,T)$.

We also assume the following conditions on the data:

$$\left\{ \begin{array}{c} \eta^0 > 0 \text{ on } \Omega,\ \eta^0 \in L^1(\Omega), \\ v_0 \in L^2(\Omega), \quad \sqrt{\rho^0}\ v_x^0 \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \quad \inf_\Omega \theta^0 > 0. \end{array} \right. \tag{11}$$

We look for a weak solution (η, v, θ) such that

$$\eta(x,t) = \eta^0(x) + \int_0^t \left(r^2 v\right)_x (x,s)\ ds, \tag{12}$$

for a.e. $x \in \Omega$ and any $t > 0$, and such that for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$

$$\int_{Q_T} \left[\phi_t v + \left(r^2 \phi_x + \frac{2\eta\phi}{r} \right) p - \frac{\mu \phi_x r^4}{\eta} v_x - 2\mu \frac{\phi \eta v}{r^2} + f\phi \right] dx\ dt = \int_\Omega \phi(0,x)\ v^0(x)\ dx, \tag{13}$$

and

$$\int_{Q_T} \left[\phi_t e + \frac{\kappa r^4 \theta_x}{\eta} \phi_x - r^2 v \sigma \phi_x - r^2 v \sigma_x \phi \right] dx\ dt = \int_\Omega \phi(0,x)\ \theta^0(x)\ dx. \tag{14}$$

Then our first result is the following

Theorem 1 *Suppose that the initial data satisfy (11) and that T is an arbitrary positive number.*

Then the problem (6)(7)(8) possesses at least one global weak solution satisfying (9) and (10) together with properties (12), (13) and (14). Moreover, the solution is unique.

For that purpose, we first prove a classical existence result in the Hölder category. We denote by $C^\alpha(\Omega)$ the Banach space of real-valued functions on Ω which are uniformly Hölder continuous with exponent $\alpha \in (0,1)$, and $C^{\alpha,\alpha/2}(Q_T)$ the Banach space of real-valued functions on $Q_T := \Omega \times (0,T)$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha(\Omega)$ (resp. $C^{\alpha,\alpha/2}(Q_T)$) will be denoted by $\|\cdot\|_\alpha$ (resp. $|||\cdot|||_\alpha$).

Theorem 2 *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta_x^0, v^0, v_x^0, v_{xx}^0, \theta^0, \theta_x^0, \theta_{xx}^0\right) \in \left(C^\alpha(\Omega)\right)^8,$$

for some $\alpha \in (0,1)$. Suppose also that $\eta^0(x) > 0$ and $\theta^0(x) > 0$ on Ω, and that the compatibility conditions

$$\theta_x^0(0) = 0,\ \theta^0(M) = \theta_\Gamma;\ v^0(0) = v^0(M) = 0,$$

hold. Then, there exists a unique solution $(\eta(x,t), v(x,t), \theta(x,t))$ with $\eta(x,t) > 0$ and $\theta(x,t) > 0$ to the initial-boundary value problem (6)(7)(8) on $Q = \Omega \times \mathbb{R}_+$ such that for any $T > 0$

$$(\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}) \in \left(C^\alpha(Q_T)\right)^{12},$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in \left(L^2(Q_T)\right)^3.$$

Then Theorem 1 will be obtained from Theorem 2 through a regularization process.

Finally we prove

Theorem 3 *Suppose that the initial data satisfy (11).*

Let (η_S, v_S, θ_S) be the asymptotic state defined in Proposition 1 (see below). Then the solution of the problem (6)(7)(8) follows the following large time behavior: there exist positive constants T_{as}, C and λ such that for $t \geqslant T_{as}$

$$\|\eta - \eta_S\|_{L^2(\Omega)} + \|v - v_S\|_{L^2(\Omega)} + \|\theta - \theta_S\|_{L^2(\Omega)} \leqslant Ce^{-\lambda t}. \tag{15}$$

Remark 1 *After the previous result, we get a natural (rough) estimate of the cooling time T_c as the inverse of the constant λ in (15), which is of qualitative nature as it depends on the initial data and the physical constants of the problem. A major improvement would be to get a more precise behaviour of the type $\theta - \theta_S - \mathcal{C}e^{-\lambda_c t} \to 0$, with a constant λ_c independent of the initial data. As stressed previously, at least in the purely thermal Lattimer model, it is possible to prove such an estimate [9].*

The previous spherical symmetric Navier-Stokes system has been the subject of a great number of studies in the past (among them [19, 15, 16, 29, 30, 31, 8, 10, 34]) in the barotropic, or the temperature dependent situations (perfect gas, possibly including radiation). Recently S.Yanagi proved the stability [35] in the case $p(\eta,\theta) = R\frac{\theta}{\eta^\gamma}$ (see also [25] and [24] for the same model in the pure onedimensional case).

In the present work, the nonlinear dependence in temperature of the state functions complicates the analysis.

The plan of the article is as follows: in section 2 we give a priori estimates sufficient to prove in section 3 global existence of a unique solution, then we give in section 4 the asymptotic behaviour of the solution for large time.

2 A priori estimates

In the spirit of [15], we first suppose that the solution is classical in the following sense

$$\left\{ \begin{array}{c} \eta \in C^1(Q_T), \quad \rho > 0, \\ v, \theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)), \quad \theta > 0, \end{array} \right. \tag{16}$$

and

$$r > 0 \ \text{ for all } t \in [0,T]. \tag{17}$$

Our first task is to prove the following regularity result

Theorem 4 *Suppose that the initial-boundary value problem (6)(7)(8) has a classical solution described by Theorem 2. Then the solution $(\eta, v, v_x, \theta, \theta_x)$ is bounded in the Hölder space $C^{1/3,1/6}(Q_T)$*

$$|||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3} \leqslant C(T),$$

where C depends on T, the physical data of the problem and the initial data. Moreover

$$0 < \underline{\eta} \leqslant \eta \leqslant \overline{\eta}, \qquad 0 < \underline{\theta} \leqslant \theta \leqslant \overline{\theta}.$$

Let N and T be arbitrary positive numbers. In all the following, we denote by $C = C(N)$ or $K = K(N)$ various positive non-decreasing functions of N, which may possibly depend on the physical constants M etc., but not on T.

Lemma 1 *Under the following condition on the data*

$$\|v^0\|_{L^2(\Omega)} + \|\eta^0\|_{L^1(\Omega)} + \|\theta^0\|_{L^1(\Omega)} \leqslant N, \tag{18}$$

1. *The mass conservation holds*

$$\int_\Omega \eta(x,t)\ dx = \int_\Omega \eta^0(x)\ dx. \tag{19}$$

2. *The following energy-entropy inequality holds*

$$\int_\Omega \left[\frac{1}{2}v^2 + \frac{A}{2(\beta-1)}\ \eta^{\beta-1}\ (\theta - \theta_\Gamma)^2\right]\ dx$$
$$+ \int_0^T \int_\Omega \left(\frac{\kappa(\eta,\theta) r^4}{\eta\theta^2}\ \theta_x^2 + \frac{\mu}{\eta\theta}\ [(r^2 v)_x]^2 \right) dx\ dt \leqslant K(N). \tag{20}$$

3. The following estimate holds

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \left\| \frac{(\theta-\theta_\Gamma)^2}{\eta^{1-\beta}} \right\|_{L^\infty(0,T;L^1(\Omega))} \leqslant K(N), \qquad (21)$$

Proof:

1. Integrating the first equation (6) gives (19).

2. Multiplying the second equation (6) by v and adding the result to the first and third equations (6), we get first

$$\left(\eta + \frac{1}{2}\, v^2 + e\right)_t = \left(\kappa \frac{r^4}{\eta}\theta_x\right)_x + \left(r^2 v\sigma\right)_x + \left(\frac{GM_0}{r}\right)_t. \qquad (22)$$

Using now an argument of Jiang [18], we define the function

$$E(\eta,\theta) = \psi(\eta,\theta) - \psi(1,\theta_\Gamma) - \psi_\eta(1,\theta_\Gamma)(\eta-1) - \psi_\theta(\eta,\theta)(\theta-\theta_\Gamma),$$

where $\psi := e - \theta s$ is the Helmholtz free energy. Elementary computations give the relations

$$\psi_\theta = -s, \quad \psi_\eta = -p \ \text{ and } \psi_{\theta\theta} = -\frac{e_\theta}{\theta}.$$

We have

$$\left(E + \frac{1}{2}\, v^2 - \frac{GM_0}{r}\right)_t = e_t + v v_t - \theta_\Gamma s_t - \psi_\eta(1,\theta_\Gamma)\, \eta_t. \qquad (23)$$

Computing the time-derivative of the entropy $s(\eta,\theta)$, we get

$$s_t = \left(\kappa \frac{r^4}{\eta}\omega_x\right)_x + \frac{\kappa r^4}{\eta\theta^2}\, \theta_x^2 + \frac{\mu}{\eta\theta}\, [(r^2 v)_x]^2, \qquad (24)$$

where $\omega := \theta^{-1}$ (note that the Gibbs-Duhem inequality $s_t + \left(\frac{q}{\theta}\right)_x \geqslant 0$ is satisfied). Plugging (22) and (24) into (23), we obtain the identity

$$\left(E + \frac{1}{2}v^2 - \frac{GM_0}{r} + \psi_\eta(1,\theta_\Gamma)\, \eta\right)_t = \theta_\Gamma \left(\kappa \frac{r^4}{\eta}\left(1 - \frac{\theta_\Gamma}{\theta}\right)\theta_x\right)_x + \left(r^2 v\sigma\right)_x$$

$$-\theta_\Gamma \frac{\kappa r^4}{\eta\theta^2}\, \theta_x^2 - \theta_\Gamma \frac{\mu}{\eta\theta}\, [(r^2 v)_x]^2. \qquad (25)$$

Integrating on Ω, we have

$$\frac{d}{dt}\int_\Omega \left(E + \frac{1}{2}v^2 - \frac{GM_0}{r} - p(1,\theta_\Gamma)\, \eta\right)\, dx + \int_\Omega \left(\theta_\Gamma \frac{\kappa r^4}{\eta\theta^2}\, \theta_x^2 + \theta_\Gamma \frac{\mu}{\eta\theta}\, [(r^2 v)_x]^2\right)\, dx = 0.$$

Integrating in time and using (19) we find

$$\int_\Omega \left(E + \frac{1}{2}v^2 - \frac{GM_0}{r}\right)\, dx + \int_0^T \int_\Omega \left(\theta_\Gamma \frac{\kappa r^4}{\eta\theta^2}\, \theta_x^2 + \theta_\Gamma \frac{\mu}{\eta\theta}\, [(r^2 v)_x]^2\right)\, dx\, dt \leqslant K(N). \qquad (26)$$

After Taylor's formula and using the fact that $\psi_{\eta\eta} = -p_\eta$, we get

$$E(\eta,\theta) - \psi(\eta,\theta) + \psi(\eta,\theta_\Gamma) + (\theta-\theta_\Gamma)\psi_\theta(\eta,\theta) = (\eta-1)^2 \int_0^1 (1-u)\psi_{\eta\eta}\left(1 + u(\eta-1), \theta\right)\, du \geqslant 0.$$

The same formula for the second argument reads

$$\psi(\eta,\theta) - \psi(\eta,\theta_\Gamma) - (\theta-\theta_\Gamma)\psi_\theta(\eta,\theta) = -(\theta-\theta_\Gamma)^2 \int_0^1 (1-u)\psi_{\theta\theta}\left(\eta, 1 + u(\theta-\theta_\Gamma)\right)\, ds.$$

As $\psi_{\theta\theta} = -e_\theta/\theta$, the right-hand side rewrites

$$(\theta-\theta_\Gamma)^2\int_0^1 (1-s)\psi_{\theta\theta}\left(1+s(\theta-\theta_\Gamma)\right)\ ds = c_V^0\left[(1+\theta-\theta_\Gamma)\log\left(1+\theta-\theta_\Gamma\right)-(\theta-\theta_\Gamma)\right]$$
$$+\frac{A}{2(\beta-1)}\ \eta^{\beta-1}\ (\theta-\theta_\Gamma)^2,$$

where the right-hand-side is not negative. Then plugging into (26), we obtain

$$\int_\Omega\left(\frac{A}{2(\beta-1)}\ \eta^{\beta-1}\ (\theta-\theta_\Gamma)^2+\frac{1}{2}v^2-\frac{GM_0}{r}\right)\ dx$$
$$+\int_0^T\int_\Omega\left(\theta_\Gamma\frac{\kappa r^4}{\eta\theta^2}\ \theta_x^2+\theta_\Gamma\frac{\mu}{\eta\theta}\ [(r^2v)_x]^2\right)\ dx\ dt\leqslant K(N).$$

Then (20) follows, by using (19) and $R_0\leqslant r\leqslant R_1$.

3. The estimate (21) follows directly from (19) and (20) □

In order to get an absolute lower bound of the specific volume, it is simpler to go back to the Eulerian formulation (1), following the strategy of [10]. Let us introduce the static (eulerian) problem ($v_S(r)\equiv 0$)

$$\left\{\begin{array}{l} p_r(\rho_S,\theta_S)=\rho F_S(r),\\ \frac{1}{r^2}\left(r^2\kappa(\rho_S,\theta_S)\theta_r\right)_r=0,\end{array}\right. \tag{27}$$

in Ω, with $F_S(r)=-G\frac{M_0}{r^2}$, the mass constraint

$$\int_\Omega r^2\rho_S(r)\ dr = M = \int_\Omega r^2\rho^0(r)\ dr, \tag{28}$$

and the boundary condition

$$(\theta_S)_r(R_0)=0,\quad \theta_S(R_1)=\theta_\Gamma. \tag{29}$$

Proposition 1 *The problem (27)(29)(28) has a unique solution $(\rho_S(r),\theta_S(r))$ given by*

$$\left\{\begin{array}{l} \theta_S(r)=\theta_\Gamma\quad \text{for } r\in\Omega,\\ \rho_S(r)=\left[\frac{(\beta-1)2GM_0}{(\beta-2)A\theta_\Gamma^2}\ \left(\frac{1}{r}-\frac{1}{r_0}\right)\right]^{\frac{1}{1-\beta}}\quad \text{for } r\in\Omega,\end{array}\right. \tag{30}$$

where the constant r_0 depends only on the data, provided that

$$KF_\infty < M < KF_1, \tag{31}$$

where K, F_∞, F_1 are positive constants given below.

Proof: One computes easily the static solution by integrating the system of ordinary differential equations (27). As $\rho_S(r)>0$ on (R_0,R_1), the constant r_0 is greater than R_1. It is implicitly defined by the mass constraint (28)

$$M = KF(r_0), \tag{32}$$

where $K := \left[\frac{(\beta-1)2GM_0}{(\beta-2)A\theta_\Gamma^2}\right]^{\frac{1}{1-\beta}}$ and $F(x) := \int_{R_0}^{R_1}\left(\frac{1}{r}-\frac{1}{x}\right)^{\frac{1}{1-\beta}} r^2dr$. One checks that $x\to F(x)$ is monotone decreasing on (R_1,∞) and that $F_1 := \lim_{x\to R_1} F(x) = \int_{R_0}^{R_1}\left(\frac{1}{r}-\frac{1}{R_1}\right)^{\frac{1}{1-\beta}} r^2dr$ and $F_\infty := \lim_{x\to\infty} F(x) = \int_{R_0}^{R_1}\left(\frac{1}{r}\right)^{\frac{1}{1-\beta}} r^2dr$ then equation (32) has a unique solution $r_0 = F^{-1}\left(\frac{M}{K}\right)$, under condition (31) □

Lemma 2 *The following bound holds for $2\alpha\leqslant q$*

$$\int_0^T \max_\Omega\left(\theta^\alpha(\cdot,t)-\theta_\Gamma^\alpha\right)^2\ dt\leqslant C, \tag{33}$$

Proof: ¿From the identity $\theta^\alpha - \theta_\Gamma^\alpha = \alpha \int_x^M \theta^{\alpha-1}\theta_y \; dy$, we get

$$|\theta^\alpha - \theta_\Gamma^\alpha| \leqslant \alpha \left(\int_\Omega \kappa \frac{r^4}{\eta\theta^2} \; \theta_x^2 dx \right)^{1/2} \left(\int_\Omega \frac{\eta\theta^2}{\kappa r^4} \; \theta^{2\alpha-2} \; dx \right)^{1/2} + v^2.$$

Using (2)(19) and (20), taking the square and integrating on $(0,T)$ we get (33) □

Now we have the classical representation formula of the specific volume (see [1])

Lemma 3 *The following formula holds*

$$\eta^{\beta-2} = \frac{\exp\left(\frac{\beta-2}{\mu}\int_0^t \Phi(x,s) \; ds\right)}{(\eta^0)^{2-\beta} + \frac{A(2-\beta)}{\mu}\int_0^t \theta^2 \exp\left(-\frac{2-\beta}{\mu}\int_0^s \Phi(x,\sigma) \; d\sigma\right) \; ds} \; , \tag{34}$$

where $\Phi(x,t) = \frac{d}{dt}\int_0^x \frac{v}{r^2} \; dy + \int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right) \; dy + \sigma(0,t)$ *and* $\sigma(x,t) \equiv \sigma(\eta(x,t), \theta(x,t))$.

Proof: Integrating the momentum equation on $[0,x]$ we find

$$\sigma(x,t) \equiv -p + \mu\frac{\eta_t}{\eta} = \Phi(x,t).$$

Multiplying by $\frac{\beta-2}{\mu} \; \eta^{\beta-2}$, we get

$$\left(\eta^{\beta-2}\right)_t = -\frac{\beta-2}{\mu}\left[\Phi(x,t) + p\right] \; \eta^{\beta-2}.$$

Integrating on $[0,t]$

$$\eta^{\beta-2} \; \exp\left(\frac{2-\beta}{\mu}\int_0^t p \; ds\right) = (\eta^0)^{\beta-2} \; \exp\left(-\frac{2-\beta}{\mu}\int_0^t \left[\Phi(x,s)\right] \; ds\right). \tag{35}$$

Multiplying by $A\theta^2$ and integrating on $[0,t]$ gives

$$\exp\left(\frac{2-\beta}{\mu}\int_0^t p \; ds\right) = 1 + \frac{A(2-\beta)}{\mu} \; (\eta^0)^{\beta-2}\int_0^t \theta^2 \exp\left(-\frac{2-\beta}{\mu}\int_0^s \Phi(x,\tau) \; d\tau\right) \; ds.$$

Plugging this into (35) gives (34) □

Proposition 2 *Under the previous condition on the data, there exists positive constants* $\underline{\eta}$ *and* $\overline{\eta}$ *depending only on* N *such that*

$$0 < \underline{\eta} \leqslant \eta(x,t) \leqslant \overline{\eta} \;\; \textit{for} \; (t,x) \in Q_T. \tag{36}$$

Proof: After an idea of [35], from the formula giving Φ we get

$$\eta\sigma(0,t) = -\eta \; \frac{d}{dt}\int_0^x \frac{v}{r^2} \; dy - \eta\int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right) \; dy - A\eta^{\beta-1}\theta^2 + \mu\eta_t$$

$$= -\left(\eta\int_0^x \frac{v}{r^2} \; dy\right)_t - \left(r^2 v\int_0^x \frac{v}{r^2} \; dy\right)_x - \eta\int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right) \; dy - A\eta^{\beta-1}\theta^2 + \mu\eta_t.$$

Integrating on $\Omega \times [0,t]$ we get

$$\frac{1}{3}(R_1^3 - R_0^3)\int_0^t \sigma(0,s) \; ds = \int_\Omega \eta^0\left(\int_0^x \frac{v^0}{{r^0}^2} \; dy\right) \; dx - \int_\Omega \eta\left(\int_0^x \frac{v}{r^2} \; dy\right) \; dx$$

$$-\int_{Q_t}\left(v^2 + A\eta^{\beta-1}\theta^2 + \eta\int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right) \; dy\right) \; dx \; ds.$$

Finally we end with the formula

$$\int_0^t \Phi(x,s)\ ds = \Psi_1(x,t) + \Psi_2(x,t),$$

with

$$\Psi_1(x,t) = \int_0^x \frac{v^0}{{r^0}^2}\ dy - \int_0^x \frac{v}{r^2}\ dy - \frac{3}{R_1^3 - R_0^3}\left[\int_\Omega \eta^0 \left(\int_0^x \frac{v^0}{{r^0}^2}\ dy\right)\ dx - \int_\Omega \eta \left(\int_0^x \frac{v}{r^2}\ dy\right)\ dx\right],$$

and

$$\Psi_2(x,t) = \int_0^t \left[-\int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right)\ dy\right.$$
$$\left. + \frac{3}{R_1^3 - R_0^3}\int_\Omega \left(v^2 + A\eta^{\beta-1}\theta^2 + \eta \int_0^x \left(\frac{2v^2}{r^3} + \frac{GM_0}{r^4}\right)\ dy\right)\ dx\right]\ ds.$$

Clearly, after the energy bound (19), one has for some $C_1 \equiv C_1(N) > 0$ and $C_2 \equiv C_2(N) > 0$

$$C_1^{-1} \leqslant \Psi_1(x,t) \leqslant C_1 \quad \text{for any } (x,t) \in Q_T, \tag{37}$$

and

$$\Psi_2(x,t) \leqslant C_2 \quad \text{for any } (x,t) \in Q_T, \tag{38}$$

Moreover for $0 \leqslant s \leqslant t < T$ and some $C_3 \equiv C_2(N) > 0$

$$C_3^{-1}(t-s)\int_\Omega \theta^2 \eta^{\beta-1}\ dx \leqslant \Psi_2(x,t) - \Psi_2(x,s) \leqslant C_3(t-s)\left(1 + \|\eta\|_\infty^{2\beta-2}\right). \tag{39}$$

Plugging (37), (38) and (39) into (34) gives

$$\|\eta\|_\infty^{2-\beta}$$
$$= \left\| e^{\frac{2-\beta}{\mu}\Psi_1(x,t)} \left\{ (\eta^0)^{\beta-2}\ e^{\frac{2-\beta}{\mu}\Psi_2(x,t)} + \frac{A(2-\beta)}{\mu}\int_0^t \theta^2 e^{-\frac{2-\beta}{\mu}\Psi_1(x,s)} e^{\frac{2-\beta}{\mu}[\Psi_2(x,t)-\Psi_2(x,s)]}\ ds \right\} \right\|_\infty$$
$$\leqslant C_4 \left(1 + \int_0^t \|\theta^2(\cdot,s)\|_\infty\ e^{-\frac{2-\beta}{\mu}C_2(t-s)\left(1+\|\eta\|_\infty^{2-2\beta}\right)}\ ds\right). \tag{40}$$

Observing that

$$\|\theta^2(\cdot,s)\|_\infty \leqslant \|\theta^2(\cdot,s) - \theta_\Gamma\|_\infty^2 + 2\theta_\Gamma \|\theta^2(\cdot,s) - \theta_\Gamma\|_\infty + \theta_\Gamma^2,$$

and that the identity $\theta(\cdot,s) - \theta_\Gamma = \left(\theta^{1/2}(\cdot,s) - \theta_\Gamma^{1/2}\right)^2 + 2\theta_\Gamma^{1/2}\left(\theta^{1/2}(\cdot,s) - \theta_\Gamma^{1/2}\right)$ implies that

$$\|\theta(\cdot,s) - \theta_\Gamma\|_\infty \leqslant 2\|\theta^{1/2}(\cdot,s) - \theta_\Gamma^{1/2}\|_\infty^2 + \theta_\Gamma,$$

after the Cauchy-Schwarz inequality, and using Lemma 2, we have

$$\|\theta^2(\cdot,s)\|_\infty \leqslant C + F(t),$$

where $F \in L^1(0,\infty)$. Plugging this estimate into (40) implies the upper bound $\eta \leqslant \overline{\eta}$.

Using a similar approach to get a lower bound, we see after (39) that

$$\|\eta\|_\infty^{\beta-2} \leqslant C_6 \left(\int_0^t \theta^2 e^{\frac{2-\beta}{\mu}[\Psi_2(x,t)-\Psi_2(x,s)]}\ ds\right)^{-1}$$
$$\leqslant C_6 \left(\int_0^t (1 - G(s))\ e^{-\frac{\beta}{\mu}C_3^{-1}(t-s)\int_\Omega \theta^2\eta^{\beta-1}dx}\ ds\right)^{-1},$$

where $G \in L^1(0,\infty)$.

Using the upper bound for $F \in L^1(0,\infty)$ η, we find

$$\|\eta\|_\infty^{-\beta} \leqslant C_6 \left(\int_0^t (1 - G(s))\ e^{-C_7^{-1}(t-s)}\ ds\right)^{-1} \leqslant C_8,$$

which gives a lower bound for η and ends the proof □

Lemma 4 *The following uniform bound holds*

$$\int_0^T \max_\Omega \; v^2(\cdot,t)\; dt \leqslant C, \tag{41}$$

Proof: From the energy estimate, we have

$$|v(x,t)| \leqslant \frac{1}{r^2}\int_\Omega |(r^2v)_x|\; dx \leqslant C\left(\int_\Omega \frac{\mu}{\eta\theta}\;[(r^2v)_x]^2\; dx\right)^{1/2}\left(\int_\Omega \frac{\eta\theta}{\mu}\; dx\right)^{1/2}.$$

Observing that the last integral is bounded by $\left(\int_\Omega \theta^2\; dx\right)^{1/2}$, taking the square and using (20), we get (41) □

Proposition 3 *The following bounds hold*

$$\|\eta_x\|_{L^\infty(0,T,L^2(\Omega))} \leqslant K(N), \tag{42}$$

$$\|\theta\eta_x\|_{L^2(Q_T)} \leqslant K(N), \tag{43}$$

Proof: ¿From the momentum equation, we have

$$\left[\frac{v}{r^2} - \mu(\log\eta)_x\right]_t = -p_x - \frac{2v^2}{r^3} + \frac{f}{r^2}.$$

Multiplying by $\frac{v}{r^2} - \mu(\log\eta)_x$ and integrating on Ω, one gets

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left[\frac{v}{r^2} - \mu(\log\eta)_x\right]^2\; dx = \int_\Omega \left(-p_\theta\theta_x - p_\eta\eta_x - \frac{2v^2}{r^3} + \frac{f}{r^2}\right)\left(\frac{v}{r^2} - \mu(\log\eta)_x\right)\; dx.$$

Integrating by parts and using Proposition 2

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left[\frac{v}{r^2} - \mu(\log\eta)_x\right]^2\; dx + \int_\Omega \mu\;\frac{A(2-\beta)}{2\eta^{4-\beta}}\;\theta^2\eta_x^2\; dx$$
$$= -\int_\Omega \frac{A\theta}{\eta^{2-\beta}}\;\theta_x\;\frac{v}{r^2}\; dx + \int_\Omega \mu A\;\frac{\theta}{\eta^{3-\beta}}\;\frac{\theta_x\eta_x}{\eta}\; dx$$
$$-\int_\Omega \frac{A(2-\beta)}{2\eta^{3-\beta}}\;\theta^2\frac{v}{r^2}\;\eta_x\; dx + \int_\Omega \frac{fv}{r^4}\; dx - \int_\Omega \mu\;\frac{f\eta_x}{r^2\eta}\; dx - \int_\Omega \frac{2v^3}{r^5}\; dx + \int_\Omega \frac{2\mu v^2}{\eta r^3}\;\eta_x\; dx =: \sum_{k=1}^{7} J_k.$$

We first observe that the second integral in the left-hand side rewrites

$$\int_\Omega \mu\;\frac{A(2-\beta)}{2\eta^{4-\beta}}\,(\theta-\theta_\Gamma)^2\,\eta_x^2\; dx + 2\theta_\Gamma\int_\Omega \mu\;\frac{A(2-\beta)}{2\eta^{4-\beta}}\,(\theta-\theta_\Gamma)\,\eta_x^2\; dx + \theta_\Gamma^2\int_\Omega \mu\;\frac{A(2-\beta)}{2\eta^{4-\beta}}\eta_x^2\; dx,$$

so we get the inequality

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left[\frac{v}{r^2} - \mu(\log\eta)_x\right]^2\; dx + \frac{1}{2}\theta_\Gamma^2\int_\Omega \eta_x^2\; dx \leqslant \sum_{k=1}^{8} J_k, \tag{44}$$

with $J_8 = C\int_\Omega (\theta-\theta_\Gamma)^2\,\eta_x^2\; dx$.

Now, after (20), Proposition 2 and (2)

$$V(t) := \int_\Omega \frac{\kappa r^4}{\eta\theta^2}\theta_x^2\; dx \in L^1(0,\infty) \;\text{ and }\; W(t) := \int_\Omega \frac{\mu}{\eta\theta}\; v_x^2\; dx \in L^1(0,\infty).$$

Let us now estimate all of the contributions of the right-hand side.

$$|J_1| \leqslant C\int_\Omega \theta|\theta_x v|\ dx \leqslant CV(t) + C\int_\Omega \frac{\theta^4}{\kappa}v^2\ dx.$$

But as $q \geqslant 2$ we get

$$|J_1| \leqslant CV(t) + C\int_\Omega \theta^2 v^2\ dx \leqslant CV(t) + C\int_\Omega (\theta-\theta_\Gamma)^2 v^2\ dx + C\int_\Omega v^2\ dx$$
$$\leqslant CV(t) + CW(t).$$

$$|J_2| \leqslant \int_\Omega \theta|\theta_x\eta_x|\ dx \leqslant \frac{\varepsilon}{2}\int_\Omega \theta^2\eta_x^2\ dx + C_\varepsilon V(t).$$

$$|J_3| \leqslant \int_\Omega \theta^2|\eta_x v|\ dx \leqslant \frac{\varepsilon}{2}\int_\Omega \theta^2\eta_x^2\ dx + C_\varepsilon\int_\Omega \theta^2 v^2\ dx \leqslant \frac{\varepsilon}{2}\int_\Omega \theta^2\eta_x^2\ dx + C_\varepsilon V(t).$$

$$J_4 = \frac{GM_0}{3}\left(\int_\Omega \frac{dx}{r^3}\right)_t.$$

$$|J_5| \leqslant C\int_\Omega |\eta_x|\ dx \leqslant C + \frac{\varepsilon}{2}\int_\Omega \eta_x^2\ dx.$$

$$|J_6| \leqslant \int_\Omega \frac{2|v|^3}{r^5}\ dx \leqslant C\max_\Omega |v| \leqslant CW(t).$$

$$|J_7| \leqslant \int_\Omega \frac{2\mu v^2}{\eta r^3}\ |\eta_x|\ dx \leqslant \int_\Omega v^2|\eta_x|\ dx \leqslant C\max_\Omega v^2\left(1+\int_\Omega \eta_x^2\ dx\right) \leqslant CW(t)\left(1+\int_\Omega \eta_x^2\ dx\right).$$

$$|J_8| \leqslant CV(t)\int_\Omega \eta_x^2\ dx.$$

Collecting all of these estimates and taking ε small enough, we obtain

$$\frac{dU}{dt} + U \leqslant C,$$

where $U(t) := \int_\Omega \frac{1}{2}\left[\frac{v}{r^2} - \mu(\log\eta)_x\right]^2\ dx$, which implies that $U(t) \leqslant C$ and then after (19) $\int_\Omega \eta_x^2\ dx \leqslant C$ and $\int_{Q_T}\theta^2\eta_x^2\ dx\ dt \leqslant C$, which ends the proof □

Proposition 4 *The following bounds hold for* $w := r^2 v$

$$\|w_x\|_{L^\infty(0,T,L^2(\Omega))} \leqslant K(N), \tag{45}$$

$$\|w\|_{L^\infty(Q_T)} \leqslant K(N), \tag{46}$$

$$\|w_{xx}\|_{L^2(Q_T)} \leqslant K(N). \tag{47}$$

Proof: ¿From the momentum equation, we have

$$w_t = r^4\left(-p + \frac{\mu}{\eta}\ w_x\right)_x + r^2 f + 2rv^2.$$

Multiplying by $-w_{xx}$ and integrating by parts on Ω, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\ w_x^2\ dx + \int_\Omega \frac{\mu r^4}{\eta}\ w_{xx}^2\ dx = \int_\Omega \frac{\mu r^4}{\eta^2}\ \eta_x w_x w_{xx}\ dx - 2\int_\Omega r v^2 w_{xx}\ dx + \int_\Omega \frac{2Ar^4}{\eta^{2-\beta}}\ \theta\theta_x w_{xx}\ dx$$
$$-\int_\Omega \frac{\beta A r^4}{\eta^{3-\beta}}\ \theta^2\eta_x w_{xx}\ dx - \int_\Omega r^2 f w_{xx}\ dx =: \sum_{k=1}^5 H_k.$$

We estimate the right-hand side as follows

$$|H_1| \leqslant C \int_\Omega |\eta_x w_x w_{xx}| \; dx \leqslant \frac{1}{2} \; \varepsilon \int_\Omega w_{xx}^2 \; dx + \frac{1}{2\varepsilon} \int_\Omega \eta_x^2 w_x^2 \; dx$$

$$\leqslant \frac{1}{2} \; \varepsilon \int_\Omega w_{xx}^2 \; dx + \frac{1}{2\varepsilon} \left(\max_\Omega w_x^2 \; \max_{[0,T]} \int_\Omega \eta_x^2 \; dx \right)$$

$$\leqslant \frac{1}{2} \; \varepsilon \int_\Omega w_{xx}^2 \; dx + \frac{C}{2\varepsilon} \; \max_\Omega w_x^2 \; dx$$

$$\leqslant \frac{1}{2} \; \varepsilon \int_\Omega w_{xx}^2 \; dx + \frac{C}{2\varepsilon} \left(\frac{\varepsilon^2}{C} \int_\Omega w_{xx}^2 \; dx + \frac{C}{\varepsilon^2} \int_\Omega w_x^2 \; dx \right)$$

$$\leqslant \varepsilon \int_\Omega w_{xx}^2 \; dx + C \int_\Omega w_x^2 \; dx.$$

$$|H_2| \leqslant C \int_\Omega v^2 |w_{xx}| \; dx \leqslant \varepsilon \int_\Omega w_{xx}^2 \; dx + C \max_\Omega v^2.$$

$$|H_3| \leqslant C \int_\Omega \theta |\theta_x w_{xx}| \; dx \leqslant \varepsilon \int_\Omega w_{xx}^2 \; dx + C \int_\Omega \theta^2 \theta_x^2 \; dx.$$

$$|H_4| \leqslant C \int_\Omega \theta^2 |\eta_x w_{xx}| \; dx \leqslant \varepsilon \int_\Omega w_{xx}^2 \; dx + C \int_\Omega \theta^2 \eta_x^2 \; dx.$$

$$|H_5| \leqslant C \int_\Omega |w_{xx}| \; dx \leqslant \varepsilon \int_\Omega w_{xx}^2 \; dx + C.$$

Using these estimates, we get the inequality

$$\frac{d}{dt} \int_\Omega \; w_x^2 \; dx + \int_\Omega w_{xx}^2 \; dx \leqslant C \left(1 + \int_\Omega (w_x^2 + \theta^2 \theta_x^2) \; dx + \max_\Omega (\theta - \theta_\Gamma)^2 + \max_\Omega v^2 \right). \tag{48}$$

Now, multiplying the momentum equation by v and and integrating by parts on Ω, we get

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \; v^2 - \frac{GM_0}{r} \right) \; dx + \int_\Omega \frac{\mu}{\eta} \; [(r^2 v)_x]^2 \; dx = - \int_\Omega p_x r^2 v \; dx$$

$$= \int_\Omega \frac{\beta A \theta^2}{\eta^{3-\beta}} \; \eta_x r^2 v \; dx - \int_\Omega \frac{2A\theta}{\eta^{2-\beta}} \; \theta_x r^2 v \; dx.$$

Then

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \; v^2 - \frac{GM_0}{r} \right) \; dx + \int_\Omega w_x^2 \; dx \leqslant \varepsilon \int_\Omega \theta^4 \eta_x^2 \; dx + C \int_\Omega \theta^2 \theta_x^2 \; dx + C \max_\Omega v^2. \tag{49}$$

Multiplying (48) by ε and adding to (49) gives

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \; v^2 - \frac{GM_0}{r} + \varepsilon w_x^2 \right) \; dx + \int_\Omega \left(\frac{1}{2} \; v^2 - \frac{GM_0}{r} + \varepsilon w_x^2 \right) \; dx$$

$$\leqslant C \left(1 + \int_\Omega \theta^2 \theta_x^2 \; dx + \max_\Omega (\theta - \theta_\Gamma)^2 + \max_\Omega v^2 + C \int_\Omega \theta^4 \eta_x^2 \; dx \right).$$

It is now routine to show that the right-hand side is bounded by $C(1+G(t))$, where $G \in L^1(0,\infty)$ provided that $q \geqslant 4$, which gives the bounds (45) and (47).

Estimate (46) then follows from (45), which ends the proof □

Remark 2 *After (36) Lemma 2 is improved: for any* $2\alpha - 1 \leqslant q$, $\int_0^T \max_\Omega \; (\theta^\alpha(\cdot, t) - \theta_\Gamma^\alpha)^2 \; dt \leqslant C$.

Now, following [18], let us define the positive quantities

$$Y(t) := \max_{0\leqslant s\leqslant t} \int_\Omega (1+\theta^q)^2\ \theta_x^2\ dx \text{ and } X(t) := \int_{Q_t} (1+\theta)(1+\theta^q)\ \theta_s^2\ dx\ ds.$$

Lemma 5 *The following inequality holds*

$$X(t) + Y(t) \leqslant C, \quad \text{for any } t \in [0,T]. \tag{50}$$

Proof: We follow the main steps of [18] (see the proof of Theorem 1.1). We first observe that

$$\left(\theta^{q+2} - \theta_\Gamma^{q+2}\right) \leqslant C \left(\int_\Omega (1+\theta^q)^2 \theta_x^2\ dx\right)^{1/2} \left(\int_\Omega \frac{\theta^{2q+2}}{(1+\theta^q)^2}\ dx\right)^{1/2} \leqslant C\left(1+Y^{1/2}\right),$$

then

$$\max_{Q_t} \theta \leqslant C\left(1+Y^{1/2}\right)^{\frac{1}{2q+4}}. \tag{51}$$

Now, from the energy equation, we have

$$e_\theta\ \theta_t + \theta p_\theta\ w_x - \frac{\mu}{\eta}\ w_x^2 = \left(\frac{\kappa r^4}{\eta}\ \theta_x\right)_x = 4r\kappa r^4 \theta_x + r^4\left(\frac{\kappa}{\eta}\ \theta_x\right)_x.$$

Defining $K(\eta,\theta) := \int_1^\theta \frac{\kappa(\eta,s)}{\eta}\ ds$, multiplying the previous equation by K_t, dividing by r^4 and integrating by parts on Ω, we get

$$\int_\Omega \left(\frac{e_\theta}{r^4}\ \theta_t + \frac{\theta p_\theta}{r^4}\ w_x - \frac{\mu}{r^4\eta}\ w_x^2 - \frac{4\kappa\theta_x}{r^3}\right) K_t\ dx + \int_\Omega \frac{\kappa}{\eta}\ \theta_x\ K_{tx}\ dx = 0.$$

Computing $K_t = K_\eta w_x + \frac{\kappa}{\eta}\ \theta_t$ and $K_{xt} = \left(\frac{\kappa}{\eta}\ \theta_x\right)_t + K_\eta w_{xx} + K_{\eta\eta} w_x \eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t$, and observing that $|K_\eta| + |K_{\eta\eta}| \leqslant C(1+\theta)(1+\theta^q)$, we can estimate all the contributions in the sum

$$\int_\Omega \left(\frac{e_\theta}{r^4}\ \theta_t + \frac{\theta p_\theta}{r^4}\ w_x - \frac{\mu}{r^4\eta}\ w_x^2 - \frac{4\kappa\theta_x}{r^3}\right)\left(K_\eta w_x + \frac{\kappa}{\eta}\ \theta_t\right)\ dx$$

$$+ \int_\Omega \frac{\kappa}{\eta}\ \theta_x \left(\left(\frac{\kappa}{\eta}\ \theta_x\right)_t + K_\eta w_{xx} + K_{\eta\eta} w_x\eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t\right)\ dx = 0.$$

Rearranging the first term in the second integral and integrating on $[0,t]$, we get

$$\int_\Omega \frac{1}{2}\ \frac{\kappa^2}{\eta^2}\ \theta_x^2\ dx - \left[\int_\Omega \frac{1}{2}\ \frac{\kappa^2}{\eta^2}\ \theta_x^2\ dx\right]_{t=0} + \int_{Q_t} \frac{\kappa}{\eta}\ \frac{e_\theta}{r^4}\ \theta_s^2\ dx\ ds$$

$$= -\int_{Q_t} \frac{e_\theta}{r^4}\ K_\eta \theta_s w_x\ dx\ ds - \int_{Q_t} \frac{4\kappa K_\eta}{r^3}\theta_x w_x\ dx\ ds - \int_{Q_t} \frac{4\kappa^2}{r^3\eta}\ \theta_x\theta_s\ dx\ ds$$

$$- \int_{Q_t}\left(\theta p_\theta\ w_x - \frac{\mu}{\eta}\ w_x^2\right)\frac{K_\eta}{r^4}\ w_x\ dx\ ds - \int_{Q_t}\left(\theta p_\theta\ w_x - \frac{\mu}{\eta}\ w_x^2\right)\ \frac{\kappa}{r^4\eta}\ \theta_s\ dx\ ds$$

$$- \int_{Q_t} \frac{\kappa}{\eta}\ K_{\eta\eta}\theta_x w_x \eta_x\ dx\ ds - \int_{Q_t} \frac{\kappa}{\eta}\left(\frac{\kappa}{\eta}\right)_\eta \theta_x\eta_x\theta_s\ dx\ ds - \int_{Q_t}\frac{\kappa}{\eta}\ K_\eta\theta_x w_{xx}\ dx\ ds =: \sum_{k=1}^{8} I_k, \tag{52}$$

where we notice that

$$\int_{Q_t} \frac{\kappa}{\eta}\ \frac{e_\theta}{r^4}\ \theta_t^2\ dx\ ds \geqslant c_0 X(t),$$

with $c_0 = \min \left\{ c_V^0 \ , \ \frac{A\kappa}{R_1^4 \overline{\eta}^{1-\beta}} \right\}$. We estimate the various integrals of the right-hand side as follows, using Cauchy-Schwarz inequality together with Propositions 3 and 4.

$$|I_1| \leqslant C \int_{Q_t} (1+\theta)(1+\theta^q) \ |\theta_s w_x| \ dx \ ds \leqslant \varepsilon X(t) + C \max_{Q_t} \ (1+\theta^{q+1}) \ \int_0^t \max_{\Omega} \ w_x^2 \ dx$$

$$\leqslant \varepsilon X(t) + C \max_{Q_t} \ (1+\theta^{q+1}),$$

using the identity $w_x^2(x,t) \leqslant C \int_\Omega w_{xx}^2 \ dx$ and Proposition 4.

$$|I_2| \leqslant C \int_{Q_t} (1+\theta^q)^2 \ |\theta_x w_x| \ dx \ ds$$

$$\leqslant C \max_{Q_t}(1+\theta^{2q}) \int_{Q_t} (\theta_x^2 + w_x^2) \ dx \ ds \leqslant C \max_{Q_t}(1+\theta^{2q}),$$

by using Proposition 4.

$$|I_3| \leqslant C \int_{Q_t} (1+\theta^q)^2 \ |\theta_x \theta_t| \ dx \ ds \leqslant \varepsilon X(t) + C \max_{Q_t}(1+\theta^{2q+1}) \int_{Q_t} \frac{1+\theta^q}{\theta^2} \ \theta_x^2 \ dx \ ds$$

$$\leqslant C \max_{Q_t}(1+\theta^{2q+1}),$$

after (20) and Proposition 4.

$$|I_4| \leqslant C \int_{Q_t} (1+\theta^q) \left(\theta^2 |w_x| + w_x^2\right) \ |w_x| \ dx \ ds$$

$$\leqslant C \max_{Q_t} \ (1+\theta^{q+2}) \ \int_0^t \max_{\Omega} w_x^2 \ dx \ ds + C \max_{Q_t} \ (1+\theta^q) \ \int_0^t \max_{\Omega} w_x^2 \int_\Omega |w_x| \ dx \ ds$$

$$\leqslant C \max_{Q_t} \ (1+\theta^{q+2}) \ \int_0^t \max_{\Omega} w_x^2 \ dx \ ds + C \max_{Q_t} \ (1+\theta^{q+2}) \ \int_0^t \max_{\Omega} w_x^2 \int_\Omega w_x^2 \ dx \ ds$$

$$\leqslant C \max_{Q_t} \ (1+\theta^{q+2}),$$

after Proposition 4.

$$|I_5| \leqslant C \int_{Q_t} (1+\theta^q) \left(\theta^2 |w_x| + w_x^2\right) \ |\theta_t| \ dx \ ds$$

$$\leqslant \varepsilon X(t) + C \int_{Q_t} (1+\theta^q) \left(\theta^4 w_x^2 + 2\theta^2 |w_x|^3 + w_x^4\right) \ dx \ ds$$

$$\leqslant \varepsilon X(t) + C \max_{Q_t} \ \theta^4 (1+\theta^q) \ \int_{Q_t} w_x^2 \ dx \ ds + C \max_{Q_t} \ \theta^2 (1+\theta^q) \ \int_{Q_t} |w_x|^3 \ dx \ ds$$

$$+ C \max_{Q_t} \ (1+\theta^q) \ \int_{Q_t} w_x^4 \ dx \ ds \leqslant C \max_{Q_t} \ (1+\theta^{q+4}),$$

after Proposition 4.

$$|I_6| \leqslant C \int_{Q_t} (1+\theta^q)^2 |\theta_x w_x \eta_x| \ dx \ ds \leqslant C \left(\int_{Q_t} (1+\theta^q)^2 \theta_x^2 \ dx \ ds \right)^{1/2} \left(\int_{Q_t} (1+\theta^q)^2 w_x^2 \eta_x^2 \ dx \ ds \right)^{1/2}$$

$$\leqslant CY(t) + C \max_{Q_t} \ (1+\theta^{\frac{2q+1}{2}}) \left(\max_{[0,T]} \int_\Omega \eta_x^2 \ dx \right)^{1/2} \left(\int_0^t \max_{\Omega} \ w_x^2 \ ds \right)^{1/2} \leqslant CY(t) + C \max_{Q_t} \ (1+\theta^{\frac{2q+1}{2}}),$$

after (19) and Propositions 3 and 4.

$$|I_7| \leqslant \int_\Omega \frac{\kappa}{\eta} \left| \left(\frac{\kappa}{\eta} \right)_\eta \theta_x \eta_x \theta_t \right| \ dx \leqslant \varepsilon X(t) + C \int_{Q_t} \left(\frac{\kappa r^4}{\eta} \ \theta_x \right)^2 (1+\theta^{2q-1}) \ \eta_x^2 \ dx \ ds$$

$$\leqslant \varepsilon X(t) + C \max_{Q_t} (1+\theta^{4q-2}) + C \max_{Q_t} (1+\theta^{2q-1}) \int_0^t \left|\frac{\kappa r^4}{\eta}\theta_x\right| \left|\left[\frac{\kappa r^4}{\eta}\ \theta_x\right]_x\right| \ dx\ ds$$

$$\leqslant \varepsilon X(t) + C \max_{Q_t} (1+\theta^{2q+1}).$$

$$|I_8| \leqslant C \int_\Omega \frac{\kappa}{\eta}\ K_\eta |\theta_x w_{xx}|\ dx \leqslant C \int_{Q_t} (1+\theta^q)^2 \theta_x^2\ dx\ ds + C \int_{Q_t} (1+\theta^q)^2 w_{xx}^2\ dx\ ds$$

$$\leqslant CY(t) + C \max_{Q_t} (1+\theta^{2q}),$$

after Proposition 4.

Plugging all of these estimates into (52) and taking ε small enough, we get

$$X + Y \leqslant C(1 + \max_{Q_t}\ \theta^{2q+1}).$$

After (51), we end with the inequality $X + Y \leqslant C\left(1 + Y^{\frac{2q+1}{2q+4}}\right)$, which implies (50) and ends the proof $\square$

Corollary 1 *For any* $T > 0$

$$\max_{[0,T]} \int_\Omega \theta_x^2\ dx \leqslant K, \tag{53}$$

and

$$\max_\Omega \theta_x^2 \in L^1(0,T). \tag{54}$$

Proof:

1. From Lemma 5, we see that $X(t) \leqslant C$, for any $t \leqslant T$. Inequality (53) follows directly.
2. The estimate (54) is a direct consequence of Lemma 5 $\square$

Proposition 5 *1. There exist two positive constants* $\underline{\theta}$ *and* $\overline{\theta}$ *depending only on* N*, such that*

$$0 < \underline{\theta} \leqslant \theta(x,t) \leqslant \overline{\theta} \quad \text{for } (x,t) \in Q_T. \tag{55}$$

2.

$$\|\eta_x\|_{L^2(Q_t)} \leqslant K(N). \tag{56}$$

Proof:

1. We have

$$\left|\log\left(\frac{e}{e_\Gamma}\right)\right| \leqslant C \int_\Omega \frac{|e_x|}{e}\ dx \leqslant C \int_\Omega \frac{e_\theta}{e}\ |\theta_x|\ dx + C \int_\Omega \frac{e_\eta}{e}\ |\eta_x|\ dx$$

$$\leqslant C \int_\Omega \frac{|\theta_x|}{e}\ dx + C \int_\Omega (|\eta_x| + |\theta_x|)\ dx,$$

where $e_\Gamma := e(\eta(M,t), \theta_\Gamma)$, observing that $e(\overline{\eta}, \theta_\Gamma) \leqslant e(\eta(M,t), \theta_\Gamma) \leqslant e(\underline{\eta}, \theta_\Gamma)$.

As the last integral is bounded after Proposition 3 and Corollary 1, we get

$$\left|\log\left(\frac{e}{e_\Gamma}\right)\right| \leqslant C + C\left(\int_\Omega \theta_x^2\ dx\right)^{1/2} \left(\int_\Omega \frac{dx}{e^2}\right)^{1/2}. \tag{57}$$

To check that the last integral is bounded, we multiply the last equation (6) by $-e^{-3}$, and integrate by parts, ending with

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\frac{dx}{e^2} + \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + \int_\Omega \frac{\kappa r^4 e_\theta}{\eta e^4}\ \theta_x^2\ dx = -\int_\Omega \frac{\kappa r^4 e_\eta}{\eta e^4}\ \theta_x \eta_x\ dx + \int_\Omega \frac{p}{e^3}\ w_x\ dx =: J_1 + J_2.$$

Let us estimate the right-hand side. We get

$$|J_1| \leqslant \varepsilon \int_\Omega \frac{\kappa r^4 e_\eta}{\eta e^4}\ \theta_x^2\ dx + C_\varepsilon \int_\Omega \eta_x^2\ dx,$$

where the last integral is in $L^1(0,T)$, and

$$\begin{aligned}|J_2| = &\int_\Omega \left[\frac{1}{e^3(\eta,\theta)} - \frac{1}{e^3(\eta,\theta_\Gamma)}\right] p(\eta,\theta) w_x\ dx + \int_\Omega \left[\frac{1}{e^3(\eta,\theta_\Gamma)} - \frac{1}{e^3(\eta_M,\theta_\Gamma)}\right] p(\eta,\theta) w_x\ dx \\ &+ \int_\Omega [p(\eta,\theta) - p(\eta,\theta_\Gamma]\ \frac{w_x}{e^3(\eta_M,\theta_\Gamma)}\ dx + \int_\Omega [p(\eta,\theta_\Gamma) - p(\eta_M,\theta_\Gamma)] \frac{w_x}{e^3(\eta_M,\theta_\Gamma)}\ dx \\ &+ \int_\Omega \frac{p(\eta_M,\theta_\Gamma)}{e^3(\eta_M,\theta_\Gamma)}\ w_x\ dx =: \sum_{j=1}^5 K_j,\end{aligned}$$

where the last integral K_5 is zero due to boundary conditions. Bounding all of these terms, using Taylor formula and Cauchy-Schwarz inequality, we get

$$|K_1| \leqslant \varepsilon \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + C_\varepsilon \int_\Omega (\theta - \theta_\Gamma)^2\, dx.$$

$$|K_2| \leqslant \varepsilon \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + C_\varepsilon \int_\Omega (\eta - \eta_M)^2\, dx.$$

$$|K_3| \leqslant \varepsilon \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + C_\varepsilon \int_\Omega (\theta - \theta_\Gamma)^2\, dx.$$

$$|K_4| \leqslant \varepsilon \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + C_\varepsilon \int_\Omega (\eta - \eta_M)^2\, dx.$$

Finally

$$|J_2| \leqslant \varepsilon \int_\Omega \frac{\mu}{\eta e^3}\ w_x^2\ dx + F(t),$$

where $F \in L^1(0,T)$.

We obtain the inequality

$$\frac{d}{dt}\int_\Omega \frac{dx}{e^2} \leqslant G(t),$$

where $G \in L^1(0,T)$. Then we conclude, using Gronwall's Lemma, which implies finally, together with Corollary 1, that the left hand side of (57) is finite. Inequality (55) then follows.

2. After (55) and Proposition 3, one gets (56) □

Proposition 6 *The following bounds hold*

$$\max_{[0,T]} \|w_t\|_{L^2(\Omega)} \leqslant C(T), \quad \|w_{xt}\|_{L^2(Q_T)} \leqslant C(T), \tag{58}$$

$$\max_{[0,T]} \|\theta_t\|_{L^2(\Omega)} \leqslant C(T), \quad \|\theta_{xt}\|_{L^2(Q_T)} \leqslant C(T), \tag{59}$$

$$\max_{[0,T]} \|w_{xx}\|_{L^2(\Omega)} \leqslant C(T), \quad \max_{[0,T]} \|\theta_{xx}\|_{L^2(\Omega)} \leqslant C(T). \tag{60}$$

Proof:

1. The first equation (6) rewrites

$$w_t = r^4 \left(-p + \frac{\mu}{\eta}\ w_x\right)_x + \frac{2w^2}{r^3} - GM_0.$$

We derivate formally this equation with respect to t (this can be made rigorous by taking finite difference and passing to the limit (see [1])), multiply by w_t and integrate by parts in x

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2} w_t^2 \, dx\right) + \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx = -\int_\Omega \frac{4\mu}{r^2} \, w w_t w_x \, dx - \int_\Omega \frac{4\mu r}{\eta} \, w_x^2 w_t \, dx$$

$$-\int_\Omega \frac{4\mu r}{\eta} \, w w_x w_{tx} \, dx + \int_\Omega \frac{4\mu r}{\eta} \, w_t w_x^2 \, dx - \int_\Omega 4\mu r w_t w_{tx} \, dx + \int_\Omega \frac{\mu r^4}{\eta^2} \, w_x^2 w_{tx} \, dx$$

$$+\int_\Omega \frac{4}{r^3} \, w w_t^2 \, dx - \int_\Omega \frac{6}{r^6} \, w^3 w_t \, dx + \int_\Omega \frac{4p\eta}{r^2} \, w w_t \, dx + \int_\Omega 4prw w_{tx} \, dx + \int_\Omega 4prw_t w_x \, dx$$

$$+\int_\Omega 4r\eta p_\eta w_x w_t \, dx + \int_\Omega 4r\eta p_\theta \theta_t w_t \, dx + \int_\Omega r^4 p_\eta w_x w_{xt} \, dx + \int_\Omega r^4 p_\theta \theta_t w_{xt} \, dx =: \sum_{j=1}^{15} D_j.$$

Let us estimate all of these terms.

$$|D_1| \leqslant C \int_\Omega |w w_t w_x| \, dx \leqslant C \int_\Omega w^2 w_t^2 \, dx + C \int_\Omega w_x^2 \, dx \leqslant C \max_\Omega w^2 \int_\Omega w_t^2 \, dx + C \int_\Omega w_{xx}^2 \, dx.$$

$$|D_2| \leqslant C \int_\Omega |w w_t w_x \eta_x| \, dx \leqslant C \int_\Omega w_x^2 w_t^2 \, dx + C \int_\Omega w_x^2 \, dx \leqslant C \max_\Omega v_x^2 \int_\Omega w_t^2 \, dx + C \int_\Omega w_{xx}^2 \, dx.$$

$$|D_3| \leqslant C \int_\Omega |w w_x w_{xt}| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \max_\Omega w^2 \int_\Omega w_{xx}^2 \, dx.$$

$$|D_4| \leqslant C \int_\Omega |w_t w_x^2| \, dx \leqslant C \int_\Omega w_x^2 w_t^2 \, dx + C \int_\Omega w_x^2 \, dx \leqslant C \max_\Omega v_x^2 \int_\Omega w_t^2 \, dx + C \int_\Omega w_{xx}^2 \, dx.$$

$$|D_5| \leqslant C \int_\Omega |w_t w_{tx}| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega w_t^2 \, dx.$$

$$|D_6| \leqslant C \int_\Omega w_x^2 |w_{tx}| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega w_x^4 \, dx$$

$$\leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega w_{xx}^2 \, dx.$$

$$|D_7| \leqslant C \int_\Omega |w_t^2 w| \, dx \leqslant C \int_\Omega w_t^2 \, dx.$$

$$|D_8| \leqslant C \int_\Omega |w_t w^3| \, dx \leqslant C + C \int_\Omega w_t^2 \, dx.$$

$$|D_9| \leqslant C \int_\Omega |w_t w| \, dx \leqslant C + C \int_\Omega w_t^2 \, dx.$$

$$|D_{10}| \leqslant C \int_\Omega |w w_{tx}| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega w^2 \, dx.$$

$$|D_{11}| \leqslant C \int_\Omega |w_x w_t| \, dx \leqslant C + C \int_\Omega w_t^2 \, dx.$$

$$|D_{12}| \leqslant C \int_\Omega |w_t w_x| \, dx \leqslant C + C \int_\Omega w_t^2 \, dx.$$

$$|D_{13}| \leqslant C \int_\Omega |w_t \theta_t| \, dx \leqslant C \int_\Omega \theta_t^2 \, dx + C \int_\Omega w_t^2 \, dx.$$

$$|D_{14}| \leqslant C \int_\Omega p_\theta |w_{xt} w_x| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega w_{xx}^2 \, dx.$$

$$|D_{15}| \leqslant C \int_\Omega p_\theta |w_{xt} \theta_t| \, dx \leqslant \varepsilon \int_\Omega r^4 \frac{\mu}{\eta} \, w_{xt}^2 \, dx + C \int_\Omega \theta_t^2 \, dx.$$

So finally, choosing ε small enough

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}w_t^2\ dx\right)+\frac{1}{2}\int_\Omega r^4\frac{\mu}{\eta}\ w_{xt}^2\ dx \leqslant C + C\int_\Omega w_t^2 dx + C\int_\Omega \left(w_{xx}^2+\theta_t^2\right)\ dx. \tag{61}$$

As the last integral in the right-hand side is in $L^1(0,T)$ after Lemma 5 and Proposition 4, we get the inequalities (58).

2. The second equation (6) rewrites

$$w_{xx} = \frac{\eta}{r^6\mu}\ w_t - \frac{2w^2}{r^5} + \frac{GM_0\eta}{\mu r^6} - \frac{1}{\eta}\ \eta_x w_x + \frac{\eta}{\mu}\ p_\theta\theta_x + \frac{\eta}{\mu}\ p_\eta\eta_x.$$

Taking the square and integrating on Ω, we get

$$\int_\Omega w_{xx}^2\ dx \leqslant C + C\int_\Omega \left(w_t^2+\theta_x^2+w_x^2+\eta_x^2+w^4\right)\ dx,$$

which, after the previous estimates, gives the first estimate (60).

3. Rewriting the third equation (6) as

$$e_\theta\theta_t = q_x - \theta p_\theta w_x + \frac{\mu}{\eta}w_x^2,$$

and derivating this equation with respect to t (this can be made rigorous as previously), and multiply by $e_\theta\theta_t$

$$\int_\Omega\left(\frac{1}{2}\ e_\theta^2\theta_t^2\right)_t\ dx = \int_\Omega e_\theta\theta_t q_{xt}\ dx - \int_\Omega (\theta p_\theta w_x)_t e_\theta\theta_t\ dx + \int_\Omega (\frac{\mu}{\eta}w_x^2)_t e_\theta\theta_t\ dx = A_1+A_2+A_3,$$

where

$$A_1 = -\int_\Omega \left(e_{\theta\eta}\eta_x\theta_t + e_{\theta\theta}\theta_x\theta_t + e_\theta\theta_{tx}\right)$$
$$\times\left(\kappa_\eta\frac{r^4}{\eta}w_x\theta_x + \kappa_\theta\frac{r^4}{\eta}\theta_t\theta_x + \frac{4r\kappa}{\eta}w\theta_x - \frac{\kappa r^4}{\eta^2}w_x\theta_x + \frac{\kappa r^4}{\eta}\theta_{xt}\right)\ dx,$$
$$A_2 = -\int_\Omega \left(p_\theta e_\theta w_x\theta_t^2 + \theta p_{\theta\eta}w_x^2 + \theta p_{\theta\theta}\theta_t w_x + \theta p_\theta w_{xt}\right)e_\theta\theta_t dx,$$
$$A_3 = \int_\Omega\left(-\frac{\mu}{\eta^2}e_\theta w_x^3\theta_t + \frac{2\mu}{\eta}e_\theta w_x w_{xt}\theta_t\right)\ dx.$$

Integrating on $(0,t)$ for $0\leqslant t\leqslant T$, we find that, for two positive constant α and β

$$\alpha\max_{[0,T]}\int_\Omega \theta_t^2 dx + \beta\int_{Q_T}\theta_{tx}^2 dx\ dt \leqslant \int_\Omega\left(\frac{1}{2}\ e_\theta^2\theta_t^2\right)_t(x,0)\ dx + \sum_{k=1}^{20}E_k, \tag{62}$$

where the E_k are the various contributions corresponding to the integrands in A_1, A_2, A_3.

Let us estimate all of these terms, using previous bounds and Cauchy-Schwarz.

$$|E_1| \leqslant C\int_{Q_T}|\eta_x\theta_t w_x\theta_x|\ dx\ dt \leqslant C\int_{Q_T}\eta_x^2 w_x^2 dx\ dt + C\int_{Q_T}\theta_t^2\theta_x^2 dx\ dt$$
$$\leqslant C\int_0^T \max_\Omega\theta_t^2\int_\Omega\theta_x^2 dx\ dt + \max_{Q_T}w_x^2\int_{Q_T}\eta_x^2 dx\ dt$$
$$\leqslant C\left(1+\int_0^T\max_\Omega\theta_t^2 dt\right) \leqslant C + C\int_{Q_T}|\theta_t\theta_{xt}|\ dx\ dt \leqslant C + C\left(\int_{Q_T}\theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_2| \leqslant C\int_{Q_T}|\eta_x\theta_t^2\theta_x|\ dx\ dt \leqslant C\int_0^T\max_\Omega\theta_t^2\left(\frac{1}{2}\int_\Omega\theta_x^2 dx + \frac{1}{2}\int_\Omega\eta_x^2 dx\right)\ dt$$

$$\leqslant C\int_0^T \max_\Omega \theta_t^2 dt \leqslant C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_3| \leqslant C\int_{Q_T} |\eta_x\theta_t w\theta_x|\ dx\ dt \leqslant C\int_{Q_T} \eta_x^2 w^2 dx\ dt + C\int_{Q_T} \theta_t^2\theta_x^2 dx\ dt$$

$$\leqslant C\int_0^T \max_\Omega \theta_t^2 \int_\Omega \theta_x^2 dx\ dt + \max_{Q_T} w^2 \int_{Q_T} \eta_x^2 dx\ dt$$

$$\leqslant C\left(1+\int_0^T \max_\Omega \theta_t^2 dt\right) \leqslant C + C\int_{Q_T} |\theta_t\theta_{xt}|\ dx\ dt \leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_4| \leqslant C\int_{Q_T} |\eta_x\theta_t w_x\theta_x|\ dx\ dt \leqslant C|E_1|.$$

$$|E_5| \leqslant C\int_{Q_T} |\eta_x\theta_t\theta_{xt}|\ dx\ dt \leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 dx\ dt + C_\varepsilon\int_{Q_T} \theta_t^2\eta_x^2 dx\ dt$$

$$\leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 w_x^2 dx\ dt + C_\varepsilon\int_0^T \max_\Omega \theta_t^2 dt \leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 w_x^2 dx\ dt + C_\varepsilon\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_6| \leqslant C\int_{Q_T} |w_x\theta_t\theta_x^2|\ dx\ dt \leqslant C\int_0^T \max_\Omega \theta_t^2\ dt + C\max_{Q_T} w_x^2 \int_{Q_T} \theta_x^2 dx\ dt$$

$$\leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_7| \leqslant C\int_{Q_T} |\theta_t^2\theta_x^2|\ dx\ dt \leqslant C\int_0^T \max_\Omega \theta_t^2\ dt \leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_8| \leqslant C\int_{Q_T} |w\theta_t\theta_x^2|\ dx\ dt \leqslant C\int_0^T \max_\Omega \theta_t^2\ dt + C\max_{Q_T} w^2 \int_{Q_T} \theta_x^2 dx\ dt$$

$$\leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_9| \leqslant C\int_{Q_T} |w_x\theta_t\theta_x^2|\ dx\ dt \leqslant C|E_6|.$$

$$|E_{10}| \leqslant C\int_{Q_T} |\eta_x\theta_t\theta_{xt}|\ dx\ dt \leqslant C|E_5|.$$

$$|E_{11}| \leqslant C\int_{Q_T} |\theta_x w_x\theta_{xt}|\ dx\ dt \leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 dx\ dt + C_\varepsilon\int_{Q_T} w_x^2\theta_x^2 dx\ dt$$

$$\leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 w_x^2 dx\ dt + C_\varepsilon.$$

$$|E_{12}| \leqslant C\int_{Q_T} |\eta_x\theta_t\theta_{xt}|\ dx\ dt \leqslant C|E_{10}|.$$

$$|E_{13}| \leqslant C\int_{Q_T} |\theta_x w_x\theta_{xt}|\ dx\ dt \leqslant C|E_{11}|.$$

$$|E_{14}| \leqslant C\int_{Q_T} |\theta_x w_x\theta_{xt}|\ dx\ dt \leqslant C|E_{11}|.$$

$$|E_{15}| \leqslant C\int_{Q_T} |w_x\theta_t^2|\ dx\ dt \leqslant C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_{16}| \leqslant C\int_{Q_T} |w_x^2\theta_t|\ dx\ dt \leqslant C\int_{Q_T} w_x^4\ dx\ dt + C\int_{Q_T} \theta_t^2\ dx\ dt$$

$$\leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_{17}| \leqslant C\int_{Q_T} |w_x\theta_t^2|\ dx\ dt \leqslant C\int_{Q_T} \theta_t^2\ dx\ dt \leqslant C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_{18}| \leqslant C\int_{Q_T} |\theta_t\theta_{xt}|\ dx \leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 dx\ dt + C_\varepsilon\int_{Q_T} \theta_t^2 dx\ dt$$

$$\leqslant \varepsilon\int_{Q_T} \theta_{xt}^2 w_x^2 dx\ dt + C_\varepsilon\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_{19}| \leqslant C\int_{Q_T} |w_x^3\theta_t|\ dx \leqslant C\int_{Q_T} w_x^6\ dx\ dt + C\int_{Q_T} \theta_t^2\ dx\ dt \leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_{20}| \leqslant C\int_{Q_T} |w_x\theta_t\theta_{xt}|\ dx\ dt \leqslant C\int_{Q_T} |\theta_t\theta_{xt}|\ dx\ dt \leqslant C|E_{18}|.$$

Finally, plugging all these estimate into (62) for ϵ small enough, we get

$$\alpha \max_{[0,T]}\int_\Omega \theta_t^2 dx + \frac{1}{2}\ \beta\int_{Q_T} \theta_{tx}^2 dx\ dt \leqslant C + C\left(\int_{Q_T} \theta_{xt}^2 dx\ dt\right)^{1/2},$$

which implies estimates (59).

4. The third equation (6) rewrites

$$\theta_{xx} = +\frac{4\eta}{r^3}\ \theta_x - \frac{\mu}{\kappa r^4}\ w_x^2 + \frac{\eta}{\kappa r^4}(e_\theta\theta_t + e_\eta w_x) + \frac{1}{\eta}\ \eta_x\theta_x.$$

Taking the square and integrating on Ω, we get

$$\int_\Omega \theta_{xx}^2\ dx \leqslant C\int_\Omega \left(\theta_x^4 + \theta_x^2 + [w_x^4 + \theta_t^2 + \eta_x^2\theta_x^2\right)\ dx.$$

Using Corollary 1, together with Proposition 4 and the first bound (60), we can bound the right-hand side, which provide us with the last estimate (60) $\square$

Lemma 6 *The following uniform bounds hold*

$$\max_{Q_T} w_x^2 \leqslant C(T),\quad \max_{Q_T} \theta_x^2 \leqslant C(T),\quad \max_{Q_T} \eta_x^2 \leqslant C(T) \tag{63}$$

Proof: The first two inequalities follow after (60). To get the last one, we derive formula (3) with respect to x and use the first two inequalities together with Propositions 2 and 5 $\square$

Proof of Theorem 4

1. We have first

$$|\eta(x,t) - \eta(x,t')| \leqslant |t-t'|^{1/2}\left(\int_0^T w_x^2\ dt\right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

After Proposition 4

$$|\eta(x,t) - \eta(x',t)| \leqslant C|x-x'|^{1/2}\left(1 + \int_\Omega \eta_x^2\ dx\right) \leqslant C|x-x'|^{1/2},$$

so we find that $\eta \in C^{1/2,1/4}(Q_T)$.

2. From Proposition 6 we have also

$$|\theta(x,t)-\theta(x,t')| \leqslant |t-t'|^{1/2}\left(\int_0^T \theta_t^2\ dt\right)^{1/2}$$

$$\leqslant C|t-t'|^{1/2}\left(\int_0^T\int_\Omega \theta_{xt}^2\ dx\ dt\right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

After Propositions 4 and 6

$$|\theta(x,t)-\theta(x',t)| \leqslant C|x-x'|^{1/2}\left(T\cdot\max_{[0,T]}\int_\Omega \theta_t^2\ dx + \int_0^T\int_\Omega \theta_{xt}^2\ dx\right) \leqslant C|x-x'|^{1/2},$$

so we find that $\theta \in C^{1/2,1/4}(Q_T)$.

As we have also after Propositions 6

$$|\theta_x(x,t)-\theta_x(x',t)| \leqslant |x-x'|^{1/2}\left(\int_\Omega \theta_{xx}^2\ dt\right)^{1/2} \leqslant |x-x'|^{1/2},$$

we deduce as in [20], using an interpolation argument of [22], that $\theta_x \in C^{1/3,1/6}(Q_T)$.

The same arguments holding verbatim for w and w_x, we have that $w, w_x \in C^{1/3,1/6}(Q_T)$, which ends the proof of Theorem 4 $\square$

3 Existence and uniqueness of solutions

In this section we prove existence of classical solution by means of the classical Leray-Schauder fixed point theorem in the same spirit as in [20, 7], then using a limiting process we will get weak solutions as in [15].

We recall the classical Leray- Schauder fixed point theorem

Theorem 5 *Let $\mathcal{B}$ be a banach space and suppose that $P : [0,1]\times\mathcal{B} \to \mathcal{B}$ has the following properties:*

- *i) For any fixed $\lambda \in [0,1]$ the map $P(\lambda,.) : \mathcal{B} \to \mathcal{B}$ is completely continuous.*
- *ii) For every bounded subset $\mathcal{S} \subset \mathcal{B}$ the family of maps $P(.,\chi) : [0,1] \to \mathcal{B},\ \chi \in \mathcal{S}$ is uniformly equicontinuous.*
- *iii) There is a bounded subset $\mathcal{S}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $P(\lambda,.), \lambda \in [0,1]$ is contained in $\mathcal{S}$.*
- *iv) $P(0,.)$ has precisely one fixed point in $\mathcal{B}$.*

Then, $P(1,.)$ has at least one fixed point in $\mathcal{B}$.

In our case $\mathcal{B}$ will be Banach space of functions $\eta, v, \theta \in \mathcal{B}$ on Q_T with $\eta, v, v_x, \theta, \theta_x \in C^{1/3,1/6}(Q_T)$ with the norm

$$|||(\eta,v,\theta)|||_{\mathcal{B}} := |||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3}.$$

For $\lambda \in [0,1]$ we define $P(\lambda,.)$ as the map which carries $\{\tilde\eta, \tilde w \equiv \tilde r^2\tilde v, \tilde\theta\} \in \mathcal{B}$ into $\{\eta, w \equiv r^2 v, \theta\} \in \mathcal{B}$ by solving the system

$$\left\{\begin{aligned} &\eta_t = w_x,\\ &w_t - \frac{\mu}{\tilde\eta}w_{xx} = -\frac{\mu}{\tilde\eta^2}\tilde\eta_x\tilde w_x - \tilde r^4\tilde p_\eta(\tilde\eta,\tilde\theta)\eta_x - \tilde r^4\tilde p_\theta(\tilde\eta,\tilde\theta)\theta_x + \lambda\tilde r^2\tilde f + 2\frac{\tilde w^2}{\tilde r},\\ &\tilde e_\theta(\tilde u,\tilde\theta)\theta_t - \frac{\tilde\kappa(\tilde\eta,\tilde\theta)\tilde r^4}{\tilde\eta}\ \theta_{xx} = \left(\frac{\tilde\kappa(\tilde\eta,\tilde\theta)\tilde r^4}{\tilde\eta}\right)_\eta \tilde\theta_x\eta_x + \frac{\tilde\kappa_\theta(\tilde\eta,\tilde\theta)}{\tilde\eta}\tilde\theta_x^2 + \frac{\mu}{\tilde\eta}\tilde w_x^2 - \tilde\theta\tilde p_\theta(\tilde\eta,\tilde\theta)\tilde w_x, \end{aligned}\right. \tag{64}$$

where $\tilde{r} = r(\tilde{\eta}) = [R_0^3 + 3\int_0^x \tilde{\eta}(y,t)\ dy]^{1/3}$, together with the boundary conditions

$$\begin{cases} w|_{x=0,M} = 0, \\ \theta_x|_{x=0} = 0,\ \theta|_{x=M} = \theta_\Gamma, \end{cases} \tag{65}$$

for $t > 0$, and initial conditions

$$\begin{cases} \eta(x,0) = (1-\lambda) + \lambda\eta_0(x), \\ v(x,0) = \lambda v_0(x), \\ \theta(x,0) = (1-\lambda)\theta_\Gamma + \lambda\theta_0(x). \end{cases} \tag{66}$$

We can consider the second and the third equations of (64) as parabolic type and apply the classical Schauder-Friedmann estimates

$$\|w\|_{1/3} + \|w_x\|_{1/3} \leqslant c\{\|\eta\|_{1/3} + \|\tilde{w}\|_{1/3} + \|\tilde{\theta}_x\|_{1/3}\}$$

$$\|\theta_x\|_{1/3} + \|\theta\|_{1/3} \leqslant c\{\|\tilde{\theta}_x\|_{1/3} + \|\tilde{w}_x\|_{1/3}\}.$$

Moreover from first (64), we get

$$\|\eta\|_{1/3} \leqslant \|w_x\|_{1/3}.$$

It implies that $P(\lambda,.) : \mathcal{B} \to \mathcal{B}$ is well defined and continuous.

Using a priori estimates from Section 2 it follows that for any $\{\tilde{\eta},\tilde{v},\tilde{\theta}\}$ from any fixed bounded subset the family $P(.,\{\tilde{\eta},\tilde{v},\tilde{\theta}\}) : [0,1] \to \mathcal{B}$ of mappings is uniformly equicontinuous.

Now, in order to verify (iii), we observe that any fixed point of P will initially satisfy original problem, therefore η and θ cannot escape from $[\underline{\eta},\overline{\eta}]$, $[\underline{\theta},\overline{\theta}]$ up to time T. This fact follows clearly from Theorem 4. To check (iv) we see by inspection that the unique fixed point of $P(0,.)$ is given by $\eta(x,t) = 1, v(x,t) = 0, \theta(x,t) = \theta_\Gamma$.

All the previous facts allow us to apply Theorem 5, which imply the existence of classical solutions of (6)-(8) in $\Omega \times (0,t^*)$. This ends the proof of Theorem 2.

Let us now consider the existence of a weak solution. From previous results it follows

- v_k converge to v in $L^p(0,t^*;C^0(\Omega))$ strongly and in $L^p(0,t^*;H^1(\Omega))$ weakly for $1 < p < \infty$,
- $v_k \to v$ a.e. in $\Omega \times [0,t^*]$ and in $L^\infty(0,t^*);L^4(\Omega))$ weakly *,
- $\partial_t v_k \to \partial_t v$ in $L^2(0,t^*,L^2(\Omega))$ weakly,
- θ_k converge to θ in $L^2(0,t^*,C^0(\Omega))$ strongly and in $L^2(0,t^*,H^1(\Omega))$ weakly ,
- $\theta_k \to \theta$ a.e. in $\Omega \times [0,t^*]$ and in $L^\infty(0,t^*;L^2(\Omega))$ weakly,
- $r_k \to r$ in $C^0(\Omega \times (0,t^*))$,
- $p_k r_k^2 \to A_1$ in $L^2(0,t^*;H^1(\Omega))$,

After the definition of $r(x,t)$, one has

$$r(x,t) = r_0(x) + \int_0^t v(x,t')dt' \text{ a. e. } \Omega \times (0,T^*),$$

then

$$r_k(x,t) - r_k(y,t) = (\int_y^x \eta_k(s,t)ds)^{1/3}$$
$$\geqslant \epsilon(x-y)\ \bigvee(x,y,t) \in \Omega \times (0,x) \times (0,T^*).$$

Then from the previous computations we get

$$r(x,t) - r(y,t) \geqslant \epsilon(x-y)\ \bigvee(x,y,t) \in \Omega \times (0,x) \times (0,T^*),$$

It implies that

$$\eta_k \to \eta \text{ a.e. in } \Omega \times [0,t^*] \text{ and } L^s(\Omega \times [0,t^*]) \text{ strongly for all } s \in [1,\infty[.$$

This implies that

- $\frac{2\eta_k}{r_k} p_k \to A_2$ strongly in $L^2(0,t^*,L^2(\Omega))$,
- $\frac{\eta_k v_k}{r_k^2} \to A_3$ strongly in $L^2(0,t^*,L^2(\Omega))$,
- $f_k \to f$ in $C^0(\Omega \times (0,t^*))$,
- $\frac{\kappa_k r_k^2 (\theta_k)_x}{\eta_k} \to A_4$ weakly in $L^2(0,t^*,H^1(\Omega))$,
- $r_k^2 v_k^2 \sigma_k \to A_5$ in $L^2(0,t^*,H^1(\Omega))$ weakly ,
- $r_k^2 v_k (\sigma_k)_x \to A_6$ in $L^\infty(0,t^*,L^2(\Omega))$ weakly *.

Then applying similar technique as in [15] it follows that

- $A_1 = pr^2$ in $L^2(0,t^*); H^1(\Omega)$,
- $A_2 = \frac{2\eta}{r} p$ in $L^2(0,t^*,L^2(\Omega))$,
- $A_3 = \frac{\eta_k v_k}{r_k^2}$ in $L^2(0,t^*,L^2(\Omega))$,
- $A_4 = \frac{\kappa r^2 (\theta)_x}{\eta}$ in $L^2(0,t^*,H^1(\Omega))$,
- $A_5 = r^2 v^2 \sigma$ in $L^2(0,t^*,H^1(\Omega))$,
- $A_6 = r^2 v (\sigma)_x$ in $L^\infty(0,t^*,L^2(\Omega))$.

Finally, we prove uniqueness for the solution.

Let $\eta_i, v_i, \theta_i,\ i = 1,2$ be two solutions of (5), and let us consider the differences: $E = \eta_1 - \eta_2$, $T = \theta_1 - \theta_2$ and $V = v_1 - v_2$. The following simple result holds

Lemma 7 *The following bounds hold*

$$|r_2^m - r_1^m| \leqslant c \int_\Omega |E|\ dx \quad \text{for any } m \in \mathbb{Z},$$

$$|\kappa(\eta_1,\theta_1) - \kappa(\eta_2,\theta_2)| \leqslant C\left(|T| + |E|\right),$$
$$|\rho_1 - \rho_2)| \leqslant C|E|,$$
$$|c_V(\eta_1,\theta_1) - c_V(\eta_2,\theta_2)| \leqslant C\left(|T| + |E|\right),$$
$$|p_\theta(\eta_1,\theta_1) - p_\theta(\eta_2,\theta_2)| \leqslant C\left(|T| + |E|\right),$$
$$|E_x| \leqslant C\left(|T| + |E|\right).$$

Proof: Using the identity

$$r_2^m - r_1^m = \frac{m}{3}\left(r_2^3 - r_1^3\right)\int_0^1 \left[r_1^3 + s(r_2^3 - r_1^3)\right]^{\frac{m-3}{3}}\ ds,$$

we have

$$|r_2^m - r_1^m| \leqslant C_m \left|r_2^3 - r_1^3\right|,$$

where $C_m = \frac{m}{3}\left(2R_1^3 - R_0^3\right)^{\frac{m-3}{3}}$, if $m \geqslant 3$, and $C_m = \frac{|m|}{3} R_0^{m-3}$, if $m < 3$. Then using the definition of $r(x,t)$, we see that

$$|r_2^m - r_1^m| \leqslant 3C_m \int_\Omega |E|\ dx.$$

The other inequalities follow in the same way from Taylor's formula together with Propositions 2 and 5, Lemma 6 and formula (3) $\square$

¿From the first equation (6) written for η_1, w_1 and η_2, w_2 subtracting, multiplying by a test function χ, integrating by part and putting $\chi = E$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega E^2 dx = \int_\Omega E W_x \ dx \leqslant \|E\|_2 \|W_x\|_2.$$

Using Cauchy Schwarz inequality for $\varepsilon > 0$

$$\frac{d}{dt}\int_\Omega E^2 dx \leqslant \varepsilon \|W_x\|_2^2 + C_\varepsilon \big\|E\|_2^2. \tag{67}$$

Denoting now by W the difference $r_2^2 v_2 - r_1^2 v_1$, rewritting the second equation (6) for w_2 and w_1, subtracting, multiplying by a test function ϕ, integrating by part and putting $\phi = W$ we obtain the following

$$\frac{1}{2}\frac{d}{dt}\int_\Omega W^2 dx + \int_\Omega \mu \ \frac{r_2^4}{\eta_2} \ W_x^2 \ dx = -\sum_{i=1}^{3} \mathcal{A}_i,$$

with

$$\begin{aligned}|\mathcal{A}_1| &= \left| \int_\Omega \left\{ r_2^4 \frac{A}{2} \frac{\eta_1^{2-\beta}(\theta_2^2 - \theta_1^2)}{\eta_2^{2-\beta}\eta_1^{2-\beta}} + \frac{\theta_2^2(\eta_1^{2-\beta} - \eta_2^{2-\beta})}{\eta_2^{2-\beta}\eta_1^{2-\beta}} \right\} W_x \ dx \right| \\ &\leqslant c\|W_x\|_2 \big(\|T\|_2 + \|E\|_2\big) \leqslant \varepsilon \|W_x\|_2^2 + C_\varepsilon \big(\|T\|_2^2 + \|E\|_2^2\big),\end{aligned}$$

where we used Lemma 7 and Cauchy Schwarz inequality for $\varepsilon > 0$.

In the same stroke

$$\begin{aligned}|\mathcal{A}_2| &= \left| \int_\Omega r_2^4 (r_2^2 - r_1^2) p_2 W_x \ dx \right| \\ &\leqslant c\|W_x\|_2 \|E\|_2 \leqslant \varepsilon \|W_x\|_2^2 + C_\varepsilon \|E\|_2^2,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{A}_3| &= \left| \int_\Omega \{ r^4 (\frac{\eta_2 - \eta_1}{\eta_2 \eta_1})(w_1)_x (w_2 - w_1)_x + (r_2^4 - r_1^4)\frac{\mu}{\eta_1} w_{1x}(w_2 - w_1)_x \} \ dx \right| \\ &\leqslant c\|E\|_2 \|W_x\|_2 \leqslant \varepsilon \|W_x\|_2^2 + C_\varepsilon \|E\|_2^2.\end{aligned}$$

So we get finally, taking ε small enough

$$\frac{d}{dt}\int_\Omega W^2 dx + \int_\Omega W_x^2 \ dx \leqslant C\big(\|T\|_2^2 + \|E\|_2^2\big). \tag{68}$$

Now, dividing the energy equation by e_θ, we have

$$\theta_t = -\frac{\theta p_\theta}{e_\theta} \ w_x + \frac{q_x}{e_\theta} + \frac{\mu}{\eta e_\theta} \ w_x^2.$$

Subtracting this equation written for η_1, w_1, θ_1 from the same for η_2, w_2, θ_2, multiplying by a test function ψ, integrating by part and putting $\psi = T$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega T^2 dx = -\int_\Omega \left[\frac{\theta_1 p_\theta(\eta_1, \theta_1)}{e_\theta(\eta_1, \theta_1)} \ w_{1x} - \frac{\theta_2 p_\theta(\eta_2, \theta_2)}{e_\theta(\eta_2, \theta_2)} \ w_{2x} \right] T \ dx$$

$$+ \int_\Omega \left[\frac{\kappa(\eta_1, \theta_1) r_1^4}{\eta_1 e_\theta(\eta_1, \theta_1)} - \frac{\kappa(\eta_2, \theta_2) r_2^4}{\eta_2 e_\theta(\eta_2, \theta_2)} \right] T \ dx + \int_\Omega \mu \left[\frac{{w_1}_x^2}{\eta_1 e_\theta(\eta_1, \theta_1)} - \frac{{w_2}_x^2}{\eta_2 e_\theta(\eta_2, \theta_2)} \right] T \ dx := -\sum_{i=1}^{3} \mathcal{B}_i.$$

Bounding the $\mathcal{B}_i$, using as previously Lemma 6 and 7 and Cauchy Schwarz inequality for $\varepsilon > 0$, we get

$$|\mathcal{B}_1| \leqslant \varepsilon \left(\|W_x\|_2^2 + \|T_x\|_2^2 \right) + C_\varepsilon \left(\|E\|_2^2 + \|T\|_2^2 \right),$$

$$|\mathcal{B}_2| \leqslant -\int_\Omega \frac{\kappa(\eta_2, \theta_2) r_2^4}{\eta_2 e_\theta(\eta_2, \theta_2)} \ T_x^2 \ dx + \varepsilon \int_\Omega T_x^2 \ dx + C_\varepsilon \left(\|E\|_2^2 + \|T\|_2^2 \right),$$

and

$$|\mathcal{B}_3| \leqslant \varepsilon \|W_x\|_2^2 + C_\varepsilon \|E\|_2^2.$$

We obtain finally

$$\frac{d}{dt}\int_\Omega T^2 dx + \int_\Omega T_x^2 \ dx \leqslant \varepsilon \int_\Omega W_x^2 \ dx + C\left(\|E\|_2^2 + \|T\|_2^2\right). \tag{69}$$

Then adding inequalities (67), (68) and (69) and choosing ε small enough, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(E^2 + W^2 + T^2\right) dx \leqslant C\left(\|E\|_2^2 + \|W\|_2^2 + \|T\|_2^2\right),$$

which clearly implies uniqueness.

4 Asymptotic behaviour

To prove the result of asymptotic stability of Theorem 3, it is more convenient, following [35], to go back to the Eulerian formulation.

1. After Proposition 1, the static solution ρ_S, θ_S is the unique solution (under condition (31)) of

$$-(p_S)_r = \rho_S \ \frac{GM_0}{r^2}, \quad \theta_S = \theta_\Gamma.$$

So we rewrite the momentum equation in (1) as

$$\rho(v_t + vv_r) = -p_r + \mu\left(\frac{w_r}{r^2}\right)_r + \rho \ \frac{(p_S)_r}{\rho_S},$$

where $w = r^2 v$. Multiplying by w and integrating on $\omega = (R_0, R_1)$, we get

$$\frac{d}{dt}\int_\omega \frac{1}{2}\rho v^2 r^2 dr + \int_\omega \mu \frac{w_r^2}{r^2} \ dr = \int_\omega p w_r \ dr + \int_\omega r^2 \rho v \ \frac{(p_S)_r}{\rho_S} \ dr.$$

Using the second law of thermodynamics $\theta ds = de + pd\eta$ and (1), we compute the Eulerian energy equality

$$\frac{d}{dt}\int_\omega \rho\left(\frac{1}{2}v^2 + e - \theta_\Gamma s\right) r^2 dr + \theta_\Gamma \int_\omega \kappa \frac{\theta_r^2}{\theta^2} \ dr + \theta_\Gamma \int_\omega \mu \frac{w_r^2}{r^2\theta} \ dr = \int_\omega \rho v \frac{(p_S)_r}{\rho_S} \ r^2 dr.$$

Applying the identity $\frac{d}{dt}\int_\omega \rho F(r) \ r^2 dr = \int_\omega \rho r^2 v F_r(r) \ dr$, with $F \equiv p_S(\rho_S, \theta_S)$, and rearranging the integrand, we obtain finally

$$\begin{aligned}\frac{d}{dt}\int_\omega \Bigg[\rho\left\{\frac{1}{2}v^2 + c_V^0 \theta_\Gamma\left(\frac{\theta}{\theta_\Gamma} - \log\frac{\theta}{\theta_\Gamma} - 1\right) + A\frac{(\theta-\theta_\Gamma)^2}{2(\beta-1)\eta^{1-\beta}}\right\} \\ -\frac{A\theta_\Gamma^2}{2(\beta-1)}\left\{\rho^{2-\beta} + (2-\beta)\rho\rho_S^{1-\beta}\right\}\Bigg] r^2 dr \\ +\theta_\Gamma \int_\omega \kappa \frac{\theta_r^2}{\theta^2} \ dr + \theta_\Gamma \int_\omega \mu \frac{w_r^2}{r^2\theta} \ dr = 0.\end{aligned} \tag{70}$$

2. The first equation (1) rewrites

$$\rho_t + \frac{1}{r^2}(\rho w)_r = 0. \tag{71}$$

Multiplying by $r^2(\rho^{1-\beta} - \rho_S^{1-\beta})$ and integrating by parts on ω, we have

$$\frac{d}{dt}\int_\omega r^2\left(\frac{\rho^{2-\beta}}{2-\beta} - \rho\rho_S^{1-\beta}\right) \ dr = \int_\omega \rho w(\rho^{1-\beta} - \rho_S^{1-\beta})_r \ dr.$$

Rearranging the integral in the right-hand side, we get

$$\frac{1}{1-\beta}\ \frac{d}{dt}\int_\omega r^2\left(\rho^{2-\beta}-(2-\beta)\rho\rho_S^{1-\beta}\right)\ dr+\int_\omega \rho^{2-\beta}w_r\ dr$$

$$+(2-\beta)\int_\omega \rho\rho_S^{-\beta}(\rho_S)_r w\ dr=0. \tag{72}$$

3. Multiplying now (71) by $\rho_S^{1-\beta}$ and integrating by parts on ω, we have after some calculations

$$\frac{1}{\beta-1}\ \frac{d}{dt}\int_\omega r^2\rho\rho_S^{1-\beta}\ dr=\int_\omega \rho\rho_S^{-\beta}(\rho_S)_r w\ dr=0. \tag{73}$$

4. Putting $a:=A\theta_\Gamma^2$, $b:=A\theta_\Gamma^2(2-\beta)$ and adding $(70)+a(72)+b(73)$, we get

$$\frac{d}{dt}\int_\omega\left[\rho\left\{\frac{1}{2}v^2+c_V^0\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma}-\log\frac{\theta}{\theta_\Gamma}-1\right)+A\frac{(\theta-\theta_\Gamma)^2}{2(\beta-1)\eta^{1-\beta}}\right\}\right.$$

$$\left.+\frac{A\theta_\Gamma^2}{2(1-\beta)}\left\{\rho^{2-\beta}-(2-\beta)\rho\rho_S^{1-\beta}+(1-\beta)\rho_S^{2-\beta}\right\}\right]r^2dr+\theta_\Gamma\int_\omega\kappa\frac{\theta_r^2}{\theta^2}\ dr+\theta_\Gamma\int_\omega\mu\frac{w_r^2}{r^2\theta}\ dr$$

$$+A\theta_\Gamma^2\int_\omega\rho^{2-\beta}w_r\ dr+A\theta_\Gamma^2(2-\beta)\int_\omega\rho\rho_S^{-\beta}(\rho_S)_r w\ dr+A\theta_\Gamma^2(2-\beta)^2\int_\omega\rho\rho_S^{-\beta}(\rho_S)_r w\ dr=0. \tag{74}$$

This implies the inequality, for a suitable positive constant C

$$\frac{d}{dt}\int_\omega\left[\rho\left\{\frac{1}{2}v^2+c_V^0\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma}-\log\frac{\theta}{\theta_\Gamma}-1\right)+A\frac{(\theta-\theta_\Gamma)^2}{2(\beta-1)\eta^{1-\beta}}\right\}\right.$$

$$\left.+\frac{A\theta_\Gamma^2}{2(1-\beta)}\left\{\rho^{2-\beta}-(2-\beta)\rho\rho_S^{1-\beta}+(1-\beta)\rho_S^{2-\beta}\right\}\right]r^2dr+C\int_\omega\left(w^2+w_r^2+\theta_r^2\right)\ dr\leqslant 0. \tag{75}$$

5. Multiplying the second equation (1) by $-\rho^{-1}\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr$, we have

$$-v_t\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr-vv_r\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr$$

$$=p_r\rho^{-1}\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr-\mu\left(\frac{w_r}{r^2}\right)_r\rho^{-1}\int_\omega(\rho-\rho_S)\ r^2dr-\ \frac{(p_S)_r}{\rho_S}\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr.$$

Integrating by parts in ω, we obtain after elementary manipulations

$$-\frac{d}{dt}\int_\omega v\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\ dr'-\int_\omega\rho v^2r^2dr+\frac{1}{2}\int_\omega v^2(\rho-\rho_S)\ r^2dr$$

$$=-\int_\omega\mu\ \frac{w_{r'}}{r'^2}\left\{\frac{\rho_{r'}}{\rho^2}\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr+\frac{r'^2}{\rho}(\rho-\rho_S)\right\}\ dr'$$

$$+\int_\omega\left(\frac{p_{r'}}{\rho}-\frac{(p_S)_{r'}}{\rho_S}\right)\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\ dr'.$$

Rearranging the right-hand side leads to

$$-\frac{d}{dt}\int_\omega v\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\ dr'$$

$$+\int_\omega\left[-\rho v^2+\frac{1}{2}v^2(\rho-\rho_S)+\frac{A(\beta-2)}{2(\beta-1)}\ \theta_\Gamma^2(\rho^{1-\beta}-\rho_S^{1-\beta})(\rho-\rho_S)\right]\ r^2dr$$

$$=\int_\omega\left(\frac{1}{2}\ A(\beta-2)\rho^{-\beta}\rho_r(\theta^2-\theta_\Gamma^2)+\frac{A\theta}{\eta^{1-\beta}}\theta_r\right)\left(\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\right)\ dr'$$

$$+\int_\omega \mu\ \frac{w_r}{r^2}(\rho-\rho_S)\ r^2dr - \int_\omega \mu\ \frac{w_r\rho_r}{r^2\rho^2}\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\ dr'.$$

Estimating the right-hand side by using Cauchy-Schwarz inequality together with the estimate $\int_\omega(\theta^2-\theta_\Gamma)^2dr \leqslant C\int_\omega \theta_r^2\ dr$, we get the inequality

$$-\frac{d}{dt}\int_\omega v\int_{R_0}^{r'}(\rho-\rho_S)\ r^2dr\ dr' + \int_\omega\left[-\rho v^2+v^2(\rho-\rho_S)+(\rho-\rho_S)^2\right]\ r^2dr$$

$$\leqslant C\int_\omega\left(w^2+w_r^2+\theta_r^2\right)\ dr, \tag{76}$$

where C is a positive constant.

6. Multiplying now (76) by a positive number ε small enough and adding the result to (75), we obtain

$$\frac{d}{dt}\int_\omega\left[\rho\left\{\frac{1}{2}v^2+c_V^0\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma}-\log\frac{\theta}{\theta_\Gamma}-1\right)+A\frac{(\theta-\theta_\Gamma)^2}{2(\beta-1)\eta^{1-\beta}}\right\}\right.$$

$$\left.+\frac{A\theta_\Gamma^2}{2(\beta-1)}\left\{\rho^{2-\beta}-(2-\beta)\rho\rho_S^{1-\beta}+(1-\beta)\rho_S^{2-\beta}\right\}-\varepsilon v\int_{R_0}^{r}(\rho-\rho_S)\ {r'}^2dr'\right]r^2dr$$

$$+C\int_\omega\left((\rho-\rho_S)^2+w^2+w_r^2+\theta_r^2\right)\ dr \leqslant 0. \tag{77}$$

Integrating this inequality with respect to time and observing that, for a suitable constant $C>0$

$$\int_\omega\left[\rho\left\{\frac{1}{2}v^2+c_V^0\theta_\Gamma\left(\frac{\theta}{\theta_\Gamma}-\log\frac{\theta}{\theta_\Gamma}-1\right)+A\frac{(\theta-\theta_\Gamma)^2}{2(\beta-1)\eta^{1-\beta}}\right\}\right.$$

$$\left.+\frac{A\theta_\Gamma^2}{2(\beta-1)}\left\{\rho^{2-\beta}-(2-\beta)\rho\rho_S^{1-\beta}+(1-\beta)\rho_S^{2-\beta}\right\}-\varepsilon v\int_{R_0}^{r}(\rho-\rho_S)\ {r'}^2dr'\right]r^2dr \geqslant C\Phi(t),$$

where

$$\Phi(t)\equiv\int_\omega\left\{(\rho-\rho_S)^2+v^2+(\theta-\theta_S)^2\right\}\ dr,$$

we ends with the inequality $\Phi(t)+\int_0^t\Phi(\tau)\ d\tau\leqslant 0$ which ends the proof of Theorem 3 □

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