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EQUIVALENCE OF NORMS OF RIESZ POTENTIAL AND FRACTIONAL MAXIMAL FUNCTION IN MORREY-TYPE SPACES

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Abstract. In this paper we find the condition on the ω which ensures the equivalency of norms of the Riesz potential and the fractional maximal function in generalized Morrey space $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α is defined by

$$M_\alpha f(x) = \sup_{r>o} |B(x,r)|^{-1+\frac{\alpha}{n}} \int_{B(x,r)} |f(y)|dy, \ 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ 0 < \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

By $A \lesssim B$ we mean that $A \le cB$ with some positive constant c independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B *are equivalent*.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \lesssim I_\alpha(|f|)(x). \tag{1.1}$$

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [9] and defined as follows: *For* $\lambda \ge 0$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))} < \infty$$

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holds .

These spaces appeared to be quite useful in the study of local behaviour of the solutions of elliptic partial differential equations.

If in place of the power function $r^{-\lambda/p}$ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive weight function ω defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,\omega}$.

Definition 1.1. Let $1 \le p \le \infty$, ω be a positive weight function ω defined on $(0,\infty)$. $f \in \mathcal{M}_{p,\omega}$ if $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\omega}} \equiv \|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \omega(r)\|f\|_{L_p(B(x,r))} < \infty$$

In the paper [2] D.R.Adams and J.Xiao have proved the following Theorem.

Theorem 1.2 (Theorem 4.2, [2])**.** *Let $1 < p < \infty$, $0 < \alpha < n$, $0 \le \lambda \le n$. If $f \in L_p^{\text{loc},+}(\mathbb{R}^n)$, then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\lambda}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\lambda}}. \tag{1.2}$$

The proof of this Theorem is not correct, because the non-correct estimation (see (4.8) in [2]) was used. In this work we show that the estimation (4.8) in [2] is wrong by giving the counterexample and we present right formulation of such estimation (see the inequality (4.3)).

We find sufficient condition on the ω which ensures the equivalency of $\mathcal{M}_{p,\omega}$-norms of the Riesz potential and the fractional maximal function. As Corollary 7.5, we obtain a correct proof of the Theorem 1.2. Our main result is presented in section 7.

2. Preliminaries

In this section we present some preliminary results on the Fefferman-Stein sharp maximal and local sharp maximal functions.

Let us denote by $f^{\#}$ and $f^{\#}_{Q_0}$ the Fefferman-Stein sharp and local sharp maximal functions. Recall that f_E denotes the mean value $f_E = (1/|E|)\int_E f(y)dy$ of an integrable function f over a set E of positive finite measure.

Definition 2.1. For $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ the sharp function $f^{\#}$ of f is defined by

$$f^{\#}(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y) - f_B|dy, \tag{2.1}$$

where the supremum is taken over all balls B containing x.

Definition 2.2. If f is integrable over fixed cube Q_0 in $\mathbb{R}^n$, the local sharp function $f^{\#}_{Q_0}$ of f relative to Q_0 is defined by

$$f^{\#}_{Q_0}(x) = \sup_{\substack{Q\subset Q_0\\ Q\ni x}} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy, \ (x \in Q_0), \tag{2.2}$$

where the supremum extends over all cubes Q that contains x and are contained in Q_0.

Proposition 2.3. ([1], Proposition 3.3) *For $0 < \alpha < n$ and any f such that $I_\alpha f$ is locally integrable,*

$$(I_\alpha f)^{\#}(x) \le c M_\alpha f(x), \ \text{all } x, \tag{2.3}$$

c independent of f and x.

Proposition 2.4. ([5], Proposition 7.4) *If f is integrable over a cube Q_0, then*

$$[(f - f_{Q_0})\chi_{Q_0}]^{**}(t) \le c \int_t^{|Q_0|} (f^{\#}_{Q_0})^*(s)\frac{ds}{s}, \ \left(0 < t < \frac{|Q_0|}{6}\right). \tag{2.4}$$

From this Proposition immediately follows next Corollary.

Corollary 2.5. *If f is integrable over a cube Q_0, then*

$$\|f - f_{Q_0}\|_{L_p(Q_0)} \le c\|f^{\#}_{Q_0}\|_{L_p(Q_0)}. \tag{2.5}$$

Proof. By Proposition 2.4

$$[(f - f_{Q_0})\chi_{Q_0}]^{*}(t) \le c \int_t^{|Q_0|} (f^{\#})^*(s)\frac{ds}{s}, \ (0 < t < |Q_0|/6)\,. \tag{2.6}$$

Since

$$\begin{aligned}
&\left(\int_0^{\frac{|Q_0|}{6}} \left([(f - f_{Q_0})\chi_{Q_0}]^*(t)\right)^p dt\right)^{\frac{1}{p}} \\
&\qquad \le c\left(\int_0^{|Q_0|}\left(\int_t^{|Q_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}\right)^p dt\right)^{\frac{1}{p}} \\
&\qquad = c\left(\int_0^{|Q_0|}\left(t^{1-\frac{1}{p'}}\int_t^{|Q_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}},
\end{aligned}$$

applying Hardy's inequality ([5], III (3.19)), we get

$$\begin{aligned}
&\left(\int_0^{\frac{|Q_0|}{6}} \left([(f - f_{Q_0})\chi_{Q_0}]^*(t)\right)^p dt\right)^{\frac{1}{p}} \\
&\qquad \le c\left(\int_0^{|Q_0|}\left((f^{\#}_{Q_0})^*(t)\right)^p dt\right)^{\frac{1}{p}} = c\|f^{\#}_{Q_0}\|_{L_p(Q_0)}.
\end{aligned}$$

But by monotonicity of $[(f - f_{Q_0})\chi_{Q_0}]^*(t)$

$$\begin{aligned}\|f - f_{Q_0}\|_{L_p(Q_0)} &= \left(\int_0^{|Q_0|} ([(f - f_{Q_0})\chi_{Q_0}]^*(t))^p\,dt\right)^{\frac{1}{p}} \\ &\le \left(\int_0^{\frac{|Q_0|}{6}} ([(f - f_{Q_0})\chi_{Q_0}]^*(t))^p\,dt\right)^{\frac{1}{p}} \\ &\quad + \left(\int_{\frac{|Q_0|}{6}}^{|Q_0|} ([(f - f_{Q_0})\chi_{Q_0}]^*(t))^p\,dt\right)^{\frac{1}{p}} \\ &\le c\left(\int_0^{\frac{|Q_0|}{6}} ([(f - f_{Q_0})\chi_{Q_0}]^*(t))^p\,dt\right)^{\frac{1}{p}} \\ &\quad + c\,[(f - f_{Q_0})\chi_{Q_0}]^*\left(\frac{|Q_0|}{6}\right)|Q_0|^{\frac{1}{p}} \\ &\le c\left(\int_0^{\frac{|Q_0|}{6}} ([(f - f_{Q_0})\chi_{Q_0}]^*(t))^p\,dt\right)^{\frac{1}{p}}.\end{aligned}$$

□

Corollary 2.6. *If f is integrable over a cube Q_0, then*

$$\|f\|_{L_p(Q_0)} \le c\|f^{\#}_{Q_0}\|_{L_p(Q_0)} + c|Q_0|^{\frac{1}{p}}|f|_{Q_0}. \tag{2.7}$$

Proof. Since

$$\|f\|_{L_p(Q_0)} \le \|f - f_{Q_0}\|_{L_p(Q_0)} + \|f_{Q_0}\|_{L_p(Q_0)},$$

By Corollary 2.5

$$\|f\|_{L_p(Q_0)} \le c\|f^{\#}_{Q_0}\|_{L_p(Q_0)} + c|Q_0|^{\frac{1}{p}}|f|_{Q_0}.$$

□

Remark 2.7. Corollary 2.6 shows that in the inequality (7.13) in Corollary 7.5 in [5] multiplier $|Q_0|^{1/p}$ in front of $|f|_{Q_0}$ had been lost.

The following Proposition is true

Proposition 2.8. *If f is integrable over a cube Q_0, then*

$$[(f - f_{Q_0})\chi_{Q_0}]^{**}(t) \le c\int_t^{|Q_0|} (f^{\#}_{Q_0})^*(s)\frac{ds}{s} + c(f^{\#}_{Q_0})^*(|Q_0|),\ (0 < t < |Q_0|)\,. \tag{2.8}$$

Proof. Indeed, taking into account monotonicity of $[(f-f_{Q_0})\chi_{Q_0}]^{**}(t)$, for t : $|Q_0|/6 \le t \le |Q_0|$ we have

$$\begin{aligned}
[(f-f_{Q_0})\chi_{Q_0}]^{**}(t) &\le [(f-f_{Q_0})\chi_{Q_0}]^{**}\left(\frac{|Q_0|}{6}\right)\\
&\le \frac{6}{|Q_0|}\int_0^{|Q_0|}[(f-f_{Q_0})\chi_{Q_0}]^{*}(t)dt\\
&= \frac{6}{|Q_0|}\int_{Q_0}|f-f_{Q_0}| \le c\inf_{x\in Q_0} f^{\#}_{Q_0}(x).
\end{aligned}$$

Since

$$f^*(t) = \sup_{|E|=t}\inf_{x\in E}|f(x)|$$

(see [8]), we arrive at

$$[(f-f_{Q_0})\chi_{Q_0}]^{**}(t) \le c(f^{\#}_{Q_0})^*(|Q_0|),\ \frac{|Q_0|}{6} \le t \le |Q_0|. \tag{2.9}$$

□

Definition 2.9. For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ the local sharp function $f^{\#}_{B_0}$ of f relative to B_0 is defined by

$$f^{\#}_{B_0}(x) = \sup_{\substack{B\subset 2B_0\\ B\ni x}} \frac{1}{|B|}\int_B |f(y)-f_B|dy,\ (x\in B_0), \tag{2.10}$$

where the supremum extends over all balls B that contains x and are contained in $2B_0$.

The following Proposition is true

Proposition 2.10. *If $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, then for any ball B_0 in $\mathbb{R}^n$*

$$[(f-f_{B_0})\chi_{B_0}]^{**}(t) \le c\int_t^{|B_0|}(f^{\#}_{B_0})^*(s)\frac{ds}{s} + c(f^{\#}_{B_0})^*(|B_0|),\ (0<t<|B_0|)\,. \tag{2.11}$$

Proof. Let $t : 0 < t < |B_0|$ and Q_0 be the cube concentric with B_0, such that $B_0 \subset Q_0 \subset 2B_0$. Then

$$\begin{aligned}
[(f-f_{B_0})\chi_{B_0}]^{**}(t) &\le [(f-f_{Q_0})\chi_{B_0}]^{**}(t) + [(f_{Q_0}-f_{B_0})\chi_{B_0}]^{**}(t)\\
&\le [(f-f_{Q_0})\chi_{B_0}]^{**}(t) + |f_{B_0}-f_{Q_0}|\\
&\le [(f-f_{Q_0})\chi_{Q_0}]^{**}(t) + \frac{1}{|B_0|}\int_{B_0}|f-f_{Q_0}|\\
&\le [(f-f_{Q_0})\chi_{Q_0}]^{**}(t) + \frac{|Q_0|}{|B_0|}\int_{Q_0}|f-f_{Q_0}|\\
&\le c[(f-f_{Q_0})\chi_{Q_0}]^{**}(t) + c\inf_{x\in Q_0}(f^{\#}_{Q_0})(x)\\
&\le c[(f-f_{Q_0})\chi_{Q_0}]^{**}(t) + c(f^{\#}_{Q_0})^*(t).
\end{aligned}$$

By Proposition 2.8 and monotonicity of $(f^{\#}_{Q_0})^*(t)$

$$\begin{aligned}[(f-f_{B_0})\chi_{B_0}]^{**}(t) &\le c\int_t^{|Q_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}+c(f^{\#}_{Q_0})^*(|Q_0|)\\ &\le \int_t^{|B_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}+\int_{|B_0|}^{|Q_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}+c(f^{\#}_{Q_0})^*(|Q_0|)\\ &\le \int_t^{|B_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}+c(f^{\#}_{Q_0})^*(|B_0|)+c(f^{\#}_{Q_0})^*(|Q_0|)\\ &\le \int_t^{|B_0|}(f^{\#}_{Q_0})^*(s)\frac{ds}{s}+c(f^{\#}_{B_0})^*(|B_0|).\end{aligned}$$

Since $f^{\#}_{Q_0}(x)\le cf^{\#}_{B_0}(x)$, $x\in B_0$, we arrive at (2.11). □

In the same way, as it has been done in the proof of Corollary 2.5, we can prove the following Corollary.

Corollary 2.11. *If $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, then any ball B_0 in $\mathbb{R}^n$*

$$\|f-f_{B_0}\|_{L_p(B_0)}\le c\|f^{\#}_{B_0}\|_{L_p(B_0)}. \tag{2.12}$$

Proof. Since by Proposition 2.10

$$[(f-f_{B_0})\chi_{B_0}]^{*}(t)\le c\int_t^{|B_0|}(f^{\#}_{B_0})^*(s)\frac{ds}{s}+c(f^{\#}_{B_0})^*(|B_0|),\ (0<t<|B_0|)\,,$$

applying Hardy's inequality ([5], III (3.19)), we get

$$\begin{aligned}\|f-f_{B_0}\|_{L_p(B_0)}&=\left(\int_0^{|B_0|}([(f-f_{B_0})\chi_{B_0}]^*(t))^p\,dt\right)^{\frac{1}{p}}\\ &\le c\left(\int_0^{|B_0|}\left(f^{\#}_{B_0})^*(t)\right)^p dt\right)^{\frac{1}{p}}=c\|f^{\#}_{B_0}\|_{L_p(B_0)}.\end{aligned}$$

□

3. L_p-estimates of Riesz potential over cubes

Let us denote by $L_p^{\mathrm{loc},+}(\mathbb{R}^n)$ the set of non-negative functions from $L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Let $1<p<\infty$, $f\in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Denote by $2Q=Q(x_0,2r)$. For any $Q\subset\mathbb{R}^n$ we have

$$\|I_\alpha f\|_{L_p(Q)}\le\|I_\alpha(f\chi_{(2Q)})\|_{L_p(Q)}+\|I_\alpha(f\chi_{\complement_{(2Q)}})\|_{L_p(Q)} \tag{3.1}$$

It's clear that $x\in Q$, $y\in {}^{\complement}(2Q)$ implies $|y-x|\sim|y-x_0|$. Therefore

$$\|I_\alpha(f\chi_{\complement_{(2Q)}})\|_{L_p(Q)}\approx c|Q|^{\frac{1}{p}}\int_{\mathbb{R}^n\setminus(2Q)}\frac{|f(y)|}{|y-x_0|^{n-\alpha}}dy. \tag{3.2}$$

Corollary 3.1. *Let* $1 < p < \infty$, $0 < \alpha < n$, $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. *Then for any cube* $Q = Q(x_0, r) \subset \mathbb{R}^n$

$$\|I_\alpha f\|_{L_p(Q)} \approx \|I_\alpha(f\chi_{(2Q)})\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus (2Q)} \frac{|f(y)|}{|y - x_0|^{n-\alpha}} dy \tag{3.3}$$

holds.

Let us estimate $\|I_\alpha(f\chi_{(2Q)})\|_{L_{p_2}(Q)}$. The following lemma is true

Lemma 3.2. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$ *and* $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then for any cube* $Q \subset \mathbb{R}^n$

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_{p_2}(Q)} \lesssim |Q|^{\frac{\alpha}{n} - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2Q)}. \tag{3.4}$$

Proof. Suppose that $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$. Then by Sobolev's theorem we have

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_{p_2}(Q)} \lesssim \|f\|_{L_{\frac{p_2 n}{n+\alpha p_2}}(2Q)}.$$

If $\frac{p_2 n}{n+\alpha p_2} = p_1$, then we arrive at (3.4). If $p_1 > \frac{p_2 n}{n+\alpha p_2}$, then applying Hölder's inequality (with exponents $\frac{p_1(n+\alpha p_2)}{p_2 n}$ and $(\frac{p_1(n+\alpha p_2)}{p_2 n})'$) we get (3.4).

Assume that $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$. Since

$$\begin{aligned} \int_Q \left(I_\alpha(f\chi_{(2Q)})(x)\right)^{p_2} dx &\le \int_0^{|Q|} \left[\left(I_\alpha(f\chi_{(2Q)})\right)^*(t)\right]^{p_2} dt \\ &\le \left[\sup_{0<t<|Q|} t^{\frac{n-\alpha}{n}} \left(I_\alpha(f\chi_{(2Q)})\right)^*(t)\right]^{p_2} \int_0^{|Q|} t^{\frac{\alpha-n}{n}p_2} dt \end{aligned} \tag{3.5}$$

Using the boundedness of I_α from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ we have

$$\int_Q \left(I_\alpha(f\chi_{(2Q)})(x)\right)^{p_2} dx \lesssim \|f\|_{L_1(2Q)}^{p_2} |Q|^{\frac{\alpha-n}{n}p_2+1} dt. \tag{3.6}$$

Therefore

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_{p_2}(Q)} \lesssim |Q|^{\frac{\alpha}{n} - \left(1 - \frac{1}{p_2}\right)} \|f\|_{L_1((2Q))}. \tag{3.7}$$

If $p_1 = 1$, then we arrive at (3.4). If $p_1 > 1$, then applying Hölder's inequality (with exponents p_1 and p_1' we get (3.4).

Suppose that $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. Let $p_0 > p_1$ be defined by $n\left(\frac{1}{p_1} - \frac{1}{p_0}\right) = \alpha$. Then by Hölder's inequality (with exponents $\frac{p_0}{p_2}$ and $(\frac{p_0}{p_2})'$) we have

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_{p_2}(Q)} \lesssim |Q|^{\frac{1}{p_2} - \frac{1}{p_0}} \|I_\alpha(f\chi_{(2Q)})\|_{L_{p_0}(Q)}. \tag{3.8}$$

Then by Sobolev's theorem we arrive at (3.4). □

The statement of the following lemma follows from (3.1), (3.2) and Lemma 3.2.

Lemma 3.3. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$, $f \in L_{p_1}^{\rm loc}(\mathbb{R}^n)$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then for any cube* $Q = Q(x_0, r) \subset \mathbb{R}^n$

$$\|I_\alpha f\|_{L_{p_2}(Q)} \le c|Q|^{\frac{1}{p_2}} \int_{\mathbb{R}^n \setminus (2Q)} \frac{|f(y)|}{|y-x_0|^{n-\alpha}} dy + c|Q|^{\frac{\alpha}{n} - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2Q)}, \quad (3.9)$$

where constant c *does not depend on* $|Q|$.

Remark 3.4. Note that statement of Lemma 3.3 for balls had been proved in [3].

Let us recall the following Theorem proved in [3]

Theorem 3.5. *Let* $0 < \alpha < n$, $0 < p < \infty$, $\frac{pn}{n+\alpha p} < 1$, *and* $f \in L_p^{\rm loc}(\mathbb{R}^n)$. *Then*

$$\|I_\alpha |f|\|_{L_p(B(x,r))} \approx r^{\frac{n}{p}} \int_{\mathbb{R}^n \setminus B(x,r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy + r^{\alpha - n\left(1-\frac{1}{p}\right)} \int_{B(x,r)} |f(y)| dy. \quad (3.10)$$

4. Relation between $\|I_\alpha f\|_{L_p(Q)}$ and $\|M_\alpha f\|_{L_p(Q)}$

By means of properties of Riesz potential and fractional maximal function obtained in previous two sections we are able to prove the following Theorem.

Theorem 4.1. *Let* $1 < p < \infty$ *and* $f \in L_p^{\rm loc,+}(\mathbb{R}^n)$. *Then for any cube* $Q = Q(x_0, r_0)$

$$\|I_\alpha f\|_{L_p(Q)} \approx \|M_\alpha f\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus Q} \frac{f(y)dy}{|y-x_0|^{n-\alpha}} \quad (4.1)$$

Proof. Let us assume that $f \ge 0$ a.e.. By Corollary 2.5 and Proposition 2.3 we obtain

$$\|I_\alpha f - (I_\alpha f)_Q\|_{L_p(Q)} \le c\|(I_\alpha f)^{\#}\|_{L_p(Q)} \le c\|M_\alpha f\|_{L_p(Q)}. \quad (4.2)$$

Since

$$\|I_\alpha f\|_{L_p(Q)} \le \|I_\alpha f - (I_\alpha f)_Q\|_{L_p(Q)} + |Q|^{\frac{1}{p}} (I_\alpha f)_Q,$$

by (4.2) we arrive at

$$\|I_\alpha f\|_{L_p(Q)} \le c\|M_\alpha f\|_{L_p(Q)} + c|Q|^{\frac{1}{p}} (I_\alpha f)_Q.$$

Applying Lemma 3.3 for $p_1 = 1$, $p_2 = 1$, we get

$$\begin{aligned}
\|I_\alpha f\|_{L_p(Q)} &\le c\|M_\alpha f\|_{L_p(Q)} + c|Q|^{\frac{1}{p}-1} |Q|^{\frac{\alpha}{n}} \|f\|_{L_p(2Q)} \\
&\quad + c|Q|^{\frac{1}{p}} \left(\int_{\mathbb{R}^n \setminus 2Q_0} \frac{f(y)dy}{|y-x_0|^{n-\alpha}} dy \right) \\
&\le c|Q|^{\frac{1}{p}} \inf_{x \in Q} M_\alpha f(x) + c|Q|^{\frac{1}{p}} \left(\int_{\mathbb{R}^n \setminus 2Q_0} \frac{f(y)dy}{|y-x_0|^{n-\alpha}} dy \right) \\
&\le c\|M_\alpha f\|_{L_p(Q)} + c|Q|^{\frac{1}{p}} \left(\int_{\mathbb{R}^n \setminus 2Q_0} \frac{f(y)dy}{|y-x_0|^{n-\alpha}} dy \right).
\end{aligned}$$

The reverse eatimate is clear from (1.1) and (3.2). □

In view of Corollary 2.11 and Remark 3.4 it is clear that the following Lemma is true.

Lemma 4.2. *Let* $1 < p < \infty$ *and* $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. *Then for any ball* $B = B(x_0, r_0)$

$$\|I_\alpha f\|_{L_p(B)} \approx \|M_\alpha f\|_{L_p(B)} + |B|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus B} \frac{f(y)dy}{|y - x_0|^{n-\alpha}} \tag{4.3}$$

Remark 4.3. Since for finite functions from $L_p^{\mathrm{loc},+}(\mathbb{R}^n)$

$$(I_\alpha f)^{\#}(x) \sim M_\alpha f(x), \text{ for any } x \in \mathbb{R}^n$$

(see [1], Proposition 3.3 and 3.4), then the inequality (4.3) could be written in the following form

$$\|I_\alpha f\|_{L_p(Q)} \approx \|(I_\alpha f)^{\#}\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{\mathbb{R}^n \setminus Q} \frac{f(y)dy}{|y - x_0|^{n-\alpha}}. \tag{4.4}$$

This inequality suggests to us that inequality (4.8) in [2] hardly to hold. Next example proves our doubt: For $0 < r < R/4$ consider the finite function

$$f(y) = |y|^{-\alpha} \chi_{B(0,R) \setminus B(0,2r)}(y).$$

It is easy to calculate that

$$\|M_\alpha f\|_{L_p(B(0,r))} \approx r^{\frac{n}{p}}$$

and

$$\|I_\alpha f\|_{L_p(B(0,r))} \approx r^{\frac{n}{p}} \ln \frac{R}{r}.$$

Thus

$$\|I_\alpha f\|_{L_p(B(0,r))} \not\approx \|M_\alpha f\|_{L_p(B(0,r))} \approx \|(I_\alpha f)^{\#}\|_{L_p(B(0,r))}.$$

Lemma 4.4. *Let* $1 < p < \infty$ *and* $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. *Then for any cube* $Q \subset \mathbb{R}^n$

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_p(Q)} \approx \|M_\alpha(f\chi_{(2Q)})\|_{L_p(Q)}. \tag{4.5}$$

Proof. Let $Q = Q(x_0, r_0)$. In view of (1.1) we need to show that

$$\|I_\alpha(f\chi_{(2Q)})\|_{L_p(Q)} \lesssim \|M_\alpha(f\chi_{(2Q)})\|_{L_p(Q)}. \tag{4.6}$$

By Theorem 4.1 we have

$$\|I_\alpha(f\chi_{2Q})\|_{L_p(Q)} \lesssim \|M_\alpha(f\chi_{2Q})\|_{L_p(Q)} + |Q|^{\frac{1}{p}} \int_{2Q \setminus Q} \frac{f(y)dy}{|y - x_0|^{n-\alpha}} \tag{4.7}$$

But if $y \in 2Q \setminus Q$, then $|y - x_0| \backsim r_0$. Hence

$$|Q|^{\frac{1}{p}} \int_{2Q \setminus Q} \frac{f(y)dy}{|y - x_0|^{n-\alpha}} \lesssim |Q|^{\frac{1}{p}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_{2Q} f(y)dy$$

$$\lesssim |Q|^{\frac{1}{p}} \inf_{x \in Q} M_\alpha(f\chi_{2Q})(x) \lesssim \|M_\alpha(f\chi_{(2Q)})\|_{L_p(Q)}.$$

□

Remark 4.5. The same statement is true with balls instead of cubes.

5. L_p-estimates of Fractional maximal function over balls

Theorem 5.1. *Let $1 < p < \infty$, and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$*

$$\|M_\alpha f\|_{L_p(B)} \lesssim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}} \left(\sup_{t \geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy \right) \tag{5.1}$$

Proof. It is obvious that for any ball $B = B(x,r)$

$$\|M_\alpha f\|_{L_p(B)} \leq \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + M_\alpha(f\chi_{\mathbb{R}^n\setminus(2B)})\|_{L_p(B)}.$$

Let y be an arbitrary point from B. If $B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\} \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then $t \geq |z-y| \geq |z-x| - |x-y| > 2r - r = r$.

On the other hand $B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\} \subset B(x,2t)$. Indeed, $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then we get $|z-x| \leq |z-y| + |y-x| \leq t + r \leq 2t$.

Hence

$$\begin{aligned} M_\alpha(f\chi_{\mathbb{R}^n\setminus(2B)})(y) &= \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap\{\mathbb{R}^n\setminus(2B)\}} f(y)dy \\ &\lesssim \sup_{t \geq r} \frac{1}{|B(x,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} f(y)dy \\ &= \sup_{t \geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy. \end{aligned}$$

Thus

$$\|M_\alpha f\|_{L_p(B)} \lesssim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}} \left(\sup_{t \geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy \right)$$

□

Theorem 5.2. *Let $1 < p < \infty$, and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$*

$$\begin{aligned} \|M_\alpha f\|_{L_p(B)} &\gtrsim |B|^{\frac{\alpha}{n} - \left(1-\frac{1}{p}\right)} \|f\|_{L_1(B)} \\ &\quad + |B|^{\frac{1}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy \right) \end{aligned} \tag{5.2}$$

Proof. Since $B(x,\frac{t}{2}) \subset B(y,t),\ t > 2r$, then

$$M_\alpha f(y) \gtrsim \sup_{t>2r} \frac{1}{|B(x,\frac{t}{2})|^{1-\frac{\alpha}{n}}} \int_{B(x,\frac{t}{2})} = \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy,$$

thus

$$\|M_\alpha f\|_{L_p(B)} \gtrsim |B|^{\frac{1}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy \right). \tag{5.3}$$

It is clear that

$$\|M_\alpha f\|_{L_p(B)} \gtrsim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)},$$

hence

$$\|M_\alpha f\|_{L_p(B)} \gtrsim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y)dy \right).$$

On the other hand, if $y \in B(x,r)$, then $B(x,t) \subset B(y,t)$ for $t > 2r$, then

$$M_\alpha(f\chi_{(2B)})(y) \gtrsim \sup_{t>2r} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap 2B(x,r)} f(y)dy \gtrsim |B|^{\frac{\alpha}{n}-1} \int_B f(y)dy.$$

Hence

$$\|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} \gtrsim |B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)} \|f\|_{L_1(B)}.$$

□

From Remark 4.5, Theorem 5.1 and Lemma 3.1 in [3] follows next statement.

Lemma 5.3. *Let $0 < \alpha < n$, $0 < p_2 < \infty$, $f \in L_{p_1}^{\rm loc}(\mathbb{R}^n)$. Moreover, let $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. Then for any ball $B = B(x,r) \subset \mathbb{R}^n$*

$$\begin{aligned} \|M_\alpha f\|_{L_{p_2}(B)} \le c|B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(y)|dy \right) \\ + c|B|^{\frac{\alpha}{n}-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2B)}, \end{aligned} \tag{5.4}$$

where constant c does not depend on $|B|$.

Lemma 5.4. *Let $0 < \alpha < n$, $0 < p_2 < \infty$, $f \in L_{p_1}^{\rm loc}(\mathbb{R}^n)$. Moreover, let $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. Then for any ball $B = B(x,r) \subset \mathbb{R}^n$*

$$\|M_\alpha f\|_{L_{p_2}(B)} \le c|B|^{\frac{1}{p_2}} \left(\sup_{t\ge r} \frac{1}{|B(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(x,t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} \right), \tag{5.5}$$

where constant c does not depend on $|B|$.

Proof. Denote by

$$\begin{aligned} M_1 :&= |B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(y)|dy \right), \\ M_2 :&= |B|^{\frac{\alpha}{n}-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2B)}. \end{aligned}$$

Applying Hölder's inequality, we get

$$M_1 \lesssim |B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(x,t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} \right).$$

On the other hand

$$|B|^{\frac{1}{p_2}} \left(\sup_{t \ge 2r} \frac{1}{|B(x,t)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(x,t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} \right)$$
$$\gtrsim |B|^{\frac{1}{p_2}} \left(\sup_{t \ge 2r} |B(x,t)|^{\frac{\alpha}{n} - \frac{1}{p_1}} \right) \|f\|_{L_{p_1}(2B)} \approx M_2.$$

Since by Lemma 5.3

$$\|M_\alpha f\|_{L_{p_2}(B)} \le M_1 + M_2,$$

we arrive at (5.5). □

Remark 5.5. Inequality (5.5) improves the inequality (22) in [4]

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\frac{n}{p_2}} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n - \alpha p_1 + 1}} \right)^{\frac{1}{p_1}}.$$

This follows since

$$\sup_{t \ge r} \frac{1}{|B(0,t)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(0,t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}}$$
$$\le \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n - \alpha p_1 + 1}} \right)^{\frac{1}{p_1}}$$

Indeed, by easy calculation and the Fubini theorem, we get

$$\sup_{t \ge r} \frac{1}{|B(0,t)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(0,t)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}}$$
$$\le \sup_{t \ge r} \frac{1}{|B(0,t)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(0,r)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}}$$
$$+ \sup_{t \ge r} \frac{1}{|B(0,t)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(0,t) \setminus B(0,r)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}}$$
$$\le \frac{1}{|B(0,r)|^{\frac{1}{p_1} - \frac{\alpha}{n}}} \left(\int_{B(0,r)} |f(y)|^{p_1} dy \right)^{\frac{1}{p_1}} + \left(\int_{\mathbb{R}^n \setminus B(0,r)} \frac{|f(x)|^{p_1}}{|x|^{n - \alpha p_1}} dx \right)^{\frac{1}{p_1}}$$
$$\lesssim \left(\int_r^\infty \left(\int_{B(0,r)} |f(y)|^{p_1} dy \right) \frac{dt}{t^{n - \alpha p_1 + 1}} \right)^{\frac{1}{p_1}}$$
$$+ \left(\int_r^\infty \left(\int_{B(0,t) \setminus B(0,r)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n - \alpha p_1 + 1}} \right)^{\frac{1}{p_1}}$$
$$\lesssim \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n - \alpha p_1 + 1}} \right)^{\frac{1}{p_1}}.$$

The following Theorem is true.

Theorem 5.6. *Let* $0<\alpha<n$, $0<p<\infty$, $\frac{pn}{n+\alpha p}<1$, *and* $f\in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. *Then for any ball* $B=B(x,r)\subset\mathbb{R}^n$

$$\begin{aligned}\|M_\alpha f\|_{L_p(B)} \approx r^{\frac{n}{p}}\left(\sup_{t>r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right)\\ + r^{\alpha-n\left(1-\frac{1}{p}\right)}\int_{B(x,r)}f(y)dy.\end{aligned}\tag{5.6}$$

Proof. In view of the Theorem 5.2 we need only to prove that

$$\begin{aligned}\|M_\alpha f\|_{L_p(B)} \lesssim r^{\frac{n}{p}}\left(\sup_{t>r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right)\\ + r^{\alpha-n\left(1-\frac{1}{p}\right)}\int_{B(x,r)}f(y)dy.\end{aligned}$$

By Theorem 5.1 and Lemma 4.4, we have

$$\|M_\alpha f\|_{L_p(B)} \lesssim \|I_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}}\left(\sup_{t\geq 2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right).$$

Taking into account Lemma 3.1 in [3], we get

$$\|M_\alpha f\|_{L_p(B)} \lesssim |B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)}\|f\|_{L_1(2B)} + |B|^{\frac{1}{p}}\left(\sup_{t\geq 2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right).$$

Then

$$\begin{aligned}\|M_\alpha f\|_{L_p(B)} &\lesssim |B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)}\|f\|_{L_1(B)} + |B|^{\frac{1}{p}}\left(\sup_{r\leq t<2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right)\\ &\quad + |B|^{\frac{1}{p}}\left(\sup_{t\geq 2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right)\\ &\approx |B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)}\|f\|_{L_1(B)} + |B|^{\frac{1}{p}}\left(\sup_{t\geq r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}f(y)dy\right).\end{aligned}$$

□

6. A note on some Hardy-type inequalities

In this section we present some results about Hardy inequality, which is needed to prove main theorem.

Lemma 6.1. *Let* $\beta>0$, $\delta>0$ *and* ω *be a positive weight function defined on* $(0,\infty)$. *Then the following inequality*

$$\sup_{r>0}\omega(r)r^\beta\int_r^\infty\frac{g(t)}{t^\delta}dt \lesssim \sup_{r>0}\omega(r)r^\beta\left(\sup_{t>r}t^{-\delta}\int_0^t g(s)ds\right)\tag{6.1}$$

holds for any non-negative measurable g *on* $(0,\infty)$ *if and only if*

$$\int_x^\infty t^{-\delta-1}\,(\psi(t))^{-1}\,dt \le c\,x^{-\delta}\,(\psi(x))^{-1},\ x>0, \tag{6.2}$$

where

$$\psi(x) = \sup_{x<s<\infty} s^{-\delta} \sup_{0<\tau<s} \omega(\tau)\tau^{\beta}.$$

Proof. Denote by

$$A := \sup_{g\ge 0} \frac{\sup_{r>0} \omega(r) r^{\beta} \int_r^\infty \frac{g(t)}{t^{\delta}} dt}{\sup_{r>0} \omega(r) r^{\beta} \left(\sup_{t>r} t^{-\delta} \int_0^t g(s)ds\right)}.$$

Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-increasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(0,t)} G(s),\ t\in(0,\infty); \tag{6.3}$$

likewise, when F is non-decreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} G(s),\ t\in(0,\infty). \tag{6.4}$$

By standart duality argument, the Fubini theorem and (6.3), we have for A

$$A = \sup_{g\ge 0}\sup_{\varphi\ge 0} \frac{\int_0^\infty \left(\int_r^\infty \frac{g(t)}{t^{\delta}} dt\right)\varphi(r)dr}{\int_0^\infty \omega^{-1}(r) r^{-\beta}\varphi(r)dr} \frac{1}{\sup_{r>0} t^{-\delta}\left(\sup_{0<s<r}\omega(s)s^{\beta}\right)\left(\int_0^t g(s)ds\right)}$$

$$= \sup_{\varphi\ge 0} \frac{1}{\int_0^\infty \omega^{-1}(r) r^{-\beta}\varphi(r)dr} \sup_{g\ge 0} \frac{\int_0^\infty t^{-\delta}\left(\int_0^t \varphi(r)dr\right) g(t)dt}{\sup_{t>0} t^{-\delta} W(t)\left(\int_0^t g(s)ds\right)},$$

where $W(t) = \sup_{0<s<t}\omega(s)s^{\beta}$. By Theorem 5.4 in [6]

$$\sup_{g\ge 0} \frac{\int_0^\infty t^{-\delta}\left(\int_0^t \varphi(r)dr\right) g(t)dt}{\sup_{t>0} t^{-\delta} W(t)\left(\int_0^t g(s)ds\right)}$$
$$\approx \int_0^\infty \sup_{t<s<\infty}\left(s^{-\delta}\int_0^s \varphi(r)dr\right) d\left(\sup_{t<s<\infty} s^{-\delta} W(s)\right)^{-1}.$$

Thus

$$A \approx \frac{\int_0^\infty \sup_{t<s<\infty}\left(s^{-\delta}\int_0^s \varphi(r)dr\right) d\left(\sup_{t<s<\infty} s^{-\delta}W(s)\right)^{-1}}{\int_0^\infty \omega^{-1}(r) r^{-\beta}\varphi(r)dr}.$$

Applying Theorem 4.1 in [7], we get

$$A \approx \sup_{x>0}\left(x^{-\delta}\int_0^x d\left(\sup_{t<s<\infty} s^{-\delta}W(s)\right)^{-1} + \int_x^\infty t^{-\delta} d\left(\sup_{t<s<\infty} s^{-\delta}W(s)\right)^{-1}\right)\sup_{0<t<x}\omega(t)t^\beta < \infty.$$

Applying (6.4), we get

$$\sup_{x>0} x^{-\delta}\int_0^x d\left(\sup_{t<s<\infty} s^{-\delta}W(s)\right)^{-1}\sup_{0<t<x}\omega(t)t^\beta = \sup_{x>0}\left(\sup_{x<s<\infty} s^{-\delta}\sup_{0<t<s}\omega(t)t^\beta\right)^{-1}\sup_{x<s<\infty} s^{-\delta}\sup_{0<t<s}\omega(t)t^\beta = 1.$$

Hence $A<\infty$ if and only if

$$\sup_{x>0}\left(\int_x^\infty t^{-\delta} d\left(\sup_{t<s<\infty} s^{-\delta}\sup_{0<\tau<s}\omega(\tau)\tau^\beta\right)^{-1}\right)\sup_{0<t<x}\omega(t)t^\beta < \infty.$$

Integrating by part, we obtain

$$\sup_{x>0}\left(\int_x^\infty t^{-\delta} d\left(\sup_{t<s<\infty} s^{-\delta}\sup_{0<\tau<s}\omega(\tau)\tau^\beta\right)^{-1}\right)\sup_{0<t<x}\omega(t)t^\beta \le c\sup_{x>0}\left(\int_x^\infty t^{-\delta-1}\left(\sup_{t<s<\infty} s^{-\delta}\sup_{0<\tau<s}\omega(\tau)\tau^\beta\right)^{-1} dt\right)\sup_{0<t<x}\omega(t)t^\beta = c\sup_{x>0} x^\delta\left(\int_x^\infty t^{-\delta-1}\left(\psi(t)\right)^{-1} dt\right) x^{-\delta}\sup_{0<t<x}\omega(t)t^\beta.$$

It is easy to see that $x^\delta\left(\int_x^\infty t^{-\delta-1}\left(\psi(t)\right)^{-1} dt\right)$ is non-decreasing function, then by (6.4) we get

$$\sup_{x>0} x^\delta\left(\int_x^\infty t^{-\delta-1}\left(\psi(t)\right)^{-1} dt\right) x^{-\delta}\sup_{0<t<x}\omega(t)t^\beta = \sup_{x>0} x^\delta\left(\int_x^\infty t^{-\delta-1}\left(\psi(t)\right)^{-1} dt\right)\psi(x).$$

Consequently, the condition

$$\int_x^\infty t^{-\delta-1}\left(\psi(t)\right)^{-1} dt \le c x^{-\delta}\left(\psi(x)\right)^{-1},\ x>0$$

is necessary and sufficient for $A<\infty$. □

7. Equivalence of norms of Riesz potential and fractional maximal function in Morrey space

In this section we find the condition on the ω which ensures the equivalency of norms of the Riesz potential and the fractional maximal function in generalized Morrey space $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

From Lemma 4.2 follows the next statement.

Theorem 7.1. *Let $1 < p < \infty$, $0 < \alpha < n$, ω be a positive weight function defined on $(0,\infty)$ and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\omega}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\omega}}$$

if and only if

$$\sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \int_{\mathbb{R}^n\setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \left(\int_{B(x,r)} (M_\alpha f(y))^p dy \right)^{\frac{1}{p}}.$$

By Theorem 3.5 and Theorem 5.6, we conclude that for small p the statement is true.

Theorem 7.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{n-\alpha}$, ω be a positive weight function defined on $(0,\infty)$ and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\omega}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\omega}}$$

if and only if

$$\sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \int_{\mathbb{R}^n\setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \left(\sup_{t>r} t^{\alpha-n} \int_{B(x,t)} f(y) dy \right).$$

The next Theorem is true.

Theorem 7.3. *Let $1 < p < \infty$, $0 < \alpha < n$, ω be a positive weight function defined on $(0,\infty)$ and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. If*

$$\int_x^\infty t^{\alpha-n-1} (\psi(t))^{-1} dt \le c x^{\alpha-n} (\psi(x))^{-1}, \; x > 0, \tag{7.1}$$

where

$$\psi(x) = \sup_{x<s<\infty} s^{\alpha-n} \sup_{0<\tau<s} \omega(\tau) \tau^{\frac{n}{p}},$$

then

$$\|I_\alpha f\|_{\mathcal{M}_{p,\omega}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\omega}}. \tag{7.2}$$

Proof. In view of (1.1) we need only to prove

$$\|I_\alpha f\|_{\mathcal{M}_{p,\omega}} \lesssim \|M_\alpha f\|_{\mathcal{M}_{p,\omega}}. \tag{7.3}$$

With regard to Lemma 4.2 and Theorem 5.1, it will suffice to show that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \int_{\mathbb{R}^n\setminus B(x,r)} \frac{f(y)}{|y-x|^{n-\alpha}} dy \\ \lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} f(y) dy \right).\end{aligned} \tag{7.4}$$

Passing to spherical coordinates, we obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \int_r^\infty \left(\int_\Sigma f(x+\rho\xi) d\xi \right) \rho^{\alpha-1} d\rho \\ \lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega(r) r^{\frac{n}{p}} \left(\sup_{t>r} t^{\alpha-n} \int_0^t \left(\int_\Sigma f(x+\rho\xi) d\xi \right) \rho^{n-1} d\rho \right),\end{aligned} \tag{7.5}$$

where Σ denotes unit sphere in $\mathbb{R}^n$.

Let us reduce this task to more simply, but equivalent one:

$$\sup_{r>0} \omega(r) r^{\frac{n}{p}} \int_r^\infty \frac{g(t)}{t^{n-\alpha}} dt \lesssim \sup_{r>0} \omega(r) r^{\frac{n}{p}} \left(\sup_{t>r} t^{\alpha-n} \int_0^t g(s) ds \right), \tag{7.6}$$

for all non-negative measurable functions g on $(0,\infty)$. But then the statement of Theorem immediately follows from Lemma 6.1. □

Theorem 7.2 and the preceding Theorem imply the following statement.

Theorem 7.4. *Let $0 < \alpha < n$, $1 < p < \frac{n}{n-\alpha}$, ω be a positive weight function defined on $(0,\infty)$ and $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$. Then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\omega}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\omega}} \tag{7.7}$$

if and only if (7.1) holds.

From the Theorem 7.3 follows the following Corollary.

Corollary 7.5. *Let $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda \le n$. If $f \in L_p^{\mathrm{loc},+}(\mathbb{R}^n)$, then*

$$\|I_\alpha f\|_{\mathcal{M}_{p,\lambda}} \sim \|M_\alpha f\|_{\mathcal{M}_{p,\lambda}}. \tag{7.8}$$

Proof. It's easy see that if $\alpha - n + \frac{n-\lambda}{p} \le 0$, then the statement immediately follows from Theorem 7.3. But if $\alpha - n + \frac{n-\lambda}{p} > 0$, then

$$\sup_{r>0} r^{-\frac{\lambda}{p}} \|M_\alpha f\|_{L_p(B(0,r))} = \infty.$$

Indeed, since

$$M_\alpha f(x) \gtrsim \sup_{t\ge 2r} t^{\alpha-n} \int_{B(0,t)} f(y) dy \gtrsim r^{\alpha-n} \int_{B(0,r)} f(y) dy, \ x \in B(0,r),$$

then

$$\|M_\alpha f\|_{L_p(B(0,r))} \gtrsim r^{\frac{n}{p}+\alpha-n} \int_{B(0,r)} f(y) dy.$$

Thus

$$\begin{aligned}\|I_\alpha f\|_{\mathcal{M}_{p,\lambda}} \gtrsim \|M_\alpha f\|_{\mathcal{M}_{p,\lambda}} &= \sup_{r>0} r^{-\frac{\lambda}{p}} \|M_\alpha f\|_{L_p(B(0,r))} \\ &\geq \sup_{r>0} r^{\alpha-n+\frac{n-\lambda}{p}} \int_{B(0,r)} f(y)dy = \infty,\end{aligned}$$

or $f = 0$ a.e. on $\mathbb{R}^n$. In both cases (7.8) holds. □

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References

[1] D.R.Adams, *A note on Riesz potentials* . Duke Math., 42 (1975), p.765-778.

[2] D.R.Adams, J.Xiao, *Nonlinear potential analysis on Morrey spaces and their capacities.* Indiana University Mathematics Journal , Vol. 53, No. 6 (2004), p.1629-1663.

[3] V.Burenkov, A.Gogatishvili, V.Guliyev, R.Mustafayev, Sufficient conditions for boundedness of the fractional integral operators in local Morrey-type spaces. Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-21.

[4] V.I.Burenkov, H.V.Guliyev, V.S.Guliyev, *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces.* J. of Comt. and Appl. Math. 208 (1) (2007), p.280-301.

[5] C.Bennett, R.Sharpley, *"Interpolation of operators".* Pure and Applied Mathematics 129, Academic Press. Inc., Boston, MA, 1988.

[6] W.D.Evans, A.Gogatishvili, B.Opic *The reverse Hardy inequality with measures* . Math. Ineq. and Appl. Vol. 11, No. 1, (2008), p.43–74.

[7] A.Gogatishvili, B.Opic, L.Pick *Weighted inequalities for Hardy-type operators involving suprema* . Collect. Math. Vol. 57, No. 3 (2006), p.227–255.

[8] V.I.Kolyada *The embedding of certain classes of functions of several variables* . Sibirsk. Mat. Zh. 14 (1973), 766-790. = Siberian Math. J. 14 (1973), 530-546.

[9] C.B.Morrey, *On the solutions of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc. 43 (1938), 126-166.

[10] E.M.Stein, *"Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals".* Princeton Univ. Press, Princeton, NJ, 1993.

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