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SPECIAL n-FORMS ON A $2n$-DIMENSIONAL VECTOR SPACE

JIŘÍ VANŽURA

Abstract. The configuration of regular 3-forms in dimension 6 is generalized to n-forms in dimension $2n$. The algebras of complex, paracomplex, and dual numbers are systematically used. The automorphism groups of all forms are determined.

In the last decade there has arisen interest in exterior forms of higher degree, first of all in forms of degree 3 (see [2] and [3]). It is known (see [4] and [1]) that on a 6-dimensional real vector space there are exactly three types (= orbits) of regular 3-forms, and that these types are closely related with 2-dimensional unital algebras. In this note we show that these forms can be generalized to higher dimensions. Namely, using 2-dimensional unital algebras we construct n-forms on a $2n$-dimensional real vector space, and investigate their properties.

We shall consider all three 2-dimensional unital, associative and commutative real algebras, namely

$$\mathbb{C} = [1, i], \quad i^2 = -1 \quad \text{algebra of complex numbers,}$$
$$\mathbb{D} = [1, d], \quad d^2 = 1 \quad \text{algebra of paracomplex numbers,}$$
$$\mathbb{E} = [1, e], \quad e^2 = 0 \quad \text{algebra of dual numbers.}$$

Let V be a $2n$-dimensional real vector space, $n \geq 3$. On this vector space we shall consider consider three endomorphisms J, D, and E, respectively. We shall assume that they satisfy

$$J^2 = -I \quad \text{(complex structure)}$$
$$D^2 = I, \ \dim\ker(D - I) = \dim\ker(D + I) = n \quad \text{(product structure)}$$
$$E^2 = 0, \ \dim\operatorname{im} E = \dim\ker E = n \quad \text{(tangent structure).}$$

If V is endowed with a complex structure J (resp. product structure D, resp. tangent structure E), we can introduce on V a structure of a $\mathbb{C}$-module (i. e. complex vector space) (resp. $\mathbb{D}$-module, resp. $\mathbb{E}$-module) in the following way

$$(a + bi)v = av + bJv \text{ (resp. } (a + bd)v = av + bDv, \text{resp. } (a + be)v = av + bEv).$$

On the other hand if V carries a structure of a $\mathbb{C}$-module (resp. $\mathbb{D}$-module, resp. $\mathbb{E}$-module), we can introduce on V a complex structure J (resp. product structure

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D, resp. tangent structure E) by the formula

$$Jv = iv, \quad \text{(resp. } Dv = dv, \quad \text{resp. } Ev = ev).$$

Writing V we shall always consider V as a real vector space. If we want to consider V as a $\mathbb{C}$-module (resp. $\mathbb{D}$-module, resp. as a $\mathbb{E}$-module), we shall write $(V,\mathbb{C})$ (resp. $(V,\mathbb{D})$, resp. $(V,\mathbb{E})$). If there is no danger of confusion, we shall very often abbreviate $W = (V,\mathbb{C})$ (resp. $W = (V,\mathbb{D})$, resp. $W = (V,\mathbb{E})$). Using the above assumptions on J (resp. D, resp. E), we can easily see that W is an n-dimensional free $\mathbb{C}$-module (resp. $\mathbb{D}$-module, resp. $\mathbb{E}$-module).

We shall consider the group

$$GL^*(V;\mathbb{C}) = \{A \in GL(V;\mathbb{R}); AJ = JA \text{ or } AJ = -JA\}.$$

We define also

$$GL^+(V;\mathbb{C}) = \{A \in GL(V;\mathbb{R}); AJ = JA\} \quad \text{and}$$
$$GL^-(V;\mathbb{C}) = \{A \in GL(V;\mathbb{R}); AJ = -JA\}.$$

We have $GL^*(V;\mathbb{C}) = GL^+(V;\mathbb{C}) \cup GL^-(V;\mathbb{C})$, and

$$GL^+(V;\mathbb{C}) \cdot GL^+(V;\mathbb{C}) = GL^+(V;\mathbb{C}),$$
$$GL^+(V;\mathbb{C}) \cdot GL^-(V;\mathbb{C}) = GL^-(V;\mathbb{C}) \cdot GL^+(V;\mathbb{C}) = GL^-(V;\mathbb{C})$$
$$GL^-(V;\mathbb{C}) \cdot GL^-(V;\mathbb{C}) = GL^+(V;\mathbb{C})$$

Along the same lines we introduce $GL^*(V;\mathbb{D})$ and $GL^*(V;\mathbb{E})$.

Let us consider a $\mathbb{C}$-module $(V,\mathbb{C})$ (resp. $\mathbb{D}$-module $(V,\mathbb{D})$, resp. $\mathbb{E}$-module $(V,\mathbb{E})$). A real form of this module is an n-dimensional real subspace $V_0 \subset V$ such that

$$V_0 + iV_0 = V \quad \text{(resp. } V_0 + dV_0 = V, \quad \text{resp. } V_0 + eV_0 = V).$$

Because the module W is free, it is easy to see that in all these three cases a real form exists.

For a k-form ω on V, $k \geq 2$, we can define a homomorphism

$$V \to \Lambda^{k-1}V^*, \quad v \mapsto \iota_v\omega = \omega(v,\cdot,\dots,\cdot).$$

The form ω is called *regular* (or *multisymplectic*) if the above homomorphism is a monomorphism.

Next for a k-form ω on V, $k \geq 3$, we can consider all endomorphisms A of V satisfying

$$\omega(Av_1, v_2, \dots, v_k) = \omega(v_1, Av_2, \dots, v_k) = \dots = \omega(v_1, v_2, \dots, Av_k).$$

It is easy to see that such endomorphisms constitute a unital associative real algebra. This algebra is commutative (which can be very easily proved). We shall denote it by the symbol $\mathcal{A}_\omega$.

If A is an endomorphism of V and ω is a k-form, we define a k-form $A^*\omega$ in the following way.

$$(A^*\omega)(v_1,\dots,v_k) = \omega(Av_1,\dots,Av_k)$$

Next we define the derivation $\mathcal{D}_A\omega \in \Lambda^kV^*$ by the formula

$$(\mathcal{D}_A\omega)(v_1,\dots,v_k) = \sum_{i=1}^{k} \omega(v_1,\dots,v_{i-1},Av_i,v_{i+1},\dots,v_k).$$

1. Forms of the complex type

W is here an n-dimensional complex vector space, and $e_1, \ldots, e_n$ is its basis.

Lemma 1.1. *Let $\theta \neq 0$ be a complex n-form. Then $\mathcal{A}_\theta = [I, J] = \{cI; c \in \mathbb{C}\}$.*

Proof. Obviously $J \in \mathcal{A}_\theta$. If $A \in \mathcal{A}_\theta$, then $AJ = JA$, which means that A is a complex endomorphism. Let $A \in A_\theta$. Then we have

$$\theta(e_1, Ae_1, e_3, \ldots, e_n) = \theta(e_1, e_1, Ae_3, \ldots, e_n) = 0.$$

Let us write $Ae_i = \sum_{j=1}^n a_{ij} e_j$. From the above equality it follows that $a_{12} = 0$. Along the same lines we can easily prove $a_{ij} = 0$ for $i \neq j$. Consequently, we have $Ae_i = a_{ii} e_i$ for $i = 1, \ldots, n$. From the equalities

$$\theta(Ae_1, e_2, \ldots, e_n) = \theta(e_1, Ae_2, \ldots, e_n) = \cdots = \theta(e_1, e_2, \ldots, Ae_n)$$

we get $a_{11} = a_{22} = \cdots = a_{nn}$. Therefore we have $A = cI$, where $c \in \mathbb{C}$. This finishes the proof. □

Let $\theta \neq 0$ be a complex n-form on W. Then we define real n-forms ω_- and $\tilde{\omega}_-$ on V by the formula

$$\theta = \tilde{\omega}_- + i\omega_-.$$

Lemma 1.2. *The automorphism J belongs to the both algebras $\mathcal{A}_{\omega_-}$ and $\mathcal{A}_{\tilde{\omega}_-}$.*

Proof. We have

$$\begin{aligned} &\tilde{\omega}_-(Jv_1, v_2, \ldots, v_n) + i\omega_-(Jv_1, v_2, \ldots, v_n) = \\ &= \theta(Jv_1, v_2, \ldots, v_n) = i\theta(v_1, v_2, \ldots, v_n) = \theta(v_1, Jv_2, \ldots, v_n) = \\ &= \tilde{\omega}_-(v_1, Jv_2, \ldots, v_n) + i\omega_-(v_1, Jv_2, \ldots, v_n). \end{aligned}$$

□

Further we have

$$\begin{aligned} &-\omega_-(v_1, v_2, \ldots, v_n) + i\tilde{\omega}_-(v_1, v_2, \ldots, v_n) = \\ &= i[\tilde{\omega}_-(v_1, v_2, \ldots, v_n) + i\omega_-(v_1, v_2, \ldots, v_n)] = i\theta(v_1, v_2, \ldots, v_n) = \\ &= \theta(Jv_1, v_2, \ldots, v_n) = \tilde{\omega}_-(Jv_1, v_2, \ldots, v_n) + i\omega_-(Jv_1, v_2, \ldots, v_n) \end{aligned}$$

Hence we get

$$\begin{aligned} &\tilde{\omega}_-(v_1, v_2, \ldots, v_n) = \omega_-(Jv_1, v_2, \ldots, v_n) \text{ and} \\ &\omega_-(v_1, v_2, \ldots, v_n) = -\tilde{\omega}_-(Jv_1, v_2, \ldots, v_n). \end{aligned}$$

This result can be reformulated in the following way.

Lemma 1.3. $\tilde{\omega}_- = \frac{1}{n}\mathcal{D}_J \omega_-, \quad \omega_- = -\frac{1}{n}\mathcal{D}_J \tilde{\omega}_-.$

Lemma 1.4. *The forms ω_- and $\tilde{\omega}_-$ are regular.*

Proof. Let us assume that $\iota_v \omega_- = 0$. We have then

$$\begin{aligned} &(\iota_v \tilde{\omega}_-)(v_1, v_2, \ldots, v_{n-1}) = \tilde{\omega}_-(v, v_1, v_2, \ldots, v_{n-1}) = \\ &= \omega_-(Jv, v_1, v_2, \ldots, v_{n-1}) = \omega_-(v, Jv_1, v_2, \ldots, v_{n-1}) = \\ &= (\iota_v \omega_-)(Jv_1, v_2, \ldots, v_{n-1}) = 0, \end{aligned}$$

which proves that $\iota_v \tilde{\omega}_- = 0$. Consequently $\iota_v \theta = 0$, and this implies that $v = 0$. We have thus proved that the form ω_- is regular. Expressing $\tilde{\omega}_-$ using ω_- and J, we find easily that $\tilde{\omega}_-$ is also regular. □

Proposition 1.1. $\mathcal{A}_{\omega_-} = \mathcal{A}_{\tilde{\omega}_-} = [I, J]$.

Proof. Let $A \in \mathcal{A}_{\omega_-}$. We have

$$\begin{gathered}\tilde{\omega}_-(Av_1, v_2, \dots, v_n) = (1/n)(\mathcal{D}_J\omega_-)(Av_1, v_2, \dots, v_n) = \\ = \omega_-(JAv_1, v_2, \dots, v_n) = \omega_-(AJv_1, v_2, \dots, v_n) = \\ = \omega_-(Jv_1, Av_2, \dots, v_n) = (1/n)(\mathcal{D}_J\omega_-)(v_1, Av_2, \dots, v_n) = \\ = \tilde{\omega}_-(v_1, Av_2, \dots, v_n),\end{gathered}$$

which shows that $\mathcal{A}_{\omega_-} \subset \mathcal{A}_{\tilde{\omega}_-}$. The converse inclusion can be proved in a similar way. For $A \in \mathcal{A}_{\omega_-}$ we have

$$\begin{gathered}\theta(Av_1, v_2, \dots, v_n) = \tilde{\omega}_-(Av_1, v_2, \dots, v_n) + i\omega_-(Av_1, v_2, \dots, v_n) = \\ = \tilde{\omega}_-(v_1, Av_2, \dots, v_n) + i\omega_-(v_1, Av_2, \dots, v_n) = \theta(v_1, Av_2, \dots, v_n).\end{gathered}$$

According to Lemma 1.1 we have $\mathcal{A}_\omega \subset [I, J]$. The converse inclusion is obvious. $\square$

Let us consider now an automorphism $A \in \mathrm{Aut}(\omega_-)$. We have

$$\begin{gathered}\omega_-(AJA^{-1}v_1, v_2, v_3, \dots, v_n) = \omega_-(JA^{-1}v_1, A^{-1}v_2, A^{-1}v_3, \dots, A^{-1}v_n) = \\ = \omega_-(A^{-1}v_1, JA^{-1}v_2, A^{-1}v_3, \dots, A^{-1}v_n) = \omega_-(v_1, AJA^{-1}v_2, v_3, \dots, v_n).\end{gathered}$$

This shows that $AJA^{-1} \in \mathcal{A}_{\omega_-}$. Consequently, there are $a, b \in \mathbb{R}$ such that $AJA^{-1} = aI + bJ$. Squaring this identity we get

$$-I = (a^2 - b^2)I + 2abJ.$$

Obviously there must be $b \neq 0$. Consequently we have $a = 0$, and then $b = \pm 1$. This means that we have $AJA^{-1} = \pm J$ or equivalently $AJ = \pm JA$. We have thus proved the following lemma.

Lemma 1.5. *Every automorphism of ω_- is a complex linear or complex antilinear mapping.*

We define

$$\mathrm{Aut}^+(\omega_-) = \mathrm{Aut}(\omega_-) \cap GL^+(V; J), \quad \mathrm{Aut}^-(\omega_-) = \mathrm{Aut}(\omega_-) \cap GL^-(V; J).$$

We have $\mathrm{Aut}(\omega_-) = \mathrm{Aut}^+(\omega_-) \cup \mathrm{Aut}^-(\omega_-)$.

Lemma 1.6. $\mathrm{Aut}^+(\omega_-) = \mathrm{Aut}^+(\tilde{\omega}_-)$.

Proof. Let $A \in \mathrm{Aut}^+(\omega_-)$. Then we have

$$\begin{gathered}\tilde{\omega}_-(Av_1, Av_2, \dots, Av_n) = \omega_-(JAv_1, Av_2, \dots, Av_n) = \omega_-(AJv_1, Av_2, \dots, Av_n) = \\ = \omega_-(Jv_1, v_2, \dots, v_n) = \tilde{\omega}_-(v_1, v_2, \dots, v_n).\end{gathered}$$

This shows that $\mathrm{Aut}^+(\omega_-) \subset \mathrm{Aut}^+(\tilde{\omega}_-)$. The converse inclusion can be proved in the same way. $\square$

If $A \in \mathrm{Aut}^+(\omega_-)$, then $A \in \mathrm{Aut}^+(\tilde{\omega}_-)$, and we find easily

$$A^*\theta = \det_{\mathbb{C}} A \cdot \theta, \quad A^*\theta = A^*\tilde{\omega}_- + iA^*\omega_- = \tilde{\omega}_- + i\omega_- = \theta.$$

We have thus shown that if $A \in \mathrm{Aut}^+(\omega_-)$, then $\det_{\mathbb{C}} A = 1$. Conversely, it can be easily seen that if $A \in GL^+(V; \mathbb{C})$ and $\det_{\mathbb{C}} A = 1$, then $A \in \mathrm{Aut}^+(\omega_-)$.

Proposition 1.2. *An automorphism $A \in GL^+(V; \mathbb{C})$ belongs to $\mathrm{Aut}^+(\omega_-)$ if and only if $\det_{\mathbb{C}} A = 1$. Consequently $\mathrm{Aut}^+(\omega_-) = SL(V; \mathbb{C}) \cong SL(n; \mathbb{C})$.*

Our next aim is to innvestigate the set $\mathrm{Aut}^-(\omega_-)$. First of all we must see that this set is not empty. We shall start with the following lemma.

Lemma 1.7. *Let $A \in \mathrm{Aut}(\omega_-)$. Then $A \in \mathrm{Aut}^-(\omega_-)$ if and only if one of the following two equivalent conditions is satisfied.*

(i) $A^*\tilde{\omega}_- = -\tilde{\omega}_-$,
(ii) $A^*\theta = -\bar{\theta}$.

Proof. Let us assume first that $A \in \mathrm{Aut}^-(\omega_-)$. Then we have

$$(A^*\tilde{\omega}_-)(v_1, v_2, \dots, v_n) = \tilde{\omega}_-(Av_1, Av_2, \dots, Av_n) = \omega_-(JAv_1, Av_2, \dots, Av_n) =$$
$$= -\omega_-(AJv_1, Av_2, \dots, Av_n) = -\omega_-(Jv_1, v_2, \dots, v_n) = -\tilde{\omega}_-(v_1, v_2, \dots, v_n).$$

On the other hand, let us suppose that $A^*\tilde{\omega}_- = -\tilde{\omega}_-$. Then we have

$$\tilde{\omega}_-(Av_1, Av_2, \dots, Av_n) = -\tilde{\omega}_-(v_1, v_2, \dots, v_n)$$
$$\omega_-(JAv_1, Av_2, \dots, Av_n) = -\omega_-(Jv_1, v_2, \dots, v_n)$$
$$\omega_-(A^{-1}JAv_1, v_2, \dots, v_n) = \omega_-(-Jv_1, v_2, \dots, v_n).$$

Because the form ω_- is regular, the last equality implies that $A^{-1}JA = -J$, which shows that $A \in \mathrm{Aut}^-(\omega_-)$.

If $A^*\tilde{\omega}_- = -\tilde{\omega}_-$, then we have

$$A^*\theta = A^*\tilde{\omega}_- + iA^*\omega = -\tilde{\omega}_- + i\omega = -\bar{\theta}.$$

The converse direction is now obvious. This finishes the proof. □

Now we shall consider the complex conjugate $\bar{W}$ of W. We recall that $\bar{W} = W$, and multiplivation by a complex number c in $\bar{W}$ is defined by the formula $c * v = \bar{c}v$. We have

$$\bar{\theta}(c * v_1, v_2, \dots, v_n) = \bar{\theta}(\bar{c}v_1, v_2, \dots, v_n) = c\bar{\theta}(v_1, v_2, \dots, v_n),$$

which shows that $\bar{\theta}$ is a complex n-form on the complex vector space $\bar{W}$.

Because W and $\bar{W}$ are n-dimensional vector spaces, there exists an isomorphism $B : W \to \bar{W}$ such that $B^*\bar{\theta} = \theta$.

Lemma 1.8. *Let $A \in GL^-(V; J)$. Then $A \in \mathrm{Aut}^-(\omega_-)$ if and only if* $\det_{\mathbb{C}}(AB) = -1$.

Proof. Let $A \in \mathrm{Aut}^-(\omega_-)$. This means that $A^*\omega_- = \omega_-$, and we can easily prove that $A^*\tilde{\omega}_- = -\tilde{\omega}_-$. Then

$$A^*\theta = A^*\tilde{\omega}_- + iA^*\omega_- = -\tilde{\omega}_- + i\omega_- = -\bar{\theta}.$$

Next we get

$$(AB)^*\theta = B^*A^*\theta = -B^*\bar{\theta} = -\theta,$$

which proves that $\det_{\mathbb{C}}(AB) = -1$.

Conversely, let us assume that $\det_{\mathbb{C}}(AB) = -1$. We have

$$(AB)^*\theta = -\theta$$
$$B^*(A^*\theta) = -\theta$$

Because B was chosen in such a way that $B^*\bar{\theta} = \theta$, it is obvious that $A^*\theta = -\bar{\theta}$. Hence we get

$$A^*\tilde{\omega}_- + iA^*\omega_- = -\tilde{\omega}_- + i\omega_-.$$

We can see that $A \in \operatorname{Aut}(\omega_-)$, and according to the previous lemma there is $A \in \operatorname{Aut}^-(\omega_-)$. □

From this lemma we can easily see that $\operatorname{Aut}^-(\omega_-) \neq \emptyset$. Now it is obvious that

$$\operatorname{Aut}^+(\omega_-) \cdot \operatorname{Aut}^+(\omega_-) = \operatorname{Aut}^+(\omega_-),$$
$$\operatorname{Aut}^+(\omega_-) \cdot \operatorname{Aut}^-(\omega_-) = \operatorname{Aut}^-(\omega_-) \cdot \operatorname{Aut}^+(\omega_-) = \operatorname{Aut}^-(\omega_-),$$
$$\operatorname{Aut}^-(\omega_-) \cdot \operatorname{Aut}^-(\omega_-) = \operatorname{Aut}^+(\omega_-).$$

Summarizing, we have the following proposition.

Proposition 1.3. *The automorphism group* $\operatorname{Aut}(\omega_-)$ *consists of two connected components* $\operatorname{Aut}^+(\omega_-)$ *and* $\operatorname{Aut}^-(\omega_-)$, *where* $\operatorname{Aut}^+(\omega_-)$ *is the connected component of the unit. The group* $\operatorname{Aut}^+(\omega_-) = SL(V;\mathbb{C}) \cong SL(n;\mathbb{C})$. *Moreover,* $\dim_{\mathbb{R}} \operatorname{Aut}(\omega_-) = 2(n^2-1)$.

2. Forms of the product type

It is obvious that the elements $\rho = (1/2)(1+d)$ and $\sigma = (1/2)(1-d)$ form a basis of $\mathbb{D}$ and that we have

$$\rho^2 = \rho, \quad \rho\sigma = 0, \quad \sigma^2 = \sigma.$$

Let us choose a real form V_0 of V. It is easy to see that every element $v \in V$ can be uniquely expressed in the form

$$v = \rho x + \sigma y, \quad \text{where } x, y \in V_0.$$

Lemma 2.1. *Vectors* $w_1 = \rho u_1 + \sigma v_1, \dots, w_k = \rho u_k + \sigma v_k$ *are linearly independent in the* $\mathbb{D}$*-module* W *if and only if the vectors* $u_1, \dots, u_k$ *are linearly independent in the vector space* V_0 *and the vectors* $v_1, \dots, v_k$ *are linearly independent in the same vector space* V_0. *If the vectors* $u_1, \dots, u_k$ *are linearly independent in* V_0 *and the vectors* $v_1, \dots, v_k$ *are linearly independent in* V_0, *then the vectors* $w_1, \dots, w_k$ *can be completed to a basis of* W.

Proof. Let us write

$$c_i = a_i\rho + b_i\sigma, \quad w_i = \rho u_i + \sigma v_i, \quad i = 1, \dots, k.$$

We have

$$c_1 w_1 + \dots + c_k w_k = \rho(a_1 u_1 + \dots + a_k u_k) + \sigma(b_1 v_1 + \dots + b_k v_k).$$

We can easily see that the vectors $w_1, \dots, w_k$ are linearly independent in the module W if and only if the vectors $u_1, \dots, u_n$ are linearly independent in V_0 and the vectors $v_1, \dots, v_n$ are linearly independent in V_0. Moreover, it is obvious that if the vectors $w_1, \dots, w_k$ are linearly independent, they can be completed to a basis. □

Lemma 2.2. *Let* θ *be a* $\mathbb{D}$*-multilinear* n*-form on* W *with values in the algebra* $\mathbb{D}$. *Then* $\mathcal{A}_\theta = [I, D] = \{cI; c \in \mathbb{D}\}$.

Proof. The proof proceeds along the same lines as the proof of Lemma 1.1. □

We take now a non-zero n-form θ on the dual module W^* to the $\mathbb{D}$-module W. We introduce real valued $\mathbb{R}$-multilinear n-forms ω_+ and $\tilde{\omega}_+$ on V by the formula

$$\theta = \tilde{\omega}_+ + e\omega_+.$$

Lemma 2.3. *The automorphism* D *belongs to the both algebras* $\mathcal{A}_{\omega_+}$ *and* $\mathcal{A}_{\tilde{\omega}_+}$.

Proof. This proof is the same as the proof of Lemma 1.2. $\square$

Lemma 2.4. *The n-forms ω_+ and $\tilde{\omega}_+$ satisfy the relations $\tilde{\omega}_+ = \frac{1}{n}\mathcal{D}_D\omega_+$ and $\omega_+ = \frac{1}{n}\mathcal{D}_D\tilde{\omega}_+$.*

Proof. Here it suffices to proceed in the same way as in the proof of Lemma 1.3. $\square$

Lemma 2.5. *The forms ω_+ and $\tilde{\omega}_+$ are regular.*

Proof. The proof is same as the proof of Lemma 1.4. $\square$

Proposition 2.1. $\mathcal{A}_{\omega_+} = \mathcal{A}_{\tilde{\omega}_+} = [I, D]$.

Proof. The proof follows the lines of the proof of Proposition 1.1. $\square$

Let us assume now that $A \in \mathrm{Aut}(\omega_+)$. Then we have

$$\begin{aligned}&\omega_+(ADA^{-1}v_1, v_2, \dots, v_n) = \omega_+(DA^{-1}v_1, A^{-1}v_2, \dots, A^{-1}v_n) = \\ &= \omega_+(A^{-1}v_1, DA^{-1}v_2, \dots, A^{-1}v_n) = \omega_+(v_1, ADA^{-1}v_2, \dots, v_n),\end{aligned}$$

which shows that $ADA^{-1} \in \mathcal{A}_{\omega_+}$. This implies that there are $a, b \in \mathbb{R}$ such that $ADA^{-1} = aI + bD$. Taking the second power of this equality, we get

$$I = (a^2 + b^2)I + 2abD.$$

Obviously, there must be either $a = 0$ or $b = 0$. Let us assume first that $b = 0$. Then we get $I = a^2 I$, which implies $a = \pm 1$. In this situation we have the following two possibilities.

$$\begin{aligned}ADA^{-1} &= I \qquad & ADA^{-1} &= -I \\ AD &= A & AD &= -A \\ D &= I & D &= -I\end{aligned}$$

But according to the assumptions concerning D neither $D = I$ nor $D = -I$ is possible. Consequently there must be $a = 0$ and $b = \pm 1$. Then $ADA^{-1} = \pm D$, and we have the following two possibilities.

$$\begin{aligned}ADA^{-1} &= D \qquad & ADA^{-1} &= -D \\ AD &= DA & AD &= -DA\end{aligned}$$

We have thus proved the following lemma.

Lemma 2.6. *Every automorphism of ω_+ is a paracomplex linear or paracomplex antilinear mapping.*

We denote

$$\begin{aligned}\mathrm{Aut}^+(\omega_+) &= \{A \in \mathrm{Aut}(\omega_+); AD = DA\}, \\ \mathrm{Aut}^-(\omega_+) &= \{A \in \mathrm{Aut}(\omega_+); AD = -DA\}.\end{aligned}$$

We have obviously $\mathrm{Aut}(\omega_+) = \mathrm{Aut}^+(\omega_+) \cup \mathrm{Aut}^-(\omega_+)$.

Lemma 2.7. $\mathrm{Aut}^+(\omega_+) = \mathrm{Aut}^+(\tilde{\omega}_+)$.

Proof. The proof follows the lines of the proof of Proposition 1.6. $\square$

Proposition 2.2. *An automorphism $A \in GL^+(V; \mathbb{D})$ belongs to $\mathrm{Aut}^+(\omega_+)$ if and only if $\det_{\mathbb{D}} A = 1$. Consequently $\mathrm{Aut}^+(\omega_+) = SL(V; \mathbb{D})$.*

Proof. Here we proceed as in the proof of Proposition 1.2. $\square$

Every endomorphism Q of the $\mathbb{D}$-module $(V, \mathbb{D})$ can be uniquely expressed in the form $Q = \rho R + \sigma S$, where R and S are real endomorphisms, i. e. endomorphisms satisfying $RV_0 \subset V_0$ and $SV_0 \subset V_0$. Then it is obvious that

$$\det_{\mathbb{D}} Q = \rho \det_{\mathbb{R}}(R|V_0) + \sigma \det_{\mathbb{R}}(S|V_0) =$$
$$= \frac{1}{2}(\det_{\mathbb{R}}(R|V_0) + \det_{\mathbb{R}}(S|V_0)) + d\frac{1}{2}(\det_{\mathbb{R}}(R|V_0) - \det_{\mathbb{R}}(S|V_0)).$$

We can see that $\det_{\mathbb{D}} Q = 1$ if and only if $\det_{\mathbb{R}}(R|V_0) = \det_{\mathbb{R}}(S|V_0) = 1$. Now we get easily the following proposition.

Proposition 2.3. *The group* $\mathrm{Aut}^+(\omega_+) = SL(V; \mathbb{D})$ *is isomorphic with the group* $SL(V_0) \times SL(V_0) \cong SL(n; \mathbb{R}) \times SL(n; \mathbb{R})$*, and consequently is connected.*

In the algebra $\mathbb{D}$ of paracomplex numbers we can introduce conjugation by the standard formula $\overline{a + db} = a - db$. This conjugation has moreless the same properties as the conjugation of complex numbers. If W is a $\mathbb{D}$-module, we can introduce the conjugate $\mathbb{D}$-module $\bar{W}$ by setting $\bar{W}$ and $c * v = \bar{c}v$. If W is an n-dimensional free $\mathbb{D}$-module, then $\bar{W}$ is also an n-dimensional free $\mathbb{D}$-module. Consequently, there exist a $\mathbb{D}$-module isomorphism $B : W \to \bar{W}$ such that $B^*\bar{\theta} = \theta$. Now, along the same lines as in the complex case, we get the following two lemmas.

Lemma 2.8. *Let* $A \in \mathrm{Aut}(\omega_+)$*. Then* $A \in \mathrm{Aut}^-(\omega_+)$ *if and only if one of the following two equivalent conditions is satisfied.*

(i) $A^*\tilde{\omega}_+ = -\tilde{\omega}_+$,
(ii) $A^*\theta = -\bar{\theta}$.

Lemma 2.9. *Let* $A \in GL^-(V; D)$*. Then* $A \in \mathrm{Aut}^-(\omega_+)$ *if and only if* $\det_{\mathbb{D}}(AB) = -1$.

Now we can easily see that there is

$$\mathrm{Aut}^+(\omega_+) \cdot \mathrm{Aut}^+(\omega_+) = \mathrm{Aut}^+(\omega_+),$$
$$\mathrm{Aut}^+(\omega_+) \cdot \mathrm{Aut}^-(\omega_+) = \mathrm{Aut}^-(\omega_+) \cdot \mathrm{Aut}^+(\omega_+) = \mathrm{Aut}^-(\omega_+),$$
$$\mathrm{Aut}^-(\omega_+) \cdot \mathrm{Aut}^-(\omega_+) = \mathrm{Aut}^+(\omega_+).$$

Summarizing, we obtain the following proposition.

Proposition 2.4. *The automorphism group* $\mathrm{Aut}(\omega_+)$ *consists of two connected components* $\mathrm{Aut}^+(\omega_+)$ *and* $\mathrm{Aut}^-(\omega_+)$*, where* $\mathrm{Aut}^+(\omega_+)$ *is the connected component of the unit. The group* $\mathrm{Aut}^+(\omega_+) = SL(V_0) \times SL(V_0) \cong SL(n; \mathbb{R}) \times SL(n; \mathbb{R})$*. Moreover* $\dim_{\mathbb{R}} \mathrm{Aut}(\omega_+) = 2(n^2 - 1)$.

According to our assumptions concerning the automorphism D of V we can write

$$V = V_+ \oplus V_-, \text{ where } V_+ = \{v \in V; Dv = v\}, \quad V_- = \{v \in V; Dv = -v\}.$$

Lemma 2.10. *Let* $v_+ \in V_+$*,* $v_- \in V_-$*, and* $v_3, \dots, v_n \in V$*. Then*

$$\omega_+(v_+, v_-, v_3, \dots, v_n) = 0.$$

Proof. We get

$$\omega_+(v_+, v_-, v_3, \dots, v_n) = \omega_+(Dv_+, v_-, v_3, \dots, v_n) =$$
$$= \omega_+(v_+, Dv_-, v_3, \dots, v_n) = -\omega_+(v_+, v_-, v_3, \dots, v_n),$$

which shows that $\omega_+(v_+, v_-, v_3, \dots, v_n) = 0$. □

If $v_1 = v_{1+} + v_{1-}, \dots, v_n = v_{n+} + v_{n-}$, then we obviously have

$$\omega_+(v_1, \dots, v_n) = \omega_+(v_{1+}, \dots, v_{n+}) + \omega_+(v_{1-}, \dots, v_{n-}).$$

Moreover it is easy to see that $\omega_+|V_+$ and $\omega_+|V_-$ are regular forms. (Otherwise the form ω_+ would be singular.) Let $\pi_+ : V \to V_+$ and $\pi_- : V \to V_-$ denote the projections. We get the following proposition.

Proposition 2.5. *For the form ω_+ we have $\omega_+ = \pi_+^*(\omega_+|V_+) + \pi_-^*(\omega_+|V_-)$.*

3. Forms of the tangent type

In this section we shall consider an n-dimensional $\mathbb{E}$-module $(V, \mathbb{E}) = W$. First we introduce the mapping

$$\rho : \mathbb{E} \to \mathbb{E}/(e) \cong \mathbb{R},$$

which is projection onto the quotient by the ideal (e). Now we are going to prove the following lemma.

Lemma 3.1. *The elements $w_1, \dots, w_k$ are linearly independent in W if and only if the vectors $Ew_1, \dots, Ew_k$ are linearly independent in V. The elements $w_1, \dots, w_n$ constitute a basis of W if and only if the vectors $Ew_1, \dots, Ew_n$ are linearly independent in V.*

Proof. Let us assume that the elements $w_1, \dots, w_k$ are linearly independent in W. Let $a_1, \dots, a_k$ be real numbers such that $a_1Ew_1 + \cdots + a_kEw_k = 0$. Then there is $E(a_1w_1 + \cdots + a_kw_k) = 0$, which means that we can find $w \in W$ such that

$$a_1w_1 + \cdots + a_kw_k = ew$$
$$ea_1w_1 + \cdots + ea_kw_k = 0$$

Because the elements $w_1, \dots, w_k$ are linearly independent in W, we have $a_1e = \cdots = a_ke = 0$, which inplies $a_1 = \cdots = a_k = 0$. We have thus proved that the vectors $Ew_1, \dots, Ew_k$ are linearly independent in V.

On the other hand, let us suppose that the vectors $Ew_1, \dots, Ew_k$ are linearly independent in V. Let $c_1, \dots, c_k \in \mathbb{E}$ be such that $c_1w_1 + \cdots + c_kw_k = 0$. Writing $c_i = c_i' + ec_i''$, $i = 1, \dots, k$, we get from the last equality $c_1'Ew_1 \cdots + c_k'Ew_k = 0$. This implies that we have $c_1' = \cdots = c_k' = 0$. Consequently, we get

$$e(c_1''w_1 + \cdots + c_k''w_k) = 0$$
$$c_1''Ew_1 + \cdots + c_k''Ew_k = 0,$$

which again implies $c_1'' = \cdots = c_k'' = 0$. We have thus proved that $w_1, \dots, w_k$ are linearly independent in the $\mathbb{E}$-module W.

If $w_1, \dots, w_n$ is a basis of W, then the elements $w_1, \dots, w_n$ are linearly independent in W, and consequently the vectors $Ew_1, \dots, Ew_n$ are linearly independent in V. Conversely, let us assume that the vectors $Ew_1, \dots, Ew_n$ are linearly independent in V. Then $w_1, \dots, w_n$ are linearly independent in W. Finally, let $w \in W$. Then we can find uniquely determined $c_1', \dots, c_n' \in \mathbb{R}$ such that

$$Ew = c_1'Ew_1 + \cdots + c_n'Ew_n.$$

Consequently, $E(w - c_1'w_1 - \cdots - c_n'w_n) = 0$, and there is $w'' \in W$ such that $w - c_1'w_1 - \cdots - c_n'w_n = Ew''$. We can find uniquely determined $c_1'', \dots, c_n'' \in \mathbb{R}$ such that $Ew'' = c_1''Ew_1 + \cdots + c_n''Ew_n$. We can see that

$$w = (c_1' + ec_1'')w_1 + \cdots + (c_n' + ec_n'')w_n$$

which proves that $w_1, \dots, w_n$ is a basis of the $\mathbb{E}$-module W. $\square$

Let θ be a non-zero n-form on W. Then, proceeding as in the proof of Lemma 1.1 we get easily the following lemma.

Lemma 3.2. *Let $\theta \neq 0$ be an n-form on the n-dimensional $\mathbb{E}$-module W. Then $\mathcal{A}_\theta = [I, E] = \{cI; c \in \mathbb{E}\}$.*

We introduce real valued n-forms ω_0 and $\tilde{\omega}_0$ by the formula

$$\theta = \tilde{\omega}_0 + e\omega_0.$$

Lemma 3.3. *The automorphism E belongs to the both algebras $\mathcal{A}_{\omega_0}$ and $\mathcal{A}_{\tilde{\omega}_0}$.*

Proof. The proof follows the lines of the proof of Lemma 1.2. $\square$

Lemma 3.4. *The n-forms ω_0 and $\tilde{\omega}_0$ satisfy the relation $\tilde{\omega}_0 = \frac{1}{n}\mathcal{D}_E\omega_0$ and $\mathcal{D}_E\tilde{\omega}_0 = 0$.*

Proof.

$$\begin{aligned} \theta(Ew_1, w_2, \dots, w_n) &= \tilde{\omega}_0(Ew_1, w_2, \dots, w_n) + e\omega_0(Ew_1, w_2, \dots, w_n), \\ \theta(Ew_1, w_2, \dots, w_n) &= e\theta(w_1, w_2, \dots, w_n) = e\tilde{\omega}_0(w_1, w_2, \dots, w_n), \end{aligned}$$

which shows $\tilde{\omega}_0(w_1, w_2, \dots, w_n) = \omega_0(Ew_1, w_2, \dots, w_n)$ and $\tilde{\omega}_0(Ew_1, w_2, \dots, w_n) = 0$. In other words $\tilde{\omega}_0 = (1/n)\mathcal{D}_E\omega_0$ and $\mathcal{D}_E\tilde{\omega}_0 = 0$. $\square$

Lemma 3.5. *The form ω_0 is regular.*

Proof. This lemma can be proved in the same way as Lemma 1.4. $\square$

Lemma 3.6. *$\mathcal{A}_{\omega_0} = [I, E]$, $[I, E] \subset \mathcal{A}_{\tilde{\omega}_0}$.*

Proof. Let $A \in \mathcal{A}_\theta$. Then we have

$$\begin{aligned} \tilde{\omega}_0(Av_1, v_2, \dots, v_n) + e\omega_0(Av_1, v_2, \dots, v_n) &= \theta(Av_1, v_2, \dots, v_n) = \\ = \theta(v_1, Av_2, \dots, v_n) &= \tilde{\omega}_0(v_1, Av_2, \dots, v_n) + e\omega_0(v_1, Av_2, \dots, v_n), \end{aligned}$$

which shows that $\mathcal{A}_\theta \subset \mathcal{A}_{\omega_0}$ and $\mathcal{A}_\theta \subset \mathcal{A}_{\tilde{\omega}_0}$. Next, let us assume that $A \in \mathcal{A}_{\omega_0}$. Then according to Lemma 3.3 there is $AE = EA$. We have then

$$\begin{aligned} \tilde{\omega}_0(Av_1, v_2, \dots, v_n) = \omega_0(EAv_1, v_2, \dots, v_n) &= \omega_0(AEv_1, v_2, \dots, v_n) = \\ = \omega_0(Ev_1, Av_2, \dots, v_n) &= \tilde{\omega}_0(v_1, Av_2, \dots, v_n), \end{aligned}$$

which shows that $A \in \mathcal{A}_{\tilde{\omega}_0}$. Consequently $A \in \mathcal{A}_\theta$, and we have $\mathcal{A}_{\omega_0} = \mathcal{A}_\theta = [I, E]$. $\square$

In the same way as in the complex case we can prove that if $A \in \operatorname{Aut}(\omega_0)$, then $AEA^{-1} \in \mathcal{A}_{\omega_0}$. Consequently there are real numbers a, b such that $AEA^{-1} = aI + bE$. Taking the square of this relation we get $0 = a^2 I + 2abE$ and this implies $a = 0$. We obtain

Lemma 3.7. *Every automorphism $A \in \operatorname{Aut}(\omega_0)$ satisfies the relation $AE = \kappa EA$ with $\kappa \in \mathbb{R}^* = \mathbb{R} - \{0\}$.*

Lemma 3.8. *Every automorphism $A \in \operatorname{Aut}(\omega_0)$ is a conformal automorphism of the form $\tilde{\omega}_0$. More precisely, if $A \in \operatorname{Aut}(\omega_0)$ satisfies $AE = \kappa EA$, then $A^*\tilde{\omega}_0 = (1/\kappa)\tilde{\omega}_0$.*

Proof. We have

$$(A^*\tilde{\omega}_0)(v_1, v_2, \ldots, v_n) = \tilde{\omega}_0(Av_1, Av_2, \ldots, Av_n) =$$
$$= \omega_0(EAv_1, Av_2, \ldots, Av_n) = (1/\kappa)\omega_0(AEv_1, Av_2, \ldots, Av_n) =$$
$$= (1/\kappa)\omega_0(Ev_1, v_2, \ldots, v_n) = (1/\kappa)\tilde{\omega}_0(v_1, v_2, \ldots, v_n).$$

□

Lemma 3.9. *For every $\kappa \in \mathbb{R}^*$ there exists an automorphism $A \in \mathrm{Aut}(\omega_0)$ such that $AE = \kappa EA$.*

Proof. We choose a basis $\beta_1, \ldots, \beta_n$ of W^* such that $\theta = \beta_1 \wedge \cdots \wedge \beta_n$ and the corresponding dual basis $e_1, \ldots, e_n$ of W. Then $e_1, \ldots, e_n, Ee_1, \ldots, Ee_n$ is a basis of the vector space V. We take the dual basis $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}$ of the vector space V^*. We find easily that

$$\beta_1 = \alpha_1 + e\alpha_{n+1}, \ldots, \beta_n = \alpha_n + e\alpha_{2n}.$$

Now we can see that

$$\theta = \alpha_1 \wedge \cdots \wedge \alpha_n + e\sum_{i=1}^{n} \alpha_1 \wedge \cdots \wedge \alpha_{i-1} \wedge \alpha_{i+n} \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_n.$$

We define now an automorphism of V by the following formulas.

$$A^*\alpha_1 = \frac{1}{\kappa}\alpha_1, A^*\alpha_2 = \alpha_2, \ldots, A^*\alpha_n = \alpha_n,$$
$$A^*\alpha_{n+1} = \alpha_{n+1}, A^*\alpha_{n+2} = \kappa\alpha_{n+2}, \ldots, A^*\alpha_{2n} = \kappa\alpha_{2n}.$$

Now it is obvious that $A^*\omega_0 = \omega_0$ and $A^*\tilde{\omega}_0 = \frac{1}{\kappa}\tilde{\omega}_0$. Hence we have $AE = \kappa EA$. □

From the above considerations we get easily the following lemma.

Lemma 3.10. *The n-form $\tilde{\omega}_0$ is decomposable.*

We can now define an epimorphism $K : \mathrm{Aut}(\omega_0) \to \mathbb{R}^*$. If $A \in \mathrm{Aut}(\omega_0)$, then there is a unique $\kappa \in \mathbb{R}^*$ such that $AE = \kappa EA$. We set $K(A) = \kappa$. Using Lemma 3.9 we obtain the short exact sequence

$$1 \to \ker K \to \mathrm{Aut}(\omega_0) \xrightarrow{K} \mathbb{R}^* \to 1$$

If $A \in \ker K$, then $AE = EA$, A is an $\mathbb{E}$-linear automorphism, $A^*\omega_0 = \omega_0$ and $A^*\tilde{\omega}_0 = \tilde{\omega}_0$. Consequently,

$$A^*\theta = A^*\tilde{\omega}_0 + eA^*\omega_0 = \tilde{\omega}_0 + e\omega_0 = \theta.$$

Hence we can see that $\ker K = SL(V; \mathbb{E}) \cong SL(n, \mathbb{E})$. The above exact sequence can now be written in the form

$$1 \to SL(V; \mathbb{E}) \to \mathrm{Aut}(\omega_0) \xrightarrow{K} \mathbb{R}^* \to 1.$$

Introducing the subsets

$$\mathrm{Aut}^+(\omega) = \{A \in \mathrm{Aut}(\omega); K(A) > 0\}, \mathrm{Aut}^-(\omega) = \{A \in \mathrm{Aut}(\omega); K(A) < 0\},$$

we have an exact sequence

$$1 \to SL(V; \mathbb{E}) \to \mathrm{Aut}^+(\omega_0) \xrightarrow{K} \mathbb{R}^+ \to 1.$$

Lemma 3.11. *The group $\mathrm{Aut}(\omega_0)$ is a semidirect product $SL(V; \mathbb{E}) \ltimes \mathbb{R}^*$. Analogously, the group $\mathrm{Aut}^+(\omega_0)$ is a semidirect product $SL(V; \mathbb{E}) \ltimes \mathbb{R}^+$.*

Proof. In the first case it suffices to find a splitting $\sigma : \mathbb{R}^* \to \mathrm{Aut}(\omega_0)$. We use the same bases as in Lemma 3.9. To $\kappa \in \mathbb{R}^*$ we assign an automorphism $\sigma(\kappa)$ defined by the formulas

$$\sigma(\kappa)e_1 = \frac{1}{\kappa}e_1, \quad \sigma(\kappa)e_2 = e_2, \quad \dots, \quad \sigma(\kappa)e_n = e_n,$$
$$\sigma(\kappa)e_{n+1} = e_{n+1}, \quad \sigma(\kappa)e_{n+2} = \kappa e_{n+2}, \quad \dots, \quad \sigma(\kappa)e_{2n} = \kappa e_{2n}.$$

It can be immediately seen that σ is a splitting. It is also obvious that $\sigma(\mathbb{R}^+) \subset \mathrm{Aut}^+(\omega_0)$, which means that the same splitting can be used also in the second case. □

We shall now investigate the group $SL(V;\mathbb{E})$. Let $A \in GL(V;\mathbb{E})$. Because A is $\mathbb{E}$-linear, it preserves the subspace $\operatorname{im} A$, and consequently induces an automorphism $\hat{A}$ of the quotient $\hat{V} = V/\operatorname{im} E$. We have the projection $\pi : V \to \hat{V}$, which satisfies $\pi(av) = \rho(a)\pi(v)$. This projection induces also a projection

$$\Lambda^n\pi : \Lambda^n_{\mathbb{E}}V \to \Lambda^n_{\mathbb{R}}\hat{V}, \quad v_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} v_n \mapsto \pi(v_1) \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} \pi(v_n).$$

We have

$$A_*(e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n) = \det_{\mathbb{E}} A \cdot e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n,$$
$$\hat{A}_*(\pi(e_1) \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} \pi(e_n)) = \det_{\mathbb{R}} \hat{A} \cdot \pi(e_1) \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} \pi(e_n),$$
$$\hat{A}_*(\pi(e_1) \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} \pi(e_n)) = \hat{A}_*(\Lambda^n\pi)(e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n) =$$
$$= (\Lambda^n\pi)A_*(e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n) = (\Lambda^n\pi)(\det_{\mathbb{E}} A \cdot e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n) =$$
$$= \rho(\det_{\mathbb{E}} A) \cdot (\Lambda^n\pi)(e_1 \wedge_{\mathbb{E}} \cdots \wedge_{\mathbb{E}} e_n) = \rho(\det_{\mathbb{E}} A) \cdot \pi(e_1) \wedge_{\mathbb{R}} \cdots \wedge_{\mathbb{R}} \pi(e_n).$$

We have thus proved the formula

$$\rho(\det_{\mathbb{E}} A) = \det_{\mathbb{R}} \hat{A}.$$

We denote Q the homomorphism assigning to an automorphism $A \in GL(V;\mathbb{E})$ the induced automorphism $\hat{A} \in GL(\hat{V})$. It is easy to see that we get a short exact sequence

$$1 \to \ker Q \to GL(V;\mathbb{E}) \xrightarrow{Q} GL(\hat{V}) \to 1.$$

Let us denote first

$$\mathcal{B} = \{B \in gl(V;\mathbb{E}); BV \subset \operatorname{im} E\}.$$

If $B \in \mathcal{B}$, then $B(\operatorname{im} E) = 0$. Namely, if $v \in \operatorname{im} E$, then $v = ev'$ for some $v' \in V$. Then $Bv = B(ev') = eB(v') = 0$. Consequently, if $B, B' \in \mathcal{B}$, then $BB' = 0$.

Every $A \in \ker Q$ can be expressed in the form $A = I + B$, where $B \in \mathcal{B}$. On the other hand, every endomorphism of the form $I+B$ with $B \in \mathcal{B}$ is an automorphism. Namely,

$$(I + B)(I - B) = I - B + B - BB = I.$$

Moreover $A = I + B$ obviously belongs to $\ker Q$. If $A = I + B$ and $A' = I + B'$ we have

$$(I + A)(I + B') = I + B + B' + BB' = I + B + B'.$$

Hence we can see that $\ker Q$ is an abelian group isomorphic with $\mathcal{B} \cong \mathbb{R}^{n^2}$.

Lemma 3.12. *The group $GL(V;\mathbb{E})$ is a semidirect product $\mathcal{B} \ltimes GL(\hat{V})$.*

Proof. We use again the same bases as in the proof of lemma 3.9. Obviously the classes $[e_1], \ldots, [e_n]$ constitute a basis of the vector space $\hat{V}$. Any automorphism $\varphi \in GL(\hat{V})$ can be expressed in the form

$$\varphi[e_i] = \sum_{j=1}^{n} a_{ij}[e_j],$$

where a_{ij} are real numbers. We define an automorphism $A \in GL(V; \mathbb{E})$ by the formulas

$$Ae_i = \sum_{j=1}^{n} a_{ij} e_j.$$

Setting $\sigma(\varphi) = A$, we get a splitting $\sigma : GL(\hat{V}) \to GL(V; \mathbb{E})$. □

Let us remind that if $v_1, v_2 \in \operatorname{im} E$, then $\theta(v_1, v_2, \ldots, v_n) = 0$. Namely, we have $v_1 = ev_1'$ and $v_2 = ev_2'$, and we get

$$\theta(v_1, v_2, v_3, \ldots, v_n) = \theta(ev_1', ev_2', v_3, \ldots, v_n) = e^2\theta(v_1', v_2', v_3, \ldots, v_n) = 0.$$

Let $A = I + B$ be again an element of $\ker Q$. We obtain

$$(A^*\theta)(v_1, v_2, \ldots, v_n) = \theta(v_1 + Bv_1, v_2 + Bv_2, \ldots, v_n + Bv_n) =$$
$$= \theta(v_1, v_2, \ldots, v_n) + \sum_{i=1}^{n} \theta(v_1, \ldots, v_{i-1}, Bv_i, v_{i+1}, \ldots, v_n) =$$
$$= \theta(v_1, v_2, \ldots, v_n) + \operatorname{tr}(B)\theta(v_1, v_2, \ldots, v_n) = (1 + \operatorname{tr}(B))\theta(v_1, v_2, \ldots, v_n).$$

We have thus proved that if $A \in \ker Q$, then $\det_{\mathbb{E}} A = 1 + \operatorname{tr}(B)$.

Using the formula $\rho(\det_{\mathbb{E}} A) = \det_{\mathbb{R}} \hat{A}$ we get another short exact sequence

$$1 \to \ker q \to SL(V; \mathbb{E}) \xrightarrow{q} SL(\hat{V}) \to 1,$$

where $q = Q|SL(V; \mathbb{E})$. Applying the last determinant formula we find easily that

$$\ker q = \mathcal{B}_0 = \{B \in \mathcal{B}; \operatorname{tr}(B) = 0\} \cong \mathbb{R}^{n^2-1}.$$

Lemma 3.13. *The group $SL(V; \mathbb{E})$ is a semidirect product $\mathcal{B}_0 \ltimes SL(\hat{V})$.*

Proof. The proof follows the same lines as the proof of Prop. 3.12. □

Summarizing we have the following proposition.

Proposition 3.1. *The automorphism group $\operatorname{Aut}(\omega_0)$ consists of two connected components $\operatorname{Aut}^+(\omega_0)$ and $\operatorname{Aut}^-(\omega_0)$, where $\operatorname{Aut}^+(\omega_0)$ is the connected component of the unit. Moreover $\dim_{\mathbb{R}} \operatorname{Aut}(\omega_0) = 2n^2 - 1$.*

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Mathematical Institute, Academy of Sciences of the Czech Republic, branch Brno, Žižkova 22, 616 62 Brno, Czech Republic