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Existence of a Weak Solution to the Navier–Stokes Equation with Navier's Boundary Condition around Striking Bodies

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Abstract

We assume that Ω^t (for $t \in [0,T]$) is a time varying domain in $\mathbb{R}^3$, which is the exterior of several compact bodies moving in a container and striking at time instants $t \in \mathcal{T}^c$, where $\mathcal{T}^c$ is a finite subset of $(0,T)$. We consider the Navier–Stokes equation with Navier's slip boundary condition and we prove its weak solvability in $Q_{(0,T)} := \{(\mathbf{x},t);\ 0<t<T,\ \mathbf{x} \in \Omega^t\}$. We show that Navier's boundary condition enables us to consider a different class of collisions than the usual no–slip Dirichlet boundary condition.

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1 Introduction

A global (in time) weak solvability of the Navier–Stokes equation with the no–slip Dirichlet boundary condition in a fixed domain $\Omega \subset \mathbb{R}^3$ is a classical result of the qualitative theory of the Navier–Stokes equation, see e.g. J. Leray [19] (1934), E. Hopf [16] (1950), O. A. Ladyzhenskaya [17] (1969), J. L. Lions [21] (1969), R. Temam [30] (1977) or G. P. Galdi [11] (2000).

The proof of the same result in a time variable domain Ω^t represents a subtler problem, especially due to the dependence of various constants in imbedding inequalities and in estimates of traces on the concrete shape of Ω^t. The first proof of the global (in time) weak solvability of the Navier–Stokes equation with the no–slip boundary condition, in a time–varying domain Ω^t with a prescribed form at each time t, was published by H. Fujita and N. Sauer [8] (1970). The authors assumed that the boundary of the variable domain Ω^t consists of a finite number of moving simple closed surfaces of the class C^3 so that the distance of any two of these surfaces is never less than $d>0$.) The result was recently generalized by J. Neustupa [22] (2007; Ω^t has an arbitrary shape and smoothness, the assumptions on Ω^t involve simulation of collisions of bodies moving in a fluid). The existence and uniqueness of a strong solution in domain Ω^t with given smooth moving boundaries was proved by O. A. Ladyzhenskaya [18] (1968; globally in time for sufficiently small data or locally in time for large data).

A series of other works, studying the Navier–Stokes equation in a time varying domain, appeared in the last decade. The works we have in mind describe the motion of one or more bodies in a fluid and the authors consider the system fluid–bodies to be interconnected so that the position of the bodies in the fluid is not known in advance. In addition to the Navier–Stokes equation and the equation of continuity (which describe the motion of the fluid), the authors also consider equations describing the motion of the bodies in the fluid in dependence on forces and torques resulting from the action of the fluid on the boundary of the bodies. Of all papers belonging to this category, let us name e.g. K. H. HOFFMANN, V. N. STAROVOITOV [14] (1999) and [15] (2000), B. DESJARDINS, M. J. ESTEBAN [3] (1999) and [4] (2000), C. CONCA, J. SAN MARTÍN, M. TUCSNAK [2] (2000), M. D. GUNZBURGER, H. C. LEE, G. SEREGIN [13] (2000), J. SAN MARTÍN, V. N. STAROVOITOV, M. TUCSNAK [23] (2002), E. FEIREISL [6] (2003), T. TAKAHASHI [27] and [28] (both 2003) T. TAKAHASHI, M. TUCSNAK [29] (2004), V. N. STAROVOITOV [25] (2005). The results presented in these papers concern 2D and 3D cases and they involve theorems on the global in time existence of weak solutions or the local in time existence of a strong solution. Some of the papers admit collisions of the bodies moving in the fluid. Another series of papers studies the motion of the interconnected system fluid–body under the assumption that the body is able to produce a certain velocity profile on its surface and it moves do to this velocity. The survey of results on these so called "self–propelled bodies" can be found in the work [11] (2002) by G. P. GALDI.

All the works cited above consider the homogeneous Dirichlet boundary condition for velocity on the boundary of domain Ω^t filled by the fluid.

V. N. STAROVOITOV [24] (2003) derived necessary conditions for the existence of a weak solution of the Navier–Stokes equation in a time variable domain Ω^t, which is an exterior of several solid bodies moving in the fluid, considering also the no–slip Dirichlet boundary condition on the surface of the bodies. The conditions show that if the bodies have boundaries of the class C^2 then they can strike only with the speed equal to zero at the instant of the collision, otherwise the weak solution cannot exist.

Motivated by this state, we study the flow of a viscous incompressible fluid around moving bodies under the assumption that the velocity of the fluid satisfies Naviers slip boundary condition on the boundary. We assume that the motion of the bodies is a priori known.

All assumptions we impose on the timevariable domain Ω^t, occupied by the fluid at time t, are in detail listed in Section 2. We prove the global in time existence of a weak solution of the initialboundary value problem

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p \;=\; \nu\Delta\mathbf{v} + \mathbf{f} \qquad \text{in } Q_{(0,T)}, \tag{1.1}$$

$$\operatorname{div}\mathbf{v} \;=\; 0 \qquad \text{in } Q_{(0,T)}, \tag{1.2}$$

$$\mathbf{v}\cdot\mathbf{n} \;=\; \mathbf{V}\cdot\mathbf{n} \qquad \text{in } \Gamma_{(0,T)}, \tag{1.3}$$

$$[\mathbb{T}_{\rm d}(\mathbf{v})\cdot\mathbf{n}]_\tau + \gamma(\mathbf{v}-\mathbf{V}) \;=\; \mathbf{0} \qquad \text{in } \Gamma_{(0,T)}, \tag{1.4}$$

$$\mathbf{v} \;=\; \mathbf{v}_0 \qquad \text{in } \Omega^0\times\{0\}, \tag{1.5}$$

where the operators div and ∇ act on the spatial variables, $Q_{(0,T)}$ denotes a space–time cylinder in $\mathbb{R}^4$ whose intersection $\Omega^t\times\{t\}$ with the time level t varies along the time axis and $\Gamma_{(0,T)}$ is the envelope of $Q_{(0,T)}$:

$$Q_{(0,T)} \;:=\; \big\{(\mathbf{x},t)\in\mathbb{R}^4;\ 0<t<T,\ \mathbf{x}\in\Omega^t\big\}, \tag{1.6}$$

$$\Gamma_{(0,T)} \;:=\; \big\{(\mathbf{x},t)\in\mathbb{R}^4;\ 0<t<T,\ \mathbf{x}\in\Gamma^t\big\}. \tag{1.7}$$

The equations (1.1), (1.2) describe the motion of a Newtonian viscous incompressible fluid in domain Ω^t. The density of the fluid is supposed to be one. The symbols $\mathbf{v}$, p, ν and $\mathbf{f}$ in equations

(1.1) and (1.2) successively denote the velocity of the fluid, the pressure, the kinematic coefficient of viscosity and the specific external body force. Condition (1.3) expresses the impermeability of Γ^t. Here $\mathbf{n}$ denotes the outer normal vector and $\mathbf{V}(\,.\,,t)$ is the velocity of "material points" on the boundary Γ^t of Ω^t. Condition (1.4) is due to H. NAVIER, who proposed in 1824 that the tangential component of the stress acting on the boundary should be proportional to the relative velocity of the fluid with respect to the material boundary. Here $\mathbb{T}_{\rm d}(\mathbf{v})$ denotes the dynamic stress tensor associated with the flow $\mathbf{v}$. It has the form $\mathbb{T}_{\rm d}(\mathbf{v}) = 2\nu\,(\nabla\mathbf{v})_s$ where $(\nabla\mathbf{v})_s$ is the symmetrized gradient of $\mathbf{v}$. The subscript τ denotes the tangential component to Γ^t. The positive constant γ is the coefficient of friction between the fluid and the boundary.

The problem is treated on a relatively general level in Sections 2-7. The definition of its weak solutions and our main result are given in Section 3. In Section 8, we consider a concrete example when Ω^t is the exterior domain of two moving bodies, striking at the time instant $t^c \in (0,T)$.

Our technique is based on the construction and estimates of the Rothe approximations. Many steps require a different approach than in the case of homogeneous Dirichlet's boundary condition. For instance, a vector function $\mathbf{u}$ from the Sobolev space $W^{1,2}_0(\Omega^t)^3$, extended by zero to $\mathbb{R}^3\smallsetminus\Omega^t$, becomes an element of $W^{1,2}(\mathbb{R}^3)^3$. Consequently, its norm in $L^6(\Omega^t)^3$ can be estimated from above by a constant times the norm in $W^{1,2}(\Omega^t)^3$, where the value of the constant is independent of Ω^t. The same consideration is, however, impossible if one assumes that $\mathbf{u}\in W^{1,2}(\Omega^t)^3$ satisfies Navier's slip boundary condition instead of Dirichlet's no–slip condition. Other difficulties arise in the part where we treat the limit transition in the nonlinear term. The standard argument based on the control of the time oscillations of the approximations and application of the Lions–Aubin lemma cannot be used in a usual way. Instead of this, we prove a strong convergence of certain local Helmoltz projections of the approximations, which turns out to be enough in order to verify the correctness of the limit transition, see Section 7. Note that the similar idea was already used by K. H. HOFFMANN, V. N. STAROVOITOV in [14] in the case of a 2D flow around a smooth body moving in a smooth tank, with the Dirichlet no–slip boundary condition on the boundary.

Notation of norms and function spaces.

- $(\,.\,,\,.\,)_{2;\,\Omega^t}$ is the scalar product and $\|\,.\,\|_{2;\,\Omega^t}$ is the norm in $L^2(\Omega^t)$ or in $L^2(\Omega^t)^3$ or in $L^2(\Omega^t)^9$, respectively. The meaning of $\|\,.\,\|_{q;\,\Omega^t}$ or $(\,.\,,\,.\,)_{2;\,\Gamma^t}$ and $\|\,.\,\|_{2;\,\Gamma^t}$ is analogous.
- $C^\infty_{0,\sigma}(\Omega^t)$ is the linear space of infinitely differentiable divergence–free vector–functions in Ω^t with a compact support in Ω^t.
- $L^q_\sigma(\Omega^t)$ is the closure of $C^\infty_{0,\sigma}(\Omega^t)$ in $L^q(\Omega^t)^3$ (for $1\le q<+\infty$).
- $W^{1,2}_\sigma(\Omega^t) := W^{1,2}(\Omega^t)^3\cap L^2_\sigma(\Omega^t)$ (with the norm $\|\,.\,\|_{1,2;\,\Omega^t}$ as in $W^{1,2}(\Omega^t)^3$)

2 General assumptions on domain Ω^t and realization of the boundary condition (1.3)

2.1 The structure of domain Ω^t. Let $T>0$. We are motivated by the following situation: K solid bodies move in the fluid in a fixed container D in the time interval $(0,T)$ so that their positions are given in advance and they do not depend on the motion of the fluid. Thus, we assume that the time variable domain Ω^t, filled by the fluid, has the form

$$\Omega^t \;=\; D\smallsetminus\cup_{k=1}^K B_k^t \qquad \text{for } 0\le t\le T, \tag{2.1}$$

where $B_1^t,\ \dots,\ B_K^t$ are compact regions occupied by the bodies at time t. (We shall further identify names of the bodies with the names of these regions.) The bodies can strike themselves or the boundary of the container at certain critical instants of time $t_1^c,\ \dots,\ t_M^c$ in the interval $(0,T)$. We denote the set of these critical times by $\mathcal{T}^c$. If $t \in \mathcal{T}^c$ then the bodies touch themselves or the boundary of D only by some points on their boundaries. Otherwise, at times $t \in [0,T] \smallsetminus \mathcal{T}^c$, the sets $B_1^t,\ \dots,\ B_K^t$ are mutually disjoint and contained in D. We assume that

(a1) D and the interiors of sets B_k^t ($k = 1, \dots, K$) are Lipschitz domains in $\mathbb{R}^3$ with piecewise C^1 boundaries.

(By a "piecewise C^1 boundary" we mean a boundary which is a union of a finite number of surfaces of the class C^1.)

Let us further denote by $\overline{\mathbf{V}}_k(t)$ the translational velocity, by $\overline{\boldsymbol{\omega}}_k(t)$ the rotational velocity and by $\overline{\mathbf{x}}_k(t)$ the center of rotation of the k–the body B_k^t at time instants $t \in (0,T) \smallsetminus \mathcal{T}^c$. Hence material points $\mathbf{x} \in B_k^t$ move with the known velocity

$$\mathbf{V}(\mathbf{x},t) \ := \ \overline{\mathbf{V}}_k(t) + \overline{\boldsymbol{\omega}}_k(t) \times \big[\mathbf{x} - \overline{\mathbf{x}}_k(t)\big] \qquad \text{for } t \in (0,T) \smallsetminus \mathcal{T}^c. \tag{2.2}$$

We assume that

(a2) $\overline{\mathbf{V}}_k$, $\overline{\boldsymbol{\omega}}_k$ and $\overline{\mathbf{x}}_k$ ($k = 1, \dots, K$) are functions from $C^2\big([0,T] \smallsetminus \mathcal{T}^c\big)^3$.

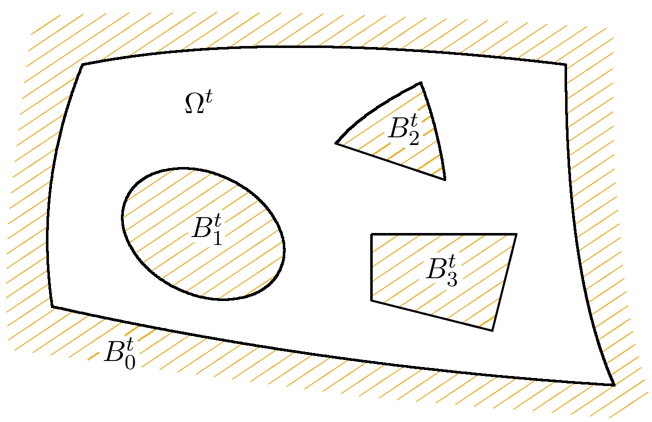

Fig. 1: A possible form of Ω^t in the case $K = 3$, $t \notin \mathcal{T}^c$ and $B_0^t \neq \emptyset$

Remark 1. In order to simplify the notation, we put $B_0^t := \mathbb{R}^3 \smallsetminus D$. Set B_0^t is clearly time–independent, we use the superscript t only in order to be consistent with the notation of the moving bodies $B_1^t,\ \dots,\ B_K^t$. Note that domain D may coincide with the whole space $\mathbb{R}^3$; in that case $B_0^t = \partial B_0^t = \emptyset$. We also extend function $\mathbf{V}$ by zero to B_0^t: $\mathbf{V}(\mathbf{x},t) := \mathbf{0}$ for $\mathbf{x} \in B_0^t$ and $0 < t < T$. Domain Ω^t can now be expressed in the form $\Omega^t = \mathbb{R}^3 \smallsetminus \cup_{k=0}^K B_k^t$. The material point in the k–th body ($k = 0, 1, \dots, K$), whose "old" position at time $t_0 \in [0,T]$ was $\mathbf{x}_0 \in B_k^{t_0}$, has a "new" position described at time t by

$$\mathbf{Y}(t;\, t_0, \mathbf{x}_0) \ = \ \mathbf{x}_0 + \int_{t_0}^t \mathbf{V}\big(\mathbf{Y}(s;\, t_0, \mathbf{x}_0), s\big)\ \mathrm{d}s \ \in \ B_k^t. \tag{2.3}$$

The function $\mathbf{Y}(t;\, t_0,\, .)$, for $t \in [0,T]$ and $t_0 \in [0,T] \smallsetminus \mathcal{T}^c$, maps the union $\cup_{k=0}^K B_k^{t_0}$ continuously onto $\cup_{k=0}^K B_k^t$. Moreover, the restriction of $\mathbf{Y}(t;\, t_0,\, .)$ to $B_k^{t_0}$ is an isometric mapping of $B_k^{t_0}$ onto B_k^t which maps $\partial B_k^{t_0}$ onto ∂B_k^t ($k = 0, \dots, K$). The restriction of $\mathbf{Y}(t;\, t_0,\, .)$ to $B_0^{t_0}$ is the identity mapping.

Remark 2. The boundary of Ω^t is denoted by Γ^t. It has the form

$$\Gamma^t \;=\; \partial D \cup \big[\cup_{k=1}^K \partial B_k^t\big] \;=\; \cup_{k=0}^K \partial B_k^t \qquad \text{for } 0 \le t \le T. \tag{2.4}$$

The sets $\partial D,\, \partial B_1^t, \dots, \partial B_K^t$ are mutually disjoint for $t \in [0,T] \smallsetminus \mathcal{T}^c$.

Remark 3. Since Ω^t is Lipschitzian for $t \in [0,T] \smallsetminus \mathcal{T}^c$, we also have $W_\sigma^{1,2}(\Omega^t) \hookrightarrow L^q(\Omega^t)^3$ for $2 \le q \le 6$ and $t \in [0,T] \smallsetminus \mathcal{T}^c$. Using the characterization of $L_\sigma^q(\Omega^t)$ (see [9, p. 111]), we can verify that $W_\sigma^{1,2}(\Omega^t) \hookrightarrow L_\sigma^q(\Omega^t)$ for these q and t.

2.2 The outer normal vector on the boundary of Ω^t. It follows from assumption (a1) that the outer normal vector field $\mathbf{n}$ is defined a.e. on Γ^t for $t \in [0,T] \smallsetminus \mathcal{T}^c$. Moreover, since Γ^t is a piecewise C^1 surface, the field $\mathbf{n}$ can be extended to the neighbourhood of Γ^t so that $\nabla \mathbf{n}$ makes sense a.e. on Γ^t for $t \in [0,T] \smallsetminus \mathcal{T}^c$. We assume that

(a3) there exists a positive constant c_1 such that

$$\int_{\Gamma^t} \boldsymbol{\phi} \cdot \nabla \mathbf{n} \cdot \boldsymbol{\phi} \, \mathrm{d}S \;\le\; c_1 \, \|\boldsymbol{\phi}\|_{2;\, \Omega^t} \big(\|\boldsymbol{\phi}\|_{2;\, \Omega^t} + \|\nabla \boldsymbol{\phi}\|_{2;\, \Omega^t} \big) \tag{2.5}$$

for $t \in [0,T] \smallsetminus \mathcal{T}^c$ and all $\boldsymbol{\phi} \in W_\sigma^{1,2}(\Omega^t)$.

Note that the integral on the left hand side of (2.5) can be estimated by a constant times $\|\boldsymbol{\phi}\|_{2;\, \Omega^t}$ times $\|\boldsymbol{\phi}\|_{1,2;\, \Omega^t}$ by means of an appropriate theorem on traces, see e.g. G. P. GALDI [9, p. 42]. Naturally, the constant in the inequality we obtain from the theorem on traces generally depends on t. Assumption (a3) thus expresses the requirement that inequality (2.5) is satisfied with constant c_1 independent of t for $t \in [0,T] \smallsetminus \mathcal{T}^c$. We shall see in Section 8 that the shape of striking bodies in the neighbourhood of points of collisions plays the decisive role in verification of condition (a3) in a concrete example.

2.3 Realization of the boundary condition (1.3) – an auxiliary function $\mathbf{a}$. In order to transform the inhomogeneous boundary condition (1.3) to the homogeneous one, we look for the solution $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{a} + \mathbf{u}$ where $\mathbf{u}$ is the new unknown function and $\mathbf{a}$ is supposed to be a known vector–function, defined in the set

$$Q^*_{[0,T] \smallsetminus \mathcal{T}^c} \;:=\; \big\{ (\mathbf{x}, t) \in \mathbb{R}^4;\; t \in [0,T] \smallsetminus \mathcal{T}^c,\; \mathbf{x} \in \overline{\Omega^t} \big\},$$

such that

$$\operatorname{div} \mathbf{a} \;=\; 0 \qquad\qquad \text{in } Q_{[0,T] \smallsetminus \mathcal{T}^c}, \tag{2.6}$$

$$\mathbf{a} \cdot \mathbf{n} \;=\; \mathbf{V} \cdot \mathbf{n} \qquad\qquad \text{in } \Gamma_{[0,T] \smallsetminus \mathcal{T}^c}. \tag{2.7}$$

Conditions (1.3) and (2.7) now imply that function $\mathbf{u}$ should satisfy the homogeneous boundary condition

$$\mathbf{u} \cdot \mathbf{n} \;=\; 0 \qquad\qquad \text{a.e. in } \Gamma_{(0,T)}. \tag{2.8}$$

We further assume that function a satisfies the five conditions listed below. The possibility of a construction of function $\mathbf{a}$, satisfying (2.6), (2.7) and these conditions, depends on domain Ω^t and its variation due to the motion and shapes of bodies $B_1^t, \dots, B_K^t$. So we attach these conditions to other assumptions on domain Ω^t and we refer to them as to (a4)-(a8).

(a4) $\mathbf{a}$ and $\partial_t\mathbf{a}$ are continuous in $Q^*_{[0,T]\smallsetminus\mathcal{T}^c}$,

(a5) $\|\mathbf{a}(\,.\,,t)\|_{1,2;\,\Omega^t} \in L^2(0,T)$ – let us denote this norm by $\theta_1(t)$,

(a6) $\|\mathbf{a}(\,.\,,t)-\mathbf{V}(\,.\,,t)\|_{2;\,\Gamma^t} \in L^2(0,T)$ – let us denote this norm by $\theta_2(t)$,

(a7) there exist functions $\theta_3 \in L^1(0,T)$, $\theta_4 \in L^2(0,T)$ and $\theta_5 \in L^1(0,T)$, continuous in $[0,T]\smallsetminus\mathcal{T}^c$, such that for $t\in[0,T]\smallsetminus\mathcal{T}^c$ and $\boldsymbol{\phi}\in W^{1,2}_\sigma(\Omega^t)$ we have

$$\left|\int_{\Omega^t}\big[\partial_t\mathbf{a}(\,.\,,t)+\mathbf{a}(\,.\,,t)\cdot\nabla\mathbf{a}(\,.\,,t)\big]\cdot\boldsymbol{\phi}\;\mathrm{d}\mathbf{x}\right| \;\leq\; \theta_3(t)\,\|\boldsymbol{\phi}\|_{2;\,\Omega^t}+\theta_4(t)\,\|\nabla\boldsymbol{\phi}\|_{2;\,\Omega^t}, \tag{2.9}$$

$$\left|\int_{\Omega^t}\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}\cdot\mathbf{a}(\,.\,,t)\;\mathrm{d}\mathbf{x}\right| \;\leq\; \tfrac{1}{10}\nu\,\|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t}+\tfrac{1}{4}\gamma\,\|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t}+\theta_5(t)\,\|\boldsymbol{\phi}\|^2_{2;\,\Omega^t}, \tag{2.10}$$

(a8) the initial–value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t;\,\vartheta,\mathbf{x}) \;=\; \mathbf{a}\big(\mathbf{X}(t;\,\vartheta,\mathbf{x}),t\big), \qquad \mathbf{X}(\vartheta;\,\vartheta,\mathbf{x})=\mathbf{x} \tag{2.11}$$

has a unique solution $\mathbf{X}(t;\,\vartheta,\mathbf{x})$, defined for $t\in[0,T]$, $\vartheta\in[0,T]$ and a.a. $\mathbf{x}\in\Omega^\vartheta$, such that the mapping $\mathbf{x}\mapsto\mathbf{X}(t;\,\vartheta,\mathbf{x})$ is a one–to–one transformation of $\Omega^\vartheta\smallsetminus\mathfrak{s}^\vartheta$ onto $\Omega^t\smallsetminus\mathfrak{s}^t$ (where $\mathfrak{s}^\vartheta$ and $\mathfrak{s}^t$ are sets of measure zero in Ω^ϑ or in Ω^t, respectively).

Remark 4. We shall often use the mapping $\mathbf{x}\mapsto\mathbf{X}(t;\,\vartheta,\mathbf{x})$ in order to transform volume integrals on Ω^ϑ to volume integrals on Ω^t. The Jacobian of this mapping equals one due to the incompressibility of flow $\mathbf{a}$.

3 A weak formulation of the initial–boundary value problem (1.1)–(1.5) and the main theorem

By analogy with $Q^*_{[0,T]\smallsetminus\mathcal{T}^c}$, we denote by $Q^*_{[0,T]}$ (respectively $Q^*_{[0,T)}$) the set of points $(\mathbf{x},t)\in\mathbb{R}^4$ such that $0\leq t\leq T$ (respectively $0\leq t<T$) and $\mathbf{x}\in\overline{\Omega^t}$.

A formal derivation of the weak formulation. Assume that $\boldsymbol{\phi}$ is an infinitely differentiable divergence–free vector–function in $Q^*_{[0,T]}$ that has a compact support in $Q^*_{[0,T)}$ and satisfies the condition $\boldsymbol{\phi}\cdot\mathbf{n}=0$ a.e. on $\Gamma_{(0,T)}$. Assume that $\mathbf{v}$ is a "sufficiently smooth" solution of (1.1)–(1.5) of the form $\mathbf{v}=\mathbf{a}+\mathbf{u}$ where $\mathbf{a}$ satisfies all the assumptions named in Section 2 and $\mathbf{u}\in L^2_\sigma(\Omega^t)$ for a.a. $t\in(0,T)$. Let us multiply equation (1.1) by function $\boldsymbol{\phi}$ and integrate on $Q_{(0,T)}$. The integral of $\{\partial_t\mathbf{u}+(\mathbf{a}\cdot\nabla)\mathbf{u}\}\cdot\boldsymbol{\phi}$ can be expressed as follows:

$$\begin{aligned}
&\int_0^T\int_{\Omega^t}\big\{\partial_t\mathbf{u}(\mathbf{x},t)+\mathbf{a}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)\big\}\cdot\boldsymbol{\phi}(\mathbf{x},t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad=\;\int_0^T\int_{\Omega^0}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{u}\big(\mathbf{X}(t;\,0,\mathbf{x}_0),t\big)\cdot\boldsymbol{\phi}\big(\mathbf{X}(t;\,0,\mathbf{x}_0),t\big)\;\mathrm{d}\mathbf{x}_0\,\mathrm{d}t\\
&\quad=\;-\int_{\Omega^0}\mathbf{u}_0(\mathbf{x}_0)\cdot\boldsymbol{\phi}(\mathbf{x}_0,0)\;\mathrm{d}\mathbf{x}_0-\int_0^T\int_{\Omega^0}\mathbf{u}\big(\mathbf{X}(t;\,0,\mathbf{x}_0),t\big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{\phi}\big(\mathbf{X}(t;\,0,\mathbf{x}_0),t\big)\;\mathrm{d}\mathbf{x}_0\,\mathrm{d}t\\
&\quad=\;-\int_{\Omega^0}\mathbf{u}_0(\mathbf{x}_0)\cdot\boldsymbol{\phi}(\mathbf{x}_0,0)\;\mathrm{d}\mathbf{x}_0-\int_0^T\int_{\Omega^t}\mathbf{u}(\mathbf{x},t)\cdot\big\{\partial_t\boldsymbol{\phi}(\mathbf{x},t)+\mathbf{a}(\mathbf{x},t)\cdot\nabla\boldsymbol{\phi}(\mathbf{x},t)\big\}\;\mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned}$$

where $\mathbf{u}_0 = \mathbf{v}_0 - \mathbf{a}(\,.\,,0)$. The integral of $\mathbf{u}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\phi}$ in Ω^t can be transformed to the integral of $\mathbf{u}\cdot\nabla\mathbf{a}\cdot\boldsymbol{\phi}$ minus the integral of $\mathbf{u}\cdot\nabla\boldsymbol{\phi}\cdot\mathbf{u}$ by means of the integration by parts. The integral of $\nabla p\cdot\boldsymbol{\phi}$ in Ω^t equals zero because the subspace of gradients of scalar functions is orthogonal to $L^2_\sigma(\Omega^t)$ in $L^2(\Omega^t)^3$. Furthermore, denoting the components of the vectors $\mathbf{a}$, $\mathbf{v}$, $\mathbf{u}$ and $\boldsymbol{\phi}$ by the same slanted letters with indices, we have

$$\begin{aligned}\int_{\Omega^t}\nu\Delta\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x} \;&=\; \int_{\Omega^t}\nu(\partial_j^2 v_i)\phi_i\,\mathrm{d}\mathbf{x} \;=\; \int_{\Gamma^t}\nu\,(\partial_j v_i)\,n_j\,\phi_i\,\mathrm{d}S - \int_{\Omega^t}\nu\,(\partial_j v_i)(\partial_j\phi_i)\,\mathrm{d}\mathbf{x}\\ &=\; \int_{\Gamma^t}\nu\,(\partial_j v_i+\partial_i v_j)\,n_j\,\phi_i\,\mathrm{d}\mathbf{x} - \int_{\Gamma^t}\nu\,(\partial_i v_j)\,n_j\,\phi_i\,\mathrm{d}\mathbf{x} - \int_{\Omega^t}\nu\,(\partial_j v_i)(\partial_j\phi_i)\,\mathrm{d}\mathbf{x}\\ &=\; -\int_{\Gamma^t}\gamma(\mathbf{v}-\mathbf{V})\cdot\boldsymbol{\phi}\,\mathrm{d}S - \int_{\Omega^t}2\nu(\nabla\mathbf{v})_s : \nabla\boldsymbol{\phi}\,\mathrm{d}\mathbf{x},\end{aligned} \tag{3.1}$$

where the subscript s denotes the symmetric part. We have used the identities $\nu\,(\partial_j v_i+\partial_i v_j)\,n_j\,\phi_i$ $= [\mathbb{T}_\mathrm{d}(\mathbf{v})\cdot\mathbf{n}]\cdot\boldsymbol{\phi} = -\gamma(\mathbf{v}-\mathbf{V})\cdot\boldsymbol{\phi}$, following from the boundary condition (1.4). Writing everywhere $\mathbf{a}+\mathbf{u}$ instead of $\mathbf{v}$, we obtain the integral equation

$$\begin{aligned}&\int_0^T\int_{\Omega^t}\big\{-(\partial_t\boldsymbol{\phi}+\mathbf{a}\cdot\nabla\boldsymbol{\phi})\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\boldsymbol{\phi}\cdot\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}+2\nu[\nabla(\mathbf{a}+\mathbf{u})]_s:\nabla\boldsymbol{\phi}\big\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &\quad+\int_0^T\int_{\Gamma^t}\gamma(\mathbf{a}+\mathbf{u}-\mathbf{V})\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t \;=\; \int_0^T\int_{\Omega^t}\mathbf{g}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_{\Omega^0}\mathbf{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x}\end{aligned} \tag{3.2}$$

where $\mathbf{g} = \mathbf{f} - \partial_t\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{a}$. Thus, we arrive at the definition:

Definition 1. Suppose that $\mathbf{u}_0 \in L^2_\sigma(\Omega^0)$ and $\mathbf{f} \in L^2(0,T;\, L^2(\Omega^t)^3)$. Put $\mathbf{g} = \mathbf{f} - \partial_t\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{a}$. We call the function $\mathbf{v} \equiv \mathbf{a}+\mathbf{u}$ a *weak solution* of the problem (1.1)–(1.5) if $\mathbf{u} \in L^2(0,T;\, W^{1,2}_\sigma(\Omega^t)) \cap L^\infty(0,T;\, L^2_\sigma(\Omega^t))$, the trace of $\mathbf{u}$ on $\Gamma_{(0,T)}$ is in $L^2(0,T;\, L^2(\Gamma^t)^3)$ and $\mathbf{u}$ satisfies (3.2) for all infinitely differentiable divergence–free vector–functions $\boldsymbol{\phi}$ in $Q^*_{[0,T]}$, with a compact support in $Q^*_{[0,T)}$, that satisfy the condition $\boldsymbol{\phi}\cdot\mathbf{n} = 0$ a.e. on $\Gamma_{(0,T)}$.

The readers can verify that if the weak solution $\mathbf{v}$ is "sufficiently smooth" and all other input data are also "sufficiently smooth" then there exists a pressure p so that the pair $\mathbf{v}$, p is a classical solution of (1.1)–(1.5).

Our main theorem, whose proof is given in Sections 5–7, reads:

Theorem 1. *Suppose that domain Ω^t satisfies all the conditions(a1)–(a3). Suppose that there exists function $\mathbf{a}$, satisfying conditions (2.6), (2.7) and (a4)–(a8) from Section 2. Then the weak solution of the problem (1.1)–(1.5), introduced in Definition 1, exists.*

Remark 5. We shall see in Section 8 that condition (a7) induces a restriction on the speed of colliding bodies in comparison with the coefficients ν and γ, if the bodies strike by C^2–surfaces.

4 An apriori energy–type estimate of a solution of the problem (1.1)–(1.5)

In this section, we present a formal derivation of the energytype inequality, assuming that $(\mathbf{v},p)$ a "sufficiently smooth" solution of (1.1)-(1.5). The formal approach has the advantage that it enables us to abstract from technical details connected with the approximations of $(\mathbf{v},p)$ and to explain clearly the basic ideas. The same energytype inequality can also be derived for appropriate approximations of $(\mathbf{v}, p)$ (see Section 7).

We are going to derive the following inequality

$$\|\mathbf{u}(\,.\,,t)\|^2_{2;\,\Omega^t} + \nu \int_0^t \|\nabla\mathbf{u}(\,.\,,s)\|^2_{2;\,\Omega^t}\,\mathrm{d}s + \gamma \int_0^t \|\mathbf{u}(\,.\,,s)\|^2_{2;\,\Gamma^t}\,\mathrm{d}s \\ \leq \|\mathbf{u}_0\|^2_{2;\,\Omega^t} + \int_0^t \omega_1(s)\,\|\mathbf{u}(\,.\,,s)\|^2_{2;\,\Omega^t}\,\mathrm{d}s + \omega_2(t) \tag{4.1}$$

where $\mathbf{u}_0 = \mathbf{u}(\,.\,,0) = \mathbf{v}_0 - \mathbf{a}^0$ and functions ω_1 and ω_2 are integrable in $(0,T)$.

Beginning with equation (1.1) (where $\mathbf{v} = \mathbf{a} + \mathbf{u}$), we multiply it by function $\mathbf{u}$ and integrate in Ω^t. We obtain

$$\int_{\Omega^t} \big\{(\partial_t\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{a})\cdot\mathbf{u} + 2\nu[\nabla(\mathbf{a}+\mathbf{u})]_s : \nabla\mathbf{u}\big\}\,\mathrm{d}\mathbf{x} \\ + \gamma\int_{\Gamma^t}(\mathbf{a}+\mathbf{u}-\mathbf{V})\cdot\mathbf{u}\,\mathrm{d}S \;=\; \int_{\Omega^t}\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} - \int_{\Omega^t}(\partial_t\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{a})\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}. \tag{4.2}$$

(We have used (3.1) and (3.2) with $\boldsymbol{\phi} = \mathbf{u}$. As usually, the integral of $\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}$ in Ω^t equals zero.)

First of all, using the transformation $\mathbf{x} \mapsto \mathbf{y} = \mathbf{X}(t+h;\,t,\mathbf{x})$ of $\Omega^t \smallsetminus \mathfrak{s}^t$ onto $\Omega^{t+h} \smallsetminus \mathfrak{s}^{t+h}$ (see condition (a8)), we can rewrite the integral of $(\partial_t\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u})\cdot\mathbf{u}$ as follows:

$$\int_{\Omega^t}\big[\partial_t\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u}\big]\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \left[\int_{\Omega^t}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\,\frac{1}{2}\,\big|\mathbf{u}\big(\mathbf{X}(\vartheta;\,t,\mathbf{x}),\vartheta\big)\big|^2\,\mathrm{d}\mathbf{x}\right]_{\vartheta=t} \\ = \lim_{h\to 0}\frac{1}{2h}\left[\int_{\Omega^t}\Big(\big|\mathbf{u}\big(\mathbf{X}(t+h;\,t,\mathbf{x}),t+h\big)\big|^2 - \big|\mathbf{u}\big(\mathbf{X}(t;\,t,\mathbf{x}),t\big)\big|^2\Big)\,\mathrm{d}\mathbf{x}\right] \\ = \lim_{h\to 0}\frac{1}{2h}\left[\int_{\Omega^{t+h}}|\mathbf{u}(\mathbf{y},t+h)|^2\,\mathrm{d}\mathbf{y} - \int_{\Omega^t}|\mathbf{u}(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x}\right] \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\int_{\Omega^t}|\mathbf{u}|^2\,\mathrm{d}\mathbf{x}.$$

Further, we successively estimate the integrals in (4.2).

- The first term on the right hand side can be estimated from below by means of assumption (a6):

$$\gamma\int_{\Gamma^t}(\mathbf{a}+\mathbf{u}-\mathbf{V})\cdot\mathbf{u}\,\mathrm{d}S \;\geq\; \frac{3\gamma}{4}\,\|\mathbf{u}\|^2_{2;\,\Gamma^t} - \gamma\,\theta_2^2(t). \tag{4.3}$$

- The integral of $(\nabla\mathbf{u})_s : \nabla\mathbf{u}$ can be treated as follows:

$$2\nu\int_{\Omega^t}(\nabla\mathbf{u})_s : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; \nu\int_{\Omega^t}|\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + \nu\int_{\Gamma^t}(\partial_i u_j)\,u_i\,n_j\,\mathrm{d}S \\ = \nu\,\|\nabla\mathbf{u}\|^2_{2;\,\Omega^t} + \nu\int_{\Gamma^t}\partial_i(u_j n_j)\,u_i\,\mathrm{d}S - \nu\int_{\Gamma^t}u_j\,(\partial_i n_j)\,u_i\,\mathrm{d}S.$$

  The second integral on the right hand side is equal to zero because $u_j n_j = 0$ a.e. on Γ^t and the integrand represents the derivative of $u_j n_j$ in the tangent direction. The third term can be estimated by means of condition (a1). Thus, we obtain

$$2\nu\int_{\Omega^t}(\nabla\mathbf{u})_s : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} \;\geq\; \frac{9\nu}{10}\,\|\nabla\mathbf{u}\|^2_{2;\,\Omega^t} - \Big[\nu c_1 + \frac{5\nu}{2}\,c_1^2\Big]\,\|\mathbf{u}\|^2_{2;\,\Omega^t}. \tag{4.4}$$

- The integral of $(\nabla\mathbf{a})_s : \nabla\mathbf{u}$ can be estimated from below by $-\frac{1}{10}\,\|\nabla\mathbf{u}\|^2_{2;\,\Omega^t} - \frac{5}{2}\theta_1^2(t)$ due to condition (a5).
- The modulus of the integral of the product $(\partial_t\mathbf{a} + \mathbf{a}^t\cdot\nabla\mathbf{a})\cdot\mathbf{u}$ in (4.2) can be estimated by means of assumption (a7), estimate (2.9):

$$\left|\int_{\Omega^t}(\partial_t\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{a})\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\right| \;\leq\; \theta_3(t)\,\|\mathbf{u}\|^2_{2;\,\Omega^t} + \frac{1}{4}\,\theta_3(t) + \frac{\nu}{10}\,\|\nabla\mathbf{u}\|^2_{2;\,\Omega^t} + \frac{5}{2\nu}\,\theta_4^2(t).$$

- The integral of $\mathbf{u}\cdot\nabla\mathbf{a}\cdot\mathbf{u}$ can be estimated by means of assumption (a7), estimate (2.10).
- Finally, the integral of $\mathbf{f}\cdot\mathbf{u}$ can be estimated by $\frac{1}{2}\,\|\mathbf{f}\|^2_{2;\,\Omega^t} + \frac{1}{2}\,\|\mathbf{u}\|^2_{2;\,\Omega^t}$.

Substituting now all these estimates to (4.2) and integrating with respect to time from 0 to t, we obtain inequality (4.1) with

$$\begin{aligned}\omega_1(t) &= 2\nu c_1 + 5\nu c_1^2 + \theta_3(t) + 2\theta_5(t) + \tfrac{1}{2},\\ \omega_2(t) &= 2\gamma\theta_2^2(t) + \tfrac{1}{4}\theta_3(t) + 5\nu\theta_1^2(t) + (5/2\nu)\,\theta_4^2(t) + \tfrac{1}{2}\,\|\mathbf{f}\|^2_{2;\,\Omega^t}.\end{aligned}$$

5 The time discretization and stationary boundary–value problems

In this section, after preliminary remarks, we define and study stationary problems obtained from (3.2) by means of the time discretization.

5.1 A partition of the interval $[0,T]$. Since the functions θ_1^2, θ_2^2, θ_3, θ_4^2 and θ_5 are integrable in $(0,T)$ and continuous in $[0,T]\smallsetminus\mathcal{T}^c$, there exists a bound $\Theta > 0$ such that to each $N\in\mathbb{N}$ there exists a partition $P_N : 0 = t_0 < t_1 < \ldots < t_N = T$ of the interval $[0,T]$ with the properties $\|P_N\| := \max_{n=1,\dots,N}\; d_n < 2T/N$ (where $d_n := t_n - t_{n-1}$) and

$$\sum_{n=1}^{N}\big[\theta_1^2(t_n) + \theta_2^2(t_n) + \theta_3(t_n) + \theta_4^2(t_n) + \theta_5(t_n)\big]\,d_n \;\leq\; \Theta. \tag{5.1}$$

We further consider number $N\in\mathbb{N}$ to be fixed in this section. Moreover, N is supposed to be "sufficiently large". (We specify in next paragraphs what it means.)

We can assume without the loss of generality that $\{t_1;\,t_2;\,\ldots;\,t_N\}\cap\mathcal{T}^c = \emptyset$.

5.2 Notation. In order to simplify the notation, we put $\Omega_n := \Omega^{t_n}$ and $\Gamma_n := \Gamma^{t_n} = \partial\Omega^{t_n}$ for $n = 0,1,\dots,N$.

Due to technical reasons, we extend function $\mathbf{a}(\,.\,,t)$ (together with its derivatives) and function $\mathbf{f}(\,.\,,t)$ by zero to $\mathbb{R}^3\smallsetminus\Omega^t$.

For $\mathbf{x}\in\Omega_n$, we denote by $[\nabla\mathbf{a}]_n(\mathbf{x})$ (respectively $\mathbf{f}_n(\mathbf{x})$) the mean value of $\nabla\mathbf{a}(\mathbf{x},\,.)$ (respectively $\mathbf{f}(\mathbf{x},\,.)$ on the time interval (t_{n-1},t_n). We put $\mathbf{g}_n(\mathbf{x}) := \mathbf{f}_n(\mathbf{x}) - [\partial_t\mathbf{a}(\mathbf{x},t_n) + \mathbf{a}(\mathbf{x},t_n)\cdot\nabla\mathbf{a}(\mathbf{x},t_n)]$. Furthermore, for $\mathbf{x}\in\Gamma_n$, we denote by $\mathbf{A}_n(\mathbf{x})$ (respectively $\mathbf{V}_n(\mathbf{x})$) the mean value of $\mathbf{a}\big(\mathbf{Y}(\,.\,;\,t_n,\mathbf{x}),\,.\big)$ (respectively $\mathbf{V}\big(\mathbf{Y}(\,.\,;\,t_n,\mathbf{x}),\,.\big)$) on (t_{n-1},t_n)

5.3 Stationary boundary value problems – the weak formulation and existence of a solution. We put $\mathbf{U}_0 := \mathbf{u}_0$ and we denote by $\mathbf{U}_n$ approximate values of the unknown function $\mathbf{u}$ on the time levels t_n ($n = 1,2,\dots,N$). On the n–th time level, we assume that $\mathbf{U}_{n-1}$ is already a known function from $L^2_\sigma(\Omega_n)$ and we look for $\mathbf{U}_n\in W^{1,2}_\sigma(\Omega_n)$ such that

$$\int_{\Omega_n} \big\{ \big[\mathbf{U}_n - \mathbf{U}_{n-1} \circ \mathbf{X}(t_{n-1};\, t_n,\, .)\big] \cdot \mathbf{\Phi} - d_n\, \mathbf{U}_n \cdot \nabla\mathbf{\Phi} \cdot \mathbf{a}(\,.\,, t_n) + d_n\, \mathbf{U}_n \cdot \nabla\mathbf{U}_n \cdot \mathbf{\Phi} \big\}\, \mathrm{d}\mathbf{x}$$
$$+ \int_{\Omega_n} 2 d_n \nu \big\{ [\nabla\mathbf{a}]_n + \nabla\mathbf{U}_n \big\}_s : \nabla\mathbf{\Phi}\, \mathrm{d}\mathbf{x}$$
$$+ \int_{\Gamma_n} d_n \gamma \big[\mathbf{A}_n + \mathbf{U}_n - \mathbf{V}_n\big] \cdot \mathbf{\Phi}\, \mathrm{d}S \;=\; \int_{\Omega^n} d_n\, \mathbf{g}_n \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{x} \tag{5.2}$$

for all $\mathbf{\Phi} \in W^{1,2}_\sigma(\Omega_n)$.

Remark 6. Equation (5.2) is a time–discretized variant of (3.2) on the time level $t = t_n$. The symbol $\mathbf{U}_{n-1} \circ \mathbf{X}(t_{n-1};\, t_n,\, .)$ denotes the composite function $\mathbf{x} \mapsto \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big)$. The difference $\mathbf{U}_n(\mathbf{x}) - \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big)$ (for $\mathbf{x} \in \Omega_n$) approximates the time derivative $(\mathrm{d}/\mathrm{d}t)\,\mathbf{u}\big(\mathbf{X}(t;\, t_n, \mathbf{x}), t\big)$ at the time $t = t_n$, multiplied by d_n.

Integral equation (5.2) represents a nonlinear boundary–value problem for the unknown function $\mathbf{U}_n$. Applying the Lions–Leray theorem, we can arrive at the next lemma on the existence of its solution (see Section 9, Appendix A1, for the complete proof):

Lemma 1. *If $n \in \{1;\, \dots;\, N\}$ and d_n is small enough then equation (5.2) has a solution $\mathbf{U}_n$ in $W^{1,2}_\sigma(\Omega_n)$.*

We further assume that all d_n (for $n = 1, \dots, N$) are as small as Lemma 1 requires, which is equivalent to N being large enough.

5.4 Estimates of solutions of the weak problem (5.2). We derive a discrete variant of the energy inequality (4.1) in this sub–section. Using $\mathbf{\Phi} = \mathbf{U}_n$ in (5.2), we obtain:

$$\frac{1}{2}\, \|\mathbf{U}_n\|^2_{2;\,\Omega_n} + \frac{1}{2} \int_{\Omega_n} \big|\mathbf{U}_n - \mathbf{U}_{n-1} \circ \mathbf{X}(t_{n-1};\, t_n,\, .)\big|^2\, \mathrm{d}\mathbf{x}$$
$$+ 2 d_n \nu \int_{\Omega_n} (\nabla\mathbf{U}_n)_s : \nabla\mathbf{U}_n\, \mathrm{d}\mathbf{x} + \int_{\Gamma_n} d_n \gamma\, |\mathbf{U}_n|^2\, \mathrm{d}S$$
$$\leq \quad \frac{1}{2}\, \|\mathbf{U}_{n-1}\|^2_{2;\,\Omega_{k-1}} + \left| d_n \int_{\Omega_n} \mathbf{g}_n \cdot \mathbf{U}_n\, \mathrm{d}\mathbf{x} \right| + \left| d_n \int_{\Omega_n} \mathbf{U}_n \cdot \nabla\mathbf{U}_n \cdot \mathbf{a}(\,.\,, t_n)\, \mathrm{d}\mathbf{x} \right|$$
$$+ \left| 2 d_n \nu \int_{\Omega_n} \big([\nabla\mathbf{a}]_n\big)_s : \nabla\mathbf{U}_n\, \mathrm{d}\mathbf{x} \right| + \left| \int_{\Gamma_n} d_n \gamma\, (\mathbf{A}_n - \mathbf{V}_n) \cdot \mathbf{U}_n\, \mathrm{d}S \right|. \tag{5.3}$$

By analogy with (4.4), we have

$$2\nu \int_{\Omega_n} (\nabla\mathbf{U}_n)_s : \nabla\mathbf{U}_n\, \mathrm{d}\mathbf{x} \;\geq\; \frac{9\nu}{10}\, \|\nabla\mathbf{U}_n\|^2_{2;\,\Omega_n} - \nu \left[c_1 + \frac{5}{2} c_1^2 \right] \|\mathbf{U}_n\|^2_{2;\,\Omega_n}. \tag{5.4}$$

The integral of $[\partial_t\mathbf{a}(\,.\,, t_n) + \mathbf{a}(\,.\,, t_n) \cdot \nabla\mathbf{a}(\,.\,, t_n)] \cdot \mathbf{U}_n$ (the part of $\mathbf{g}_n \cdot \mathbf{U}_n$) can be estimated by means of assumption (a7), inequality (2.9):

$$\left| \int_{\Omega_n} \big[\partial_t\mathbf{a}(\,.\,, t_n) + \mathbf{a}(\,.\,, t_n) \cdot \nabla\mathbf{a}(\,.\,, t_n)\big] \cdot \mathbf{U}_n\, \mathrm{d}\mathbf{x} \right| \;\leq\; \theta_3(t_n)\, \|\mathbf{U}_n\|_{2;\,\Omega_n} + \theta_4(t_n)\, \|\nabla\mathbf{U}_n\|_{2;\,\Omega_n}$$
$$\leq \quad \frac{\nu}{10}\, \|\nabla\mathbf{U}_n\|^2_{2;\,\Omega_n} + \frac{5}{2\nu}\, \theta_4^2(t_n) + \frac{1}{2}\, \theta_3(t_n)\, \big[\|\mathbf{U}_n\|^2_{2;\,\Omega_n} + 1\big]. \tag{5.5}$$

The integral of $\mathbf{f}_n \cdot \mathbf{U}_n$ (the part of $\mathbf{g}_n \cdot \mathbf{U}_n$) can be obviously estimated as follows:

$$\left| \int_{\Omega_n} \mathbf{f}_n \cdot \mathbf{U}_n \, \mathrm{d}\mathbf{x} \right| \;\leq\; \frac{1}{2} \int_{t_{n-1}}^{t_n} \|\mathbf{f}(\,.\,,t)\|_{2;\,\Omega^t}^2 \, \mathrm{d}t + \frac{1}{2} \, \|\mathbf{U}_n\|_{2;\,\Omega_n}^2. \tag{5.6}$$

The integral of $\mathbf{U}_n \cdot \nabla \mathbf{U}_n \cdot \mathbf{a}(\,.\,, t_n)$ can be estimated by means of inequality (2.10):

$$\left| \int_{\Omega_n} \mathbf{U}_n \cdot \nabla \mathbf{U}_n \cdot \mathbf{a}(\,.\,, t_n) \, \mathrm{d}\mathbf{x} \right| \;\leq\; \frac{\nu}{10} \, \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^2 + \frac{\gamma}{4} \, \|\mathbf{U}_n\|_{2;\,\Gamma_n}^2 + \theta_5(t_n) \, \|\mathbf{U}_n\|_{2\;\Omega_n}^2. \tag{5.7}$$

The integral of $\big([\nabla \mathbf{a}]_n\big)_s : \nabla \mathbf{U}_n$ can be estimated by means of assumption (a5):

$$\left| 2\nu \int_{\Omega_n} \big([\nabla \mathbf{a}]_n\big)_s : \nabla \mathbf{U}_n \, \mathrm{d}\mathbf{x} \right| \;\leq\; \frac{\nu}{10} \, \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^2 + 10\nu \, \overline{\theta_{1n}^2}, \tag{5.8}$$

where $\overline{\theta_{1n}^2}$ denotes the mean value of θ_1^2 on (t_{n-1}, t_n). (By analogy, just below we also use $\overline{\theta_{2n}^2}$ for the mean value of θ_2^2.) Finally, the integral of $(\mathbf{A}_n - \mathbf{V}_n) \cdot \mathbf{U}_n$ on Γ_n can be estimated by means of assumption (a6):

$$\begin{aligned}
&\left| \gamma \int_{\Gamma_n} \big(\mathbf{A}_n(\mathbf{x}_n) - \mathbf{V}_n(\mathbf{x}_n)\big) \cdot \mathbf{U}_n(\mathbf{x}_n) \, \mathrm{d}S(\mathbf{x}_n) \right| \\
&= \left| \frac{\gamma}{d_n} \int_{t_{n-1}}^{t_n} \int_{\Gamma_n} \big[\mathbf{a}\big(\mathbf{Y}(t;\, t_n, \mathbf{x}_n), t\big) - \mathbf{V}\big(\mathbf{Y}(t;\, t_n, \mathbf{x}_n), t\big)\big] \cdot \mathbf{U}_n(\mathbf{x}_n) \, \mathrm{d}S(\mathbf{x}_n) \, \mathrm{d}t \right| \\
&\leq \frac{\gamma}{4} \, \|\mathbf{U}_n\|_{2;\,\Gamma_n}^2 + \frac{\gamma}{d_n} \int_{t_{n-1}}^{t_n} \int_{\Gamma_n} \big|\mathbf{a}\big(\mathbf{Y}(t;\, t_n, \mathbf{x}_n), t\big) - \mathbf{V}\big(\mathbf{Y}(t;\, t_n, \mathbf{x}_n), t\big)\big|^2 \, \mathrm{d}S(\mathbf{x}_n) \, \mathrm{d}t \\
&= \frac{\gamma}{4} \, \|\mathbf{U}_n\|_{2;\,\Gamma_n}^2 + \frac{\gamma}{d_n} \int_{t_{n-1}}^{t_n} \int_{\Gamma^t} \big|\mathbf{a}(\mathbf{x}, t) - \mathbf{V}(\mathbf{x}, t)\big|^2 \, \mathrm{d}S(\mathbf{x}) \, \mathrm{d}t \;=\; \frac{\gamma}{4} \, \|\mathbf{U}_n\|_{2;\,\Gamma_n}^2 + \gamma \, \overline{\theta_{2n}^2}.
\end{aligned} \tag{5.9}$$

Substituting now estimates (5.4)–(5.9) to (5.3), summing for $n = 1, \dots, j$ (where $1 \leq j \leq N$, $j \in \mathbb{N}$), and multiplying by two, we obtain the inequality

$$\begin{aligned}
&\|\mathbf{U}_j\|_{2;\,\Omega_j}^2 + \sum_{n=1}^{j} \big\|\mathbf{U}_n - \mathbf{U}_{n-1} \circ \mathbf{X}(t_{n-1};\, t_n,\, .)\big\|_{2;\,\Omega_n}^2 + \nu \sum_{n=1}^{j} d_n \, \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^2 \\
&\quad + \gamma \sum_{n=1}^{j} d_n \, \|\mathbf{U}_n\|_{2;\,\Gamma_n}^2 \;\leq\; \|\mathbf{U}_0\|_{2;\,\Omega_0}^2 + \sum_{n=1}^{j} \omega_{1n} \, \|\mathbf{U}_n\|_{2;\,\Omega_n}^2 + \sum_{n=1}^{j} \omega_{2n},
\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}
\omega_{1n} &= 2d_n \left(c_1\nu + \frac{5}{2}\, c_1^2 \nu + 1\right) + 2d_n \, \theta_3(t_n) + 2d_n \, \theta_5(t_n), \\
\omega_{2n} &= \frac{5}{\nu} \, d_n \, \theta_4^2(t_n) + \int_{t_{n-1}}^{t_n} \|\mathbf{f}(\,.\,,t)\|_{2;\,\Omega^t}^2 \, \mathrm{d}t + 20 d_n \nu \overline{\theta_{1n}^2} + 2\gamma d_n \, \overline{\theta_{2n}^2}.
\end{aligned}$$

Remark 7. It follows from the definition of $\overline{\theta_{1n}^2}$, $\overline{\theta_{2n}^2}$, and from (5.1) that the function $\lambda_N(s) := \omega_{1n}$ (for $t_{n-1} < s \leq t_n$) is integrable on $(0, T)$ and

$$\int_0^T \lambda_N(s) \, \mathrm{d}s \;\leq\; 2\left(c_1\nu + \frac{5}{2} c_1^2 \nu + 1\right) T + \int_0^T \theta_3(s) \, \mathrm{d}s + 2\Theta \;:=\; c_3, \tag{5.11}$$

$$\sum_{n=1}^{N} \omega_{2n} \;\le\; \frac{5}{\nu}\,\Theta + \int_0^T \bigl[\|\mathbf{f}(\,.\,,s)\|^2_{2;\,\Omega^s} + 20\nu\theta_1^2(s) + 2\gamma\theta_2^2(s)\bigr]\; \mathrm{d}s \;:=\; c_4. \tag{5.12}$$

Both c_3 and c_4 are independent of N. Furthermore, $\omega_{1n} \to 0$ as $N \to +\infty$ uniformly with respect to $n \in \{1;\,\dots;\,N\}$.

6 Non–stationary approximations and their weak convergence

We define

$$\mathbf{u}^N(\mathbf{x},t) := \begin{cases} \mathbf{U}_n(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_n, \\ \mathbf{0} & \text{if } \mathbf{x} \in \mathbb{R}^3 \smallsetminus \Omega_n, \end{cases} \qquad \mathbb{U}^N(\mathbf{x},t) := \begin{cases} \nabla\mathbf{U}_n(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_n, \\ \mathbb{O} & \text{if } \mathbf{x} \in \mathbb{R}^3 \smallsetminus \Omega_n, \end{cases}$$

$$\mathbf{u}^N_*(\mathbf{x},t) := \mathbf{u}^N\bigl(\mathbf{Y}(t_n;\,t,\mathbf{x}),t\bigr) = \mathbf{U}_n\bigl(\mathbf{Y}(t_n;\,t,\mathbf{x})\bigr) \quad \text{if } \mathbf{x} \in \Gamma^t$$

for $t_{n-1} < t \le t_n$ and $n = 1,\dots,N$.

The values of $\mathbf{u}^N(\,.\,,t)$ and $\mathbf{U}_n$ at the points $\mathbf{Y}(t_n;\,t,\mathbf{x}) \in \Gamma_n$ are understood in the sense of traces of functions defined a.e. in Ω_n. The definition of functions $\mathbf{u}^N(\,.\,,t)$ and $\mathbb{U}^N(\,.\,,t)$ not only in Ω_n, but also in $\mathbb{R}^3 \smallsetminus \Omega_n$, is necessary because these functions will be later used in (3.2) and integrated in Ω^t, which generally differs from Ω_n by more than only a set of measure zero.

6.1 Estimates of the sequences $\{\mathbf{u}^N\}$, $\{\mathbb{U}^N\}$ and $\{\mathbf{u}^N_*\}$. Inequalities (5.10) and (5.12) imply that if N is so large that $\omega_{1n} < \frac{1}{2}$ for all $n \in \{1;\,\dots;\,N\}$ then

$$\begin{aligned} &\frac{1}{2}\,\|\mathbf{u}^N(\,.\,,t)\|^2_{2;\,\mathbb{R}^3} + \nu\int_0^t \|\mathbb{U}^N(\,.\,,s)\|^2_{2;\,\mathbb{R}^3}\;\mathrm{d}s + \gamma\int_0^t \|\mathbf{u}^N_*(\,.\,,s)\|^2_{2;\,\Gamma^s}\,\mathrm{d}s \\ &\qquad \le\; \|\mathbf{u}_0\|^2_{2;\,\Omega^0} + \int_0^t \lambda_N(s)\,\|\mathbf{u}^N(\,.\,,s)\|^2_{2;\,\mathbb{R}^3}\,\mathrm{d}s + c_4. \end{aligned} \tag{6.1}$$

Applying Gronwall's lemma and estimate (5.11), we deduce that

$$\begin{aligned} \|\mathbf{u}^N(\,.\,,t)\|^2_{2;\,\mathbb{R}^3} \;&\le\; 2\bigl(\|\mathbf{u}_0\|^2_{2;\,\Omega^0} + c_4\bigr) + 4\bigl(\|\mathbf{u}_0\|^2_{2;\,\Omega^0} + c_4\bigr)\int_0^t \lambda_N(s)\;\exp\Bigl(2\int_s^t \lambda_N(\sigma)\,\mathrm{d}\sigma\Bigr)\;\mathrm{d}s \\ &\le\; 2\bigl(\|\mathbf{u}_0\|^2_{2;\,\Omega^0} + c_4\bigr)\,\bigl[1 + 2c_3\,\mathrm{e}^{2c_3}\bigr] \;:=\; c_5 \end{aligned} \tag{6.2}$$

for $0 \le t \le T$. Using inequality (6.2) in (6.1), we obtain

$$\nu\int_0^T \|\mathbb{U}^N(\,.\,,s)\|^2_{2;\,\mathbb{R}^3}\;\mathrm{d}s + \gamma\int_0^T \|\mathbf{u}^N_*(\,.\,,s)\|^2_{2;\,\Gamma^s}\;\mathrm{d}s \;\le\; c_3c_5 + c_4 + 2c_5 \;:=\; c_6. \tag{6.3}$$

Constants c_5 and c_6 are independent of N. We can reversely derive from inequalities (6.2) and (6.3) that

$$\|\mathbf{U}_n\|_{2;\,\Omega_n} \;\le\; c_5 \qquad (n = 1,\dots,N), \tag{6.4}$$

$$\nu\sum_{n=1}^{N} d_n\,\|\nabla\mathbf{U}_n\|^2_{2;\,\Omega_n} + \gamma\sum_{n=1}^{N} d_n\,\|\mathbf{U}_n\|^2_{2;\,\Gamma_n} \;\le\; c_6. \tag{6.5}$$

6.2 Weak convergence of subsequences. Estimates (6.2) and (6.3) imply that there exist subsequences of $\{\mathbf{u}^N\}$, $\{\mathbb{U}^N\}$ and $\{\mathbf{u}_*^N\}$ (we shall denote them again by $\{\mathbf{u}^N\}$, $\{\mathbb{U}^N\}$ and $\{\mathbf{u}_*^N\}$) and functions $\mathbf{u} \in L^\infty(0,T;\, L^2(\mathbb{R}^3)^3)$, $\mathbb{U} \in L^2(0,T;\, L^2(\mathbb{R}^3)^9)$ and $\mathbf{u}_* \in L^2(\Gamma_{(0,T)})^3$ such that

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{weakly–}* \text{ in } L^\infty(0,T;\, L^2(\mathbb{R}^3)^3) \qquad \text{for } N \to +\infty, \tag{6.6}$$

$$\mathbb{U}^N \rightharpoonup \mathbb{U} \qquad \text{weakly in } L^2(0,T;\, L^2(\mathbb{R}^3)^9) \qquad \text{for } N \to +\infty, \tag{6.7}$$

$$\mathbf{u}_*^N \rightharpoonup \mathbf{u}_* \qquad \text{weakly in } L^2(\Gamma_{(0,T)})^3 \qquad \text{for } N \to +\infty. \tag{6.8}$$

The next lemma brings the information on relations between $\mathbf{u}$, $\mathbb{U}$ and $\mathbf{u}_*$. The lemma is proved in Section 9, Appendix A2.

Lemma 2. *a)* $\mathbb{U} = \nabla\mathbf{u}$ *in the sense of distributions in* $Q_{(0,T)}$,

b) $\mathbf{u} \in L^2(0,T;\, W_\sigma^{1,2}(\Omega^t))$,

c) $\mathbf{u}_* = \mathrm{tr}(\mathbf{u})$ *on* $\Gamma_{(0,T)}$ *(where* $\mathrm{tr}(\mathbf{u})$ *denotes the trace of the function* $\mathbf{u}\big|_{Q_{(0,T)}}$ *on* $\Gamma_{(0,T)}$*).*

6.3 Substitution of the approximations to integral equation (3.2). The approximations $\mathbf{u}^N$ (represented by $\mathbf{u}_*^N$ on Γ^t) naturally satisfy the integral equation (3.2) with a certain error $\mathcal{E}^N(\boldsymbol{\phi})$. Thus, if we denote by $\mathcal{I}(\mathbf{u}^N, \mathbf{u}_*^N, \boldsymbol{\phi})$ the the left hand side of (3.2) (where we use $\mathbf{u}^N$ in Ω^t and $\mathbf{u}_*^N$ on Γ^t), we have

$$\mathcal{I}(\mathbf{u}^N, \mathbf{u}_*^N, \boldsymbol{\phi}) \;=\; \int_0^T \int_{\Omega^t} \mathbf{g} \cdot \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_{\Omega^0} \mathbf{u}_0 \cdot \boldsymbol{\phi}(\,.\,,0)\, \mathrm{d}\mathbf{x} + \mathcal{E}^N(\boldsymbol{\phi}). \tag{6.9}$$

The following Lemma 3 provides the information on the asymptotic behaviour of $\mathcal{E}^N(\boldsymbol{\phi})$ as $N \to +\infty$. The proof is given in Section 9, Appendix A3.

Lemma 3. *Given a test function* $\boldsymbol{\phi}$ *as in Definition 1, we have* $\lim_{N\to+\infty} \mathcal{E}^N(\boldsymbol{\phi}) = 0$.

Let us now deal with the left hand side of (6.9). It can be split to the sum $\mathcal{I}_1(\mathbf{u}^N, \boldsymbol{\phi}) + \ldots + \mathcal{I}_4(\mathbf{u}^N, \boldsymbol{\phi}) + \mathcal{I}_5(\mathbf{u}_*^N, \boldsymbol{\phi})$, where

$$\begin{aligned}
\mathcal{I}_1(\mathbf{u}^N, \boldsymbol{\phi}) &:= -\int_0^T \int_{\Omega^t} (\partial_t\boldsymbol{\phi} + \mathbf{a}\cdot\nabla\boldsymbol{\phi}) \cdot \mathbf{u}^N \, \mathrm{d}\mathbf{x}\, \mathrm{d}t, \\
\mathcal{I}_2(\mathbf{u}^N, \boldsymbol{\phi}) &:= -\int_0^T \int_{\Omega^t} \mathbf{u}^N \cdot \nabla\boldsymbol{\phi} \cdot \mathbf{a}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t, \\
\mathcal{I}_3(\mathbf{u}^N, \boldsymbol{\phi}) &:= \int_0^T \int_{\Omega^t} \mathbf{u}^N \cdot \mathbb{U}^N \cdot \boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t, \\
\mathcal{I}_4(\mathbf{u}^N, \boldsymbol{\phi}) &:= \int_0^T \int_{\Omega^t} 2\nu\, [\nabla(\mathbf{a} + \mathbf{u}^N)]_s : \nabla\boldsymbol{\phi}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t, \\
\mathcal{I}_5(\mathbf{u}_*^N, \boldsymbol{\phi}) &:= \int_0^T \int_{\Gamma^t} \gamma(\mathbf{a} + \mathbf{u}_*^N - \mathbf{V}) \cdot \boldsymbol{\phi}\, \mathrm{d}S\, \mathrm{d}t.
\end{aligned}$$

Using the types of convergence named in (6.6)–(6.8) and statements a) and c) of Lemma 2, we can deduce that $\mathcal{I}_i(\mathbf{u}^N, \boldsymbol{\phi}) \to \mathcal{I}_i(\mathbf{u}, \boldsymbol{\phi})$ for $i = 1,\, 2,\, 3$ and $\mathcal{I}_5(\mathbf{u}_*^N, \boldsymbol{\phi}) \to \mathcal{I}_5(\mathbf{u}, \boldsymbol{\phi})$ as $N \to +\infty$. Thus, passing with N to $+\infty$ in (6.9), we obtain the identity

$$\int_0^T \int_{\Omega^t} \big\{ -(\partial_t\boldsymbol{\phi} + \mathbf{a}\cdot\nabla\boldsymbol{\phi}) \cdot \mathbf{u} - \mathbf{u}\cdot\nabla\boldsymbol{\phi}\cdot\mathbf{a} + 2\nu\, [\nabla(\mathbf{a}+\mathbf{u})]_s : \nabla\boldsymbol{\phi} \big\}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \lim_{N\to+\infty} \mathcal{I}_3(\mathbf{u}^N, \boldsymbol{\phi})$$

$$+\int_0^T\int_{\Gamma^t}\gamma(\mathbf{a}+\mathbf{u}-\mathbf{V})\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t \;=\; \int_0^T\int_{\Omega^t}\mathbf{g}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_{\Omega^0}\mathbf{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\mathbf{x}. \tag{6.10}$$

In order to show that $\mathbf{u}$ is a weak solution of the problem (1.1)–(1.5), it remains to verify that

$$\lim_{N\to+\infty}\mathcal{I}_3(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; \int_0^T\int_{\Omega^t}\mathbf{u}\cdot\nabla\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{6.11}$$

7 The limit process in the nonlinear term $\mathcal{I}_3(\mathbf{u}^N,\boldsymbol{\phi})$

The existence of the limit on the left hand side of (6.11) follows from (6.10). Thus, it is sufficient to check the value of the limit only for an arbitrary subsequence of $\{\mathbf{u}^N,\mathbb{U}^N\}$. The limit in (6.11) is not standard due to the variability of domain Ω^t and the test function $\boldsymbol{\phi}$, which is required to have only the normal component equal to zero on $\Gamma_{(0,T)}$. We prove the validity of (6.11) (for a subsequence of $\{\mathbf{u}^N,\mathbb{U}^N\}$) in this section. At first we successively explain in sub–sections 7.1–7.3 that it is sufficient to prove (6.11) with certain modified functions $\boldsymbol{\phi}^*$, $\boldsymbol{\phi}^{**}$ and $\boldsymbol{\phi}_j^{**}$ (for $j=1,\dots,J$) instead of the original function $\boldsymbol{\phi}$.

7.1 Definition of function $\boldsymbol{\phi}^*$. Recall that $\mathcal{T}^c=\{t_1^c;\,\dots;\,t_M^c\}\subset(0,T)$ is the family of critical time instants when the bodies moving in container D collide. Let $\epsilon_1>0$ be given. Then, due to (6.2) and (6.3), there exists $\kappa>0$ so small that

$$\begin{aligned}&\left|\sum_{m=1}^M\int_{t_m^c-\kappa}^{t_m^c+\kappa}\int_{\Omega^t}\mathbf{u}^N\cdot\mathbb{U}^N\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right|\\ &\quad\le\;\sqrt{c_5}\,c_8\sum_{m=1}^M\int_{t_m^c-\kappa}^{t_m^c+\kappa}\|\mathbb{U}^N(\,.\,,t)\|_{2;\,\Omega^t}\,\mathrm{d}\mathbf{t}\;\le\;c_8\sqrt{\frac{2\kappa c_5c_6}{\nu}}\;<\;\epsilon_1\end{aligned} \tag{7.1}$$

for all $N\in\mathbb{N}$ sufficiently large. (Here c_8 is the maximum of $|\boldsymbol{\phi}|$ on $\mathbb{R}^3_+\times[0,T]$.) Let η be an infinitely differentiable cut–off function of variable t defined on the interval $[0,T]$, with values in $[0,1]$, such that

$$\eta(t)\;:=\;\begin{cases}1 & \text{if }\ \mathrm{dist}(t;\mathcal{T}^c)\ge\kappa,\\ 0 & \text{if }\ \mathrm{dist}(t;\mathcal{T}^c)\le\frac12\kappa.\end{cases} \tag{7.2}$$

The function $\boldsymbol{\phi}^*(\mathbf{x},t):=\eta(t)\,\boldsymbol{\phi}(\mathbf{x},t)$ equals zero for $t\in[0,T]$ such that $\mathrm{dist}(t;\mathcal{T}^c)\le\frac12\kappa$ and

$$\left|\int_0^T\int_{\Omega^t}\mathbf{u}^N\cdot\mathbb{U}^N\cdot(\boldsymbol{\phi}-\boldsymbol{\phi}^*)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right|\;<\;\epsilon_1$$

due to (7.1). Since ϵ_1 can be chosen arbitrarily small, it is sufficient to prove (6.11) with function $\boldsymbol{\phi}^*$ instead of $\boldsymbol{\phi}$.

7.2 Definition of function $\boldsymbol{\phi}^{**}$. Since domain Ω^t is Lipschitzian for each $t\in[0,T\smallsetminus\mathcal{T}^c$, it has a cone property (see [1, p. 66]) and $W^{1,2}_\sigma(\Omega^t)\hookrightarrow L^6(\Omega^t)^3$. Moreover, if we restrict ourselves to times $t\in I(\kappa)$, where

$$I(\kappa)\;:=\;\big\{t\in[0,T];\ \mathrm{dist}(t;\mathcal{T}^c)>\tfrac12\kappa\big\},$$

then the cone parameters in the definition of the cone property of domain Ω^t can be chosen to be independent of t. Hence the constant in the corresponding imbedding inequality also becomes independent of t, see [1, p. 103]. Consequently,

$$\|\mathbf{u}^N(\,.\,,t)\|_{6;\,\mathbb{R}^3} \;\leq\; C\,\big(\|\mathbf{u}^N(\,.\,,t)\|_{2;\,\mathbb{R}^3} + \|\mathbb{U}^N(\,.\,,t)\|_{2;\,\mathbb{R}^3}\big)$$

for all $t \in I(\kappa)$. From this information and from (6.4), we deduce that the product $\mathbf{u}^N \cdot \mathbb{U}^N$ belongs to the space $L^2\big(I(\kappa);\, L^1(\mathbb{R}^3)^3\big) \cap L^1\big(I(\kappa);\, L^{3/2}(\mathbb{R}^3)^3\big)$. By interpolation, we obtain the inclusion $\mathbf{u}^N \cdot \mathbb{U}^N \in L^r\big(I(\kappa);\, L^s(\mathbb{R}^3)^3\big)$ for $r \geq 1$, $s \geq 1$ such that $2/r + 3/s = 4$. Particularly, $\mathbf{u}^N \cdot \mathbb{U}^N \in L^{5/4}\big(I(\kappa);\, L^{5/4}(\mathbb{R}^3)^3\big)$.

Function $\boldsymbol{\phi}^*$ can be approximated by infinitely differentiable divergence–free vector–functions with a compact support in $Q_{[0,T)}$ with an arbitrary accuracy in the norm of the space $L^{5/4}\big(I(\kappa);\, L^{5/4}(\Omega^t)^3\big)$. Hence, given $\epsilon_2 > 0$, there exists such a vector–function $\boldsymbol{\phi}^{**}$ that satisfies

$$\left| \int_0^T \int_{\Omega^t} \mathbf{u}^N \cdot \mathbb{U}^N \cdot \boldsymbol{\phi}^*\, \mathrm{d}\mathbf{x} - \int_0^T \int_{\Omega^t} \mathbf{u}^N \cdot \mathbb{U}^N \cdot \boldsymbol{\phi}^{**}\, \mathrm{d}\mathbf{x} \right| \;<\; \epsilon_2$$

for all $N \in \mathbb{N}$ sufficiently large. Since ϵ_2 can be chosen to be arbitrarily small, we can prove (6.11) only with the function $\boldsymbol{\phi}^{**}$ instead of $\boldsymbol{\phi}$ (respectively instead of $\boldsymbol{\phi}^*$).

7.3 Partition of function $\boldsymbol{\phi}^{**}$. Let $J \in \mathbb{N}$. We denote $\tau_j = jT/m$ (for $j = 0, \dots, J$). There exist $J+1$ infinitely differentiable functions $\varphi_0, \dots, \varphi_J$ on $[0,T]$ with their values in the interval $[0,1]$ such that $\operatorname{supp}\varphi_0 \subset I_0 := [\tau_0, \tau_1)$, $\operatorname{supp}\varphi_j \subset I_j := (\tau_{j-1}, \tau_{j+1})$ (for $j = 1, \dots, J-1$), $\operatorname{supp}\varphi_J \subset I_J := (\tau_{J-1}, \tau_J]$ and $\sum_{j=0}^J \varphi_j(t) = 1$ for $0 \leq t \leq T$. Now we put $\boldsymbol{\phi}_j^{**} := \varphi_j\, \boldsymbol{\phi}^{**}$ (for $j = 0, 1, \dots, J$). The functions $\boldsymbol{\phi}_j^{**}$ are divergence–free, they have compact supports in Q_{I_j} and $\sum_{j=0}^J \boldsymbol{\phi}_j^{**} = \boldsymbol{\phi}^{**}$ in $Q_{[0,T]}$.

Denote by $G_{(j)}$ the set $\big\{\mathbf{x} \in \mathbb{R}^3;\ \exists\, t \in I_j : (\mathbf{x},t) \in \operatorname{supp}\varphi_j^{**}\big\}$. If J is large enough then the distance between $G_{(j)}$ and Γ^t is greater than one half of the distance between $\operatorname{supp}\boldsymbol{\phi}^{**}$ and $\Gamma_{[0,T]}$ for all $t \in I_j$. Thus, there exists a bounded open set $\Omega_{(j)}$ in $\mathbb{R}^3$ with the boundary of the class $C^{1,1}$ such that $G_{(j)} \subset \Omega_{(j)} \subset \overline{\Omega}_{(j)} \subset \Omega^t$ for all $t \in I_j$. So, we conclude that in order to prove (6.11), it is sufficient to treat it separately with $\boldsymbol{\phi} = \boldsymbol{\phi}_j^{**}$ (for $j = 0, 1, \dots, J$) and to show that

$$\lim_{N\to+\infty} \int_{I_j} \int_{\Omega_{(j)}} \mathbf{u}^N \cdot \nabla\mathbf{u}^N \cdot \boldsymbol{\phi}_j^{**}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \;=\; \int_{I_j} \int_{\Omega_{(j)}} \mathbf{u} \cdot \nabla\mathbf{u} \cdot \boldsymbol{\phi}_j^{**}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t. \tag{7.3}$$

(Since $G_{(j)} \times I_j \subset \cup_{n=1}^N \Omega_n \times (t_{n-1}, t_n]$ for sufficiently large N, we can write $\nabla\mathbf{u}^N$ instead of $\mathbb{U}^N$ in (7.3).)

7.4 The local Helmholtz decomposition of function $\mathbf{u}^N$. We denote by P_σ^j the Helmholtz projection in $L^2(\Omega_{(j)})^3$. Put $\mathbf{w}_j^N := P_\sigma^j \mathbf{u}^N$. The function $(I - P_\sigma^j)\mathbf{u}^N$ has the form $\nabla\varphi_j^N$ for an appropriate scalar function φ_j^N. (7.3) can now be written as

$$\begin{aligned} \lim_{N\to+\infty} \int_{I_j} \int_{\Omega_{(j)}} \Big[&\mathbf{w}_j^N \cdot \nabla\mathbf{w}_j^N \cdot \boldsymbol{\phi}_j^{**} + \mathbf{w}_j^N \cdot \nabla^2\varphi_j^N \cdot \boldsymbol{\phi}_j^{**} + \nabla\varphi_j^N \cdot \nabla\mathbf{w}_j^N \cdot \boldsymbol{\phi}_j^{**} \\ &+ \nabla\varphi_j^N \cdot \nabla^2\varphi_j^N \cdot \boldsymbol{\phi}_j^{**} \Big]\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \;=\; \int_{I_j} \int_{\Omega_{(j)}} \mathbf{u} \cdot \nabla\mathbf{u} \cdot \boldsymbol{\phi}_j^{**}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t. \end{aligned} \tag{7.4}$$

Since $\nabla\varphi_j^N \cdot \nabla^2\varphi_j^N = \nabla\big(\frac{1}{2}|\nabla\varphi_j^N|^2\big)$ and $\boldsymbol{\phi}_j^{**}(\,.\,,t) \in L^2_\sigma(\Omega_j^*)$, the integral of $\nabla\varphi_j^N \cdot \nabla^2\varphi_j^N \cdot \boldsymbol{\phi}_j^{**}$ on $\Omega_{(j)}$ equals zero.

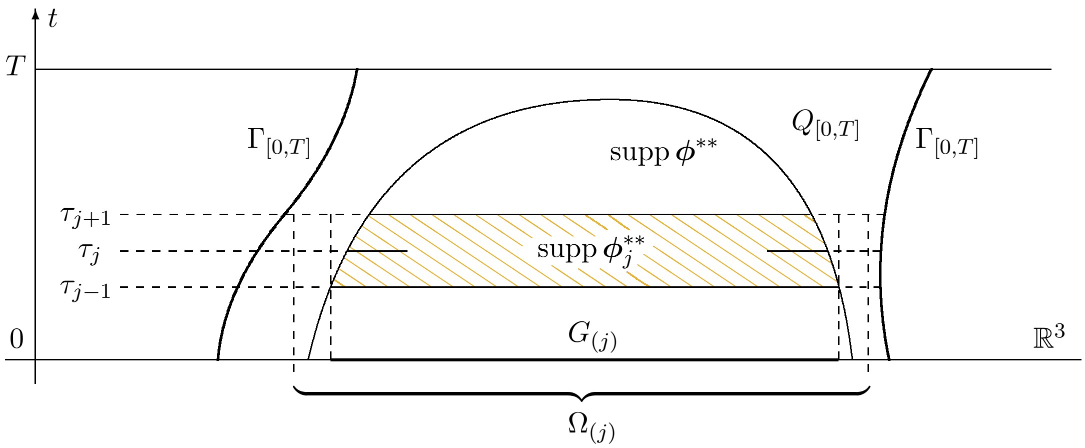

Fig. 2: Sets $Q_{[0,T]}$, $\operatorname{supp}\boldsymbol{\phi}^{**}$, $\operatorname{supp}\boldsymbol{\phi}_j^{**}$, $G_{(j)}$ and $\Omega_{(j)}$

The convergence (6.6) and (6.7), the coincidence of $\mathbb{U}^N$ with $\nabla\mathbf{u}^N$ on $\Omega_{(j)}\times I_j$ and the boundedness of operator P_σ^j in $L^2(\Omega_{(j)})^3$ and in $W^{1,2}(\Omega_{(j)})^3$ imply that

$$\mathbf{w}_j^N \rightharpoonup \mathbf{w}_j := P_\sigma^j\mathbf{u}, \quad \text{and} \quad \nabla\varphi_j^N \rightharpoonup \nabla\varphi_j := (I-P_\sigma^j)\mathbf{u} \quad \text{for } N\to+\infty \tag{7.5}$$

weakly in $L^2(I_j;\, W^{1,2}(\Omega_{(j)})^3)$ and weakly–$*$ in $L^\infty(I_j;\, L^2_\sigma(\Omega_{(j)}))$.

7.5 Strong convergence of a subsequence of $\{\mathbf{w}_j^N\}$. We are going to show that there exists a subsequence of $\{\mathbf{w}_j^N\}$ that tends to $\mathbf{w}_j$ strongly in $L^2(I_j;\, L^2_\sigma(\Omega_{(j)}))$ as $N\to+\infty$. We shall therefore use the next lemma, see J. L. Lions [21, Theorem 5.2].

Lemma 4. *Let $0<\alpha<\frac{1}{2}$ and let H_0, H and H_1 be Hilbert spaces such that $H_0 \hookrightarrow\hookrightarrow H \hookrightarrow H_1$. Let $\mathcal{H}^\alpha(\mathbb{R};\, H_0,\, H_1)$ denote the Banach space $\big\{w\in L^2(\mathbb{R};\, H_0);\ |\vartheta|^\alpha\,\hat{w}(\vartheta)\in L^2(\mathbb{R};\, H_1)\big\}$ with the norm*

$$|||w|||_{\alpha;\,\mathbb{R}} := \big(\|w\|^2_{L^2(\mathbb{R};\,H_0)} + \|\,|\vartheta|^\alpha\,\hat{w}(\vartheta)\|^2_{L^2(\mathbb{R};\,H_1)}\big)^{1/2}.$$

(Here $\hat{w}(\vartheta)$ is the Fourier transform of $w(t)$.) Let $\mathcal{H}^\alpha(a,b;\, H_0,\, H_1)$ further denote the Banach space of restrictions of functions from $\mathcal{H}^\alpha(\mathbb{R};\, H_0,\, H_1)$ onto the interval (a,b), with the norm

$$|||w|||_{\alpha;\,(a,b)} := \inf\ |||z|||_{\alpha;\,\mathbb{R}}$$

where the infimum is taken over all $z\in\mathcal{H}^\alpha(\mathbb{R};\, H_0,\, H_1)$ such that $z=w$ a.e. in (a,b). Then $\mathcal{H}^\alpha(0,T;\, H_0,\, H_1)\hookrightarrow\hookrightarrow L^2(a,b;\, H)$.

Let $j\in\{1;\dots;M\}$ be fixed. We shall use Lemma 4 with $(a,b)=I_j$, $H_0=W^{1,2}_\sigma(\Omega_{(j)})$, $H=L^2_\sigma(\Omega_{(j)})$ and $H_1=W^{-1,2}_{0,\sigma}(\Omega_{(j)})$. (Here $W^{-1,2}_{0,\sigma}(\Omega_{(j)})$ denotes the dual to $W^{1,2}_{0,\sigma}(\Omega_{(j)})$, where $W^{1,2}_{0,\sigma}(\Omega_{(j)})$ is the closure of $C^\infty_{0,\sigma}(\Omega_{(j)})$ in $W^{1,2}(\Omega_{(j)})^3$. The norm in $W^{-1,2}_{0,\sigma}(\Omega_{(j)})$ will be denoted by $\|\,.\,\|_{-1,2;\,\Omega_{(j)}}$.)

We claim that $\{\mathbf{w}_j^N\}$ is bounded in the space $\mathcal{H}^\alpha(I_j;\, H_0,\, H_1)$.

The boundedness of $\{\mathbf{w}_j^N\}$ in $L^2(I_j;\, H_0)$ follows from (6.2), (6.3), from the coincidence of $\mathbb{U}^n$ with $\nabla\mathbf{u}^n$ on $\Omega_{(j)}\times I_j$ and from the boundedness of operator P_σ^j in $L^2(\Omega_{(j)})^3$ and in $W^{1,2}(\Omega_{(j)})^3$.

Thus, we only need to verify that the sequence $\{|\vartheta|^\alpha\, \hat{\mathbf{w}}_j^N\}$ is bounded in the space $L^2(I_j;\, H_1)$, i.e. in $L^2(I_j;\, W^{-1,2}_{0,\sigma}(\Omega_{(j)}))$. Let $\mathbf{z}_j^N$ be an extension by zero of $\mathbf{w}_j^N$ from the time interval I_j onto $\mathbb{R}$. Then

$$\hat{\mathbf{z}}_j^N(\vartheta) \;=\; \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi\,\mathrm{i}\,t\vartheta}\, \mathbf{w}_j^N(t)\, \mathrm{d}t \;=\; \sum_{n\in\Lambda_j^N} \int_{t_{n-1}}^{t_n} \mathrm{e}^{-2\pi\,\mathrm{i}\,t\vartheta}\, P_\sigma^j \mathbf{U}_n\, \mathrm{d}t \tag{7.6}$$

where Λ_j^N is the set of such indices $n \in \{1;\dots;N\}$ that $[\mathbb{R}^3 \times (t_{n-1}, t_n)] \cap \operatorname{supp} \boldsymbol{\phi}_j^{**} \neq \emptyset$. Λ_j^N has the form $\Lambda_j^N = \{l; l+1;\dots;q\}$ where $1 \leq l \leq q \leq N$. Calculating the integrals in (7.6), we obtain

$$\begin{aligned} \hat{\mathbf{z}}_j^N(\vartheta) \;&=\; \sum_{n=l}^{q} \frac{1}{2\pi\mathrm{i}\vartheta} \left[\mathrm{e}^{-2\pi\,\mathrm{i}\,t_{n-1}\vartheta} - \mathrm{e}^{-2\pi\,\mathrm{i}\,t_n\vartheta}\right] P_\sigma^j \mathbf{U}_n \\ &=\; \frac{1}{2\pi\mathrm{i}\vartheta} \left[\mathrm{e}^{-2\pi\,\mathrm{i}\,t_{l-1}}\, P_\sigma^j \mathbf{U}_l - \mathrm{e}^{-2\pi\,\mathrm{i}\,t_q}\, P_\sigma^j \mathbf{U}_q\right] + \frac{1}{2\pi\mathrm{i}\vartheta} \sum_{n=l+1}^{q} \mathrm{e}^{-2\pi\,\mathrm{i}\,t_{n-1}\vartheta}\, [P_\sigma^j \mathbf{U}_n - P_\sigma^j \mathbf{U}_{n-1}]. \end{aligned}$$

Since $\Omega_{(j)} \subset \Omega^s$ for all $s \in I_j$, we also have $\Omega_{(j)} \subset \Omega_n$ for all $n \in \Lambda_j^N$ (if N is large enough). If $|\vartheta| \leq 1$ then, using (7.6) and (6.4), we can estimate the norm of $|\vartheta|^\alpha\, \hat{\mathbf{z}}_j^N(\vartheta)$ in $W^{-1,2}_{0,\sigma}(\Omega_{(j)})$ as follows:

$$\big\|\, |\vartheta|^\alpha\, \hat{\mathbf{z}}_j^N(\vartheta) \big\|_{-1,2;\,\Omega_{(j)}} \;\leq\; C(\Omega_{(j)})\, |\vartheta|^\alpha \sum_{n=l}^{q} d_n\, \|\mathbf{U}_n\|_{2:\,\Omega_{(j)}} \;\leq\; C(\Omega_{(j)})\, |\vartheta|^\alpha. \tag{7.7}$$

If $|\vartheta| > 1$ then we must proceed more subtly:

$$\begin{aligned} \big\|\, |\vartheta|^\alpha\, \hat{\mathbf{z}}_j^N(\vartheta) \big\|_{-1,2;\,\Omega_{(j)}} \;&\leq\; \frac{|\vartheta|^{\alpha-1}}{2\pi} \left(\|P_\sigma^j \mathbf{U}_l\|_{-1,2;\,\Omega_{(j)}} + \|P_\sigma^j \mathbf{U}_q\|_{-1,2;\,\Omega_{(j)}}\right) \\ &\quad + \frac{|\vartheta|^{\alpha-1}}{2\pi} \sum_{n=l+1}^{q} \|P_\sigma^j \mathbf{U}_n - P_\sigma^j \mathbf{U}_{n-1}\|_{-1,2;\,\Omega_{(j)}} \\ &\leq\; C(\Omega_{(j)})\, |\vartheta|^{\alpha-1} \left(\|\mathbf{U}_l\|_{2:\,\Omega_{(j)}} + \|\mathbf{U}_q\|_{2:\,\Omega_{(j)}}\right) \\ &\quad + \frac{|\vartheta|^{\alpha-1}}{2\pi} \sum_{n=l+1}^{q} \sup_{\psi_n} \frac{1}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \left| \int_{\Omega_{(j)}} (\mathbf{U}_n - \mathbf{U}_{n-1}) \cdot \boldsymbol{\psi}_n\, \mathrm{d}\mathbf{x} \right| \end{aligned} \tag{7.8}$$

where the supremum is taken over all $\boldsymbol{\psi}_n \in W^{1,2}_{0,\sigma}(\Omega_{(j)})$ such that $\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}} > 0$.

The sum in (7.8) can be split to two parts and estimated by $\mathcal{Z}_1 + \mathcal{Z}_2$ where

$$\begin{aligned} \mathcal{Z}_1 \;&=\; \sum_{n=l+1}^{q} \sup_{\psi_n} \frac{1}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \left| \int_{\Omega_{(j)}} \left[\mathbf{U}_n(\mathbf{x}) - \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big)\right] \cdot \boldsymbol{\psi}_n(\mathbf{x})\, \mathrm{d}\mathbf{x} \right|, \\ \mathcal{Z}_2 \;&=\; \sum_{n=l+1}^{q} \sup_{\psi_n} \frac{1}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \left| \int_{\Omega_{(j)}} \left[\mathbf{U}_{n-1}(\mathbf{x}) - \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big)\right] \cdot \boldsymbol{\psi}_n(\mathbf{x})\, \mathrm{d}\mathbf{x} \right|. \end{aligned}$$

The estimate of $\mathcal{Z}_1$. Function $\boldsymbol{\psi}_n$, extended by zero to $\mathbb{R}^3_+ \smallsetminus \Omega_{(j)}$, belongs to $W^{1,2}_\sigma(\Omega_n)$. Hence the integral of $\left[\mathbf{U}_n(\mathbf{x}) - \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big)\right] \cdot \boldsymbol{\psi}_n(\mathbf{x})$ on $\Omega_{(j)}$ equals the integral of the same function in Ω_n and it can be therefore expressed by means of (5.2). Thus, $\mathcal{Z}_1$ can be estimated:

$$\begin{aligned}
\mathcal{Z}_1 \quad &\leq \quad \sum_{n=l+1}^{q} \sup_{\psi_n} \frac{1}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \bigg| d_n \int_{\Omega_n} \mathbf{U}_n \cdot \nabla \boldsymbol{\psi}_n \cdot \mathbf{a}(\,.\,, t_n)\; \mathrm{d}\mathbf{x} \\
&- \quad d_n \int_{\Omega_n} \mathbf{U}_n \cdot \nabla \mathbf{U}_n \cdot \boldsymbol{\psi}_n \; \mathrm{d}\mathbf{x} - 2 d_n \int_{\Omega_n} \nu \big\{ [\nabla \mathbf{a}]_n + \nabla \mathbf{U}_n \big\}_s : \nabla \boldsymbol{\psi}_n \; \mathrm{d}\mathbf{x} \\
&- \quad d_n \int_{\Gamma_n} \gamma \left[\mathbf{A}_k + \mathbf{U}_n - \mathbf{V}_n \right] \cdot \boldsymbol{\psi}_n \; \mathrm{d}S + d_n \int_{\Omega_n} \mathbf{g}_n \cdot \boldsymbol{\psi}_n \; \mathrm{d}\mathbf{x} \bigg|.
\end{aligned}$$

The surface integral on Γ_n equals zero because the function $\boldsymbol{\psi}_n$ is zero on Γ_n. All other terms on the right hand side can be estimated by $C(\Omega_{(j)})$ by means of (6.4), (6.5), standard inequalities based on the Sobolev imbedding theorem (applied in $\Omega_{(j)}$) and the Hölder inequality. Let us show the procedure in greater detail, for example, in the case of the terms containing the product $\mathbf{U}_n \cdot \nabla \mathbf{U}_n \cdot \boldsymbol{\psi}_n$:

$$\begin{aligned}
&\sum_{n=l+1}^{q} \sup_{\psi_n} \frac{1}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \left| d_n \int_{\Omega_n} \mathbf{U}_n \cdot \nabla \mathbf{U}_n \cdot \boldsymbol{\psi}_n \; \mathrm{d}\mathbf{x} \right| \\
&\quad \leq \quad \sum_{n=l+1}^{q} \frac{\|\boldsymbol{\psi}_n\|_{6;\,\Omega_{(j)}}}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \, d_n \, \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n} \, \|\mathbf{U}_n\|_{2;\,\Omega_n}^{1/2} \, \|\mathbf{U}_n\|_{6;\,\Omega_n}^{1/2} \\
&\quad \leq \quad C(\Omega_{(j)}) \sum_{n=l+1}^{q} d_n \left(\|\nabla \mathbf{U}_n\|_{2;\,\Omega_n} \, \|\mathbf{U}_n\|_{2;\,\Omega_n} + \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^{3/2} \, \|\mathbf{U}_n\|_{2;\,\Omega_n}^{1/2} \right) \\
&\quad \leq \quad C(\Omega_{(j)}, c_5)\, d_n \left[\left(\sum_{n=l+1}^{q} \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^2 \right)^{1/2} N^{1/2} + \left(\sum_{n=l+1}^{q} \|\nabla \mathbf{U}_n\|_{2;\,\Omega_n}^2 \right)^{3/4} N^{1/4} \right] \\
&\quad \leq \quad C(\Omega_{(j)}, c_5, c_6).
\end{aligned}$$

The estimate of $\mathcal{Z}_2$. In order to estimate $\mathcal{Z}_2$, we use the identities

$$\begin{aligned}
\mathbf{U}_{n-1}(\mathbf{x}) - \mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\, t_n, \mathbf{x})\big) \;&=\; \int_{t_{n-1}}^{t_n} \frac{\mathrm{d}}{\mathrm{d}\xi}\, \mathbf{U}_{n-1}\big(\mathbf{X}(\xi;\, t_n, \mathbf{x})\big)\; \mathrm{d}\xi \\
&=\; \int_{t_{n-1}}^{t_n} \mathbf{a}\big(\mathbf{X}(\xi;\, t_n, \mathbf{x}), \xi\big) \cdot \nabla \mathbf{U}_{n-1}\big(\mathbf{X}(\xi;\, t_n, \mathbf{x})\big)\; \mathrm{d}\xi.
\end{aligned}$$

Then the sum $\mathcal{Z}_2$ can be estimated by means of condition (a4) and (6.5) as follows:

$$\begin{aligned}
\mathcal{Z}_2 \quad &\leq \quad C(\Omega_{(j)}) \sup_{\psi_n} \frac{\|\boldsymbol{\psi}_n\|_{6;\,\Omega_{(j)}}}{\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega_{(j)}}} \sum_{n=l+1}^{q} \left[\int_{t_{n-1}}^{t_n} \int_{\Omega_{(j)}} \big| \nabla \mathbf{U}_{n-1}\big(\mathbf{X}(\xi;\, t_n, \mathbf{x})\big) \big|^2 \, \mathrm{d}\mathbf{x}\, \mathrm{d}\xi \right]^{1/2} \\
&\qquad \cdot \left[\int_{t_{n-1}}^{t_n} \left(\int_{\Omega_{(j)}} \big| \mathbf{a}\big(\mathbf{X}(\xi;\, t_n, \mathbf{x}), \xi\big) \big|^3 \, \mathrm{d}\mathbf{x} \right)^{2/3} \mathrm{d}\xi \right]^{1/2} \\
&\leq \quad C(\Omega_{(j)}) \left[\sum_{n=l+1}^{q} \int_{t_{n-1}}^{t_n} \int_{\Omega_{k-1}} \big| \nabla \mathbf{U}_{n-1}(\mathbf{x}) \big|^2 \, \mathrm{d}\mathbf{x}\, \mathrm{d}\xi \right]^{1/2} \left[\int_0^T \|\mathbf{a}(\,.\,, \xi)\|_{1,2;\,\Omega^\xi}^2 \, \mathrm{d}\xi \right]^{1/2} \\
&\leq \quad C\big(\Omega_{(j)}, c_6, \textstyle\int_0^T \theta_1(t)\, \mathrm{d}t\big).
\end{aligned}$$

Application of Lemma 4. Substituting the estimates of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ to (7.8), we finally obtain

$$\big\| |\vartheta|^\alpha \, \hat{\mathbf{z}}_j^N(\vartheta) \big\|_{-1,2;\,\Omega_{(j)}} \;\leq\; C\big(\Omega_{(j)}, c_5, c_6, \textstyle\int_0^T \theta_1(t)\,\mathrm{d}t\big)\; |\vartheta|^{\alpha-1}. \tag{7.9}$$

The constant $C\big(\Omega_{(j)}, c_5, c_6, \int_0^T \theta_1(t)\,\mathrm{d}t\big)$ is independent of N. Recall that inequality (7.9) holds for $|\vartheta| > 1$. Since the exponent α satisfies $0 < \alpha < \frac{1}{2}$, the right hand side of (7.9) is integrable on $(-\infty, -1) \cup (1, +\infty)$ with power 2. This, together with (7.7), implies that the sequence $\{|\vartheta|^\alpha \, \hat{\mathbf{z}}_j^N(\vartheta)\}$ is bounded in $L^2(\mathbb{R};\, W^{-1,2}_{0,\sigma}(\Omega_{(j)}))$. This further implies that the sequence $\{\mathbf{w}_j^N\}$ is bounded in $\mathcal{H}^\alpha\big(I_j;\, W^{1,2}_\sigma(\Omega_{(j)}),\, W^{-1,2}_{0,\sigma}(\Omega_{(j)})\big)$. This space is reflexive, hence there exists a subsequence (we denote it again by $\{\mathbf{w}_j^N\}$) which converges weakly in $\mathcal{H}^\alpha\big(I_j;$ $W^{1,2}_\sigma(\Omega_{(j)}), W^{-1,2}_{0,\sigma}(\Omega_{(j)})\big)$. Due to (7.5), the limit must be $\mathbf{w}_j$. Applying now Lemma 4, we have: $\mathbf{w}_j^N \longrightarrow \mathbf{w}_j = P^j_\sigma \mathbf{u}$ strongly in $L^2\big(I_j;\, L^2(\Omega_{(j)})^3\big)$.

7.6 Completion of the proof of Theorem 1. This strong convergence, together with the weak convergence (7.5), enables us to pass to the limit in the first three terms on the left hand side of (7.4). The procedure is standard (see e.g. J. L. LIONS [21] or R. TEMAM [30]), therefore we omit the details. Using also the identity

$$\int_{\Omega_{(j)}} (\nabla\varphi_j \cdot \nabla)\nabla\varphi_j \cdot \boldsymbol{\phi}_j^{**} \,\mathrm{d}\mathbf{x} \;=\; 0,$$

following from the inclusion $\boldsymbol{\phi}_j^{**} \in L^2_\sigma(\Omega_{(j)})$ and from the fact that $(\nabla\varphi_j \cdot \nabla)\nabla\varphi_j$ equals $\nabla\big(\frac{1}{2}|\nabla\varphi_j|^2\big)$, we can verify the validity of (7.4), and consequently also the validity of (6.11). This confirms that $\mathbf{u}$ is a weak solution of the problem (1.1)–(1.5). The proof of Theorem 1 is completed.

8 Example: The flow around two striking bodies with ball–shaped front surfaces

The geometrical configuration. We assume that two compact bodies B_1^t and B_2^t move in $\mathbb{R}^3$ in the time interval $[0,T]$ and they strike at the time instant $t^c \in (0,T)$. Thus, the time–variable domain Ω^t has the form $\Omega^t = \mathbb{R}^3 \smallsetminus (B_1^t \cup B_2^t)$ and set $\mathcal{T}^c$ of critical times in $(0,T)$, when the considered bodies collide, is the one point set $\mathcal{T}^c = \{t^c\}$. We assume that conditions (a1) and (a2) from Section 2 are fulfilled (with $D = \mathbb{R}^3$ and $K = 2$). Furthermore, we assume that

(a0) bodies B_1^t and B_2^t touch themselves at time t^c by material points $P_1^t \in \partial B_1^t$ and $P_2^t \in \partial B_2^t$, in whose neighbourhoods the surfaces of B_1^t and B_2^t coincide with surfaces S_1^t and S_2^t of the balls with the radii R_1 and R_2.

We can deduce from these assumptions that there exists $\tau > 0$ such that for t in the time interval $(t^c - \tau, t^c + \tau)$:

- The shortest line segment ℓ^t connecting B_1^t and B_2^t has the end points on surfaces S_1^t and S_2^t.
- There exists a Cartesian coordinate system y_1^t, y_2^t, y_3^t such that ℓ^t is a subset of the y_3^t–axis, the origin O^t is in the middle of ℓ^t and the transformation $y_i^t = U_{ij}^t x_j + V_i^t$ $(i,j = 1,2,3)$ between the Cartesian coordinates x_1, x_2, x_3 and y_1^t, y_2^t, y_3^t is smooth: i.e. the entries of the 3×3 unitary matrix $U^t = (U_{ij}^t)$ and the components of the vector $\mathbf{V}^t = (V_1^t, V_2^t, V_3^t)$ are functions from $C^2\big([0,t^c) \cup (t^c, T]\big)$, continuous on $[0,T]$.

- The length δ^t of line segment ℓ^t, as a function of variable t, is continuous on $[0,T]$ and such that $\delta^t = 0$ for $t = t^c$ and $\delta^t > 0$ for $t \in [0,t^c) \cup (t^c,T]$. Moreover, it belongs to $C^2\big([0,t^c) \cup (t^c,T]\big)$.
- There exists $r > 0$ so that the graph of the function $y_3^t = g_1(y_1^t, y_2^t, \delta^t)$ (respectively $y_3^t = -g_2(y_1^t, y_2^t, \delta^t)$), where

  $$g_k(y_1^t, y_2^t, \delta^t) \;:=\; \tfrac{1}{2}\delta^t + R_k - \sqrt{R_k^2 - (y_1^t)^2 - (y_2^t)^2} \quad \text{for } k = 1,\, 2 \text{ and } (y_1^t)^2 + (y_2^t)^2 \le r^2,$$

  is a subset of $\partial B_1^t \cap S_1^t$ (respectively $\partial B_2^t \cap S_2^t$), containing point P_1^t (respectively P_2^t), for $t^c - \tau < t < t^c + \tau$. We further denote the mentioned graphs by S_{1c}^t (respectively S_{2c}^t).
- The integral of $\boldsymbol{\phi} \cdot \nabla \mathbf{n} \cdot \boldsymbol{\phi}$ on Γ^t on the left–hand side of inequality (2.5) in condition (a3) can be split to the integral on $S_{1c}^t \cup S_{2c}^t$, where $\nabla \mathbf{n}$ is negative semi–definite, and the integral on $\Gamma^t \smallsetminus (S_{1c}^t \cup S_{2c}^t)$, where $\nabla \mathbf{n}$ is bounded and one can use the continuity of the operator of traces from $W_\sigma^{1,2}(\Omega^t)$ into $L^2\big(\Gamma^t \smallsetminus (S_{1c}^t \cup S_{2c}^t)\big)^3$ with a constant in the corresponding inequality independent of t. Thus, we can verify that condition (a3) holds in our concrete considered situation.

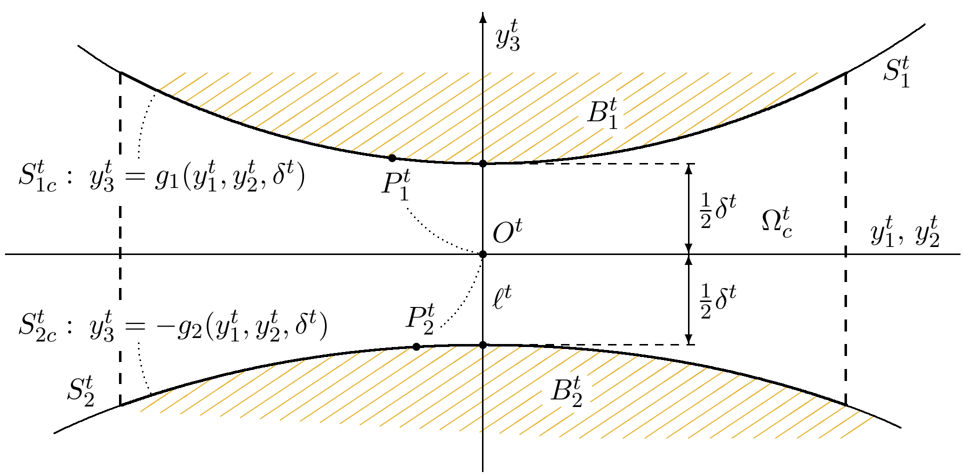

Fig. 3: The shapes of bodies B_1^t and B_2^t near the point of the collision at times t close to the instant t^c of the collision

We denote by Ω_c^t the critical sub–domain of Ω^t, where the collision occurs, namely $\Omega_c^t := \big\{\mathbf{y}^t = (y_1^t, y_2^t, y_3^t) \in \mathbb{R}^3;\ (y_1^t)^2 + (y_2^t)^2 < r^2 \text{ and } -g_2(y_1^t, y_2^t, \delta^t) < y_3^t < g_1(y_1^t, y_2^t, \delta^t)\big\}$.

In order to apply Theorem 1 in this geometrical configuration, we need to construct function $\mathbf{a}$, satisfying the equation of continuity (2.6), the boundary condition (2.7) and conditions (a1)–(a5) formulated in Section 2. **We confine ourselves only to the definition of function $\mathbf{a}$ in domain Ω_c^t for $t \in (t^c - \tau, t^c) \cup (t^c, t^c + \tau)$.** We consider an appropriate extension of $\mathbf{a}$ to the set $Q^*_{[0,t^c) \cup (t^c,T]}$ to be only a matter of standard techniques and therefore we do not describe it here. In accordance with this philosophy, we will sketch the verification of conditions (2.6), (2.7) and (a1)–(a5) only for the part of function $\mathbf{a}$, defined in Ω_c^t.

9.1 Definition of the auxiliary function $\mathbf{a}$ in Ω_c^t. At first we define vectorial potentials $\mathbf{w}_1$ and $\mathbf{w}_2$ as functions of the spatial variables $\mathbf{y}^t = (y_1^t, y_2^t, y_3^t)$:

$$\mathbf{w}_k(\mathbf{y}^t,\delta^t,\dot\delta^t) \;:=\; \pm\dot\delta^t\left(\frac{y_2^t y_3^t}{2g_k(y_1^t,y_2^t,\delta^t)},\, -\frac{y_1^t y_3^t}{2g_k(y_1^t,y_2^t,\delta^t)},\, 0\right) \tag{8.1}$$

for $k = 1,\, 2$ in the closure of Ω_c^t, where the sign "+" holds for $k = 2$ and "−" holds for $k = 1$. Further, we define vectorial potential $\mathbf{w}$ as an interpolation between $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{w}$ coincides with $\mathbf{w}_1$ in the neighbourhood of surface S_1^t and with $\mathbf{w}_2$ in the neighbourhood of S_2^t:

$$\mathbf{w}(\mathbf{y}^t,\delta^t,\dot\delta^t) \;:=\; \mathbf{w}_1(\mathbf{y}^t,\delta^t,\dot\delta^t)\,\eta(\mathbf{y}^t,\delta^t) + \mathbf{w}_2(\mathbf{y}^t,\delta^t,\dot\delta^t)\,\big[1-\eta(\mathbf{y}^t,\delta^t)\big], \tag{8.2}$$

where

$$\eta(\mathbf{y}^t,\delta^t) := \zeta\left(\frac{y_3^t + g_2(y_1^t,y_2^t,\delta^t)}{g_1(y_1^t,y_2^t,\delta^t)+g_2(y_1^t,y_2^t,\delta^t)}\right);$$

ζ is an infinitely differentiable cut–off function on the interval $[0,1]$, such that $0\le\zeta(s)\le 1$ for $0\le s\le 1$, $\zeta(s)=0$ for $0\le s\le\frac14$ and $\zeta(s)=1$ for $\frac34\le s\le 1$. Now we define

$$\widetilde{\mathbf{a}}(\mathbf{y}^t,\delta^t,\dot\delta^t) \;:=\; \mathbf{curl}\,\mathbf{w}(\mathbf{y}^t,\delta^t,\dot\delta^t); \tag{8.3}$$

$\mathbf{curl}$ being calculated with respect to the spatial variable $\mathbf{y}^t$. Formula (8.2) yields

$$\widetilde{\mathbf{a}} \;=\; \eta\,\widetilde{\mathbf{a}}_1 + [1-\eta]\,\widetilde{\mathbf{a}}_2 + \dot\delta^t\,\big[\nabla\eta\times\mathbf{w}_1 - \nabla\eta\times\mathbf{w}_2\big] \tag{8.4}$$

where $\widetilde{\mathbf{a}}_1 := \mathbf{curl}\,\mathbf{w}_1^t$, $\widetilde{\mathbf{a}}_2 := \mathbf{curl}\,\mathbf{w}_2^t$ and ∇ is also considered with respect to variable $\mathbf{y}^t$. Finally, function $\mathbf{a}$ arises from $\widetilde{\mathbf{a}}$ by means of the transformation

$$\mathbf{a}(\mathbf{x},t) \;:=\; (U^t)^T\cdot\,\widetilde{\mathbf{a}}\big(U^t\!\cdot\mathbf{x}+\mathbf{V}^t,\,\delta^t,\,\dot\delta^t\big). \tag{8.5}$$

9.2 Conditions (2.6) and (2.7). The mapping $G^t(\mathbf{x}) := \mathbf{y}^t = U^t\!\cdot\mathbf{x}+\mathbf{V}^t$ represents an isometry of $G_{-1}^t(\Omega_c^t)$ onto Ω_c^t, smoothly depending on t for $t\in[0,t^c)\cup(t^c,T]$. Thus, we can verify (2.6), (2.7) and (a1)–(a5) directly for function $\widetilde{\mathbf{a}}(\mathbf{y}^t,\delta^t,\dot\delta^t)$, considering $\mathbf{y}^t\in\Omega_c^t$. Moreover, due to the smooth dependence of matrix U^t and vector $\mathbf{V}^t$ on t, we can choose the coordinate system y_1^t, y_2^t, y_3^t to be the new reference frame for our calculations at each time and consequently, not to take into account the dependence of the coordinates y_1^t, y_2^t, y_3^t on t any more. Thus, we shall further write only $\mathbf{y}$ or y_1, y_2, y_3 instead of $\mathbf{y}^t$ or y_1^t, y_2^t, y_3^t.

Function $\widetilde{\mathbf{a}}$ is divergence–free because it is defined to be a curl of the vectorial potential $\mathbf{w}$.

Naturally, condition (2.7) for function $\widetilde{\mathbf{a}}$ makes sense only on $S_{1c}^t\cup S_{2c}^t$. Function $\widetilde{\mathbf{a}}$ coincides with $\widetilde{\mathbf{a}}_1$ in the neighbourhood of surface S_1^t and it coincides with $\widetilde{\mathbf{a}}_2$ in the neighbourhood of S_2^t. Calculating the curl of $\mathbf{w}_1$, we obtain

$$\widetilde{\mathbf{a}}_1(\mathbf{y},\delta^t,\dot\delta^t) \;=\; \dot\delta^t\cdot\left(-\frac{y_1}{2g_1},\, -\frac{y_2}{2g_1},\, \frac{y_3}{g_1} - \frac{y_1y_3(\partial_1g_1)+y_2y_3(\partial_2g_1)}{2g_1^2}\right)$$

where g_1 abbreviates $g_1(y_1,y_2,\delta^t)$. On S_{1c}^t, the outer normal vector $\mathbf{n}$ equals $\big(\partial_1g_1,\partial_2g_1,-1\big)/\sqrt{(\partial_1g_1)^2+(\partial_2g_1)^2+1}$ because $y_3=g_1$. Hence we have

$$\begin{aligned}\widetilde{\mathbf{a}}\cdot\mathbf{n}\big|_{S_{1c}^t} \;&=\; \widetilde{\mathbf{a}}_1\cdot\mathbf{n}\big|_{S_{1c}^t} \;=\; \dot\delta^t\left(-\frac{y_1}{2g_1},\, -\frac{y_2}{2g_1},\, 1-\frac{y_1\,\partial_1g_1+y_2\,\partial_2g_1}{2g_1}\right)\cdot\mathbf{n}\\ &=\; \frac{(-\dot\delta^t)}{\sqrt{(\partial_1g_1)^2+(\partial_2g_1)^2+1}} \;=\; \big(0,\,0,\,\dot\delta^t\big)\cdot\mathbf{n}.\end{aligned} \tag{8.6}$$

Since the velocity $\mathbf{V}$ of material points on surface S^t_{1c}, expressed in the reference frame $y_1,\, y_2,\, y_3$, equals the sum of the vertical component $(0,\,0,\,\dot{\delta}^t)$ and a component tangential to S^t_{1c} (due to the rotation of body B^t_1), the right hand side of (8.6) equals $\mathbf{V}\cdot\mathbf{n}$. Thus, condition (2.7) holds on surface S^t_{1c}. The validity of (2.7) on S^t_{2c} follows in the same way.

9.3 Conditions (a4)–(a6) and (a8). Since the verification of all the conditions is lengthy and technical, we choose only the first term $\eta\widetilde{\mathbf{a}}_1$ in the sum on the right hand side of (8.4) and we sketch the procedure for this term. Moreover, it is advantageous to work in the cylindrical coordinates ρ, φ, y_3, where $\rho=(y_1^2+y_2^2)^{1/2}$. The form of function $\widetilde{\mathbf{a}}_1$ in the cylindrical coordinates is

$$\widetilde{\mathbf{a}}_1(\mathbf{y},\delta^t,\dot{\delta}^t)\ =\ \big(\widetilde{a}_{1\rho},\,\widetilde{a}_{1\varphi},\,\widetilde{a}_{13}\big)\ =\ \dot{\delta}^t\cdot\bigg(-\frac{\rho}{2g_1},\ 0,\ -\frac{y_3\rho\,g_1'}{2g_1^2}+\frac{y_3}{g_1}\bigg),$$

where g_1 is now considered to be a function of the variables ρ and δ^t: $g_1(\rho,\delta^t)=\frac{1}{2}\delta^t+R_1-\sqrt{R_1^2-\rho^2}$ (for $\rho\le r$).

The gradient of $\eta\widetilde{\mathbf{a}}_1$ is $\nabla(\eta\widetilde{\mathbf{a}}_1)=\eta\nabla\widetilde{\mathbf{a}}_1+\nabla\eta\otimes\widetilde{\mathbf{a}}_1$. Calculating the non–zero cylindrical components of $\nabla\widetilde{\mathbf{a}}_1$ and $\nabla\eta$ and substituting there the explicit forms of the derivatives $(g_1)'$ and $(g_2)'$, we obtain

$$\begin{aligned}
\partial_\rho\widetilde{a}_{1\rho} &= -\frac{\dot{\delta}^t}{2g_1}+\frac{\dot{\delta}^t\rho^2}{2g_1^2\sqrt{R_1^2-\rho^2}}\\
\partial_\rho\widetilde{a}_{13} &= -\frac{y_3\dot{\delta}^t\rho(4R_1^2-3\rho^2)}{2g_1^2\,(R_1^2-\rho^2)^{3/2}}+\frac{y_3\dot{\delta}^t\rho^3}{g_1^3\,(R_1^2-\rho^2)},\\
\partial_3\widetilde{a}_{13} &= -\frac{\dot{\delta}^t\rho\,g_1'}{2g_1^2}+\frac{\dot{\delta}^t}{g_1},\\
\partial_\rho\eta &= \zeta'\bigg(\frac{y_3+g_2}{g_1+g_2}\bigg)\bigg[-\frac{\rho\,(y_3-g_1)}{(g_1+g_2)^2\sqrt{R_2^2-\rho^2}}-\frac{\rho\,(y_3+g_2)}{(g_1+g_2)^2\sqrt{R_1^2-\rho^2}}\bigg],\\
\partial_3\eta &= \zeta'\bigg(\frac{y_3+g_2}{g_1+g_2}\bigg)\frac{1}{g_1+g_2}.
\end{aligned}$$

Using these formulas, one can show that $\eta\widetilde{\mathbf{a}}_1$ and $\partial_t(\eta\widetilde{\mathbf{a}}_1)$ are continuous in $\{(\mathbf{y},t)\in\mathbb{R}^4;\ t\in[0,T]\smallsetminus\mathcal{T}^c,\ \mathbf{y}\in\Omega^t_c\cup S^t_{1c}\cup S^t_{2c}\}$ and the norms $\|\eta\widetilde{\mathbf{a}}_1\|_{1,2;\,\Omega^t_c}$ and $\|\eta\widetilde{\mathbf{a}}_1-\mathbf{V}\|_{2;\,S^t_{1c}\cup S^t_{2c}}$ are square integrable (as functions of t) in $(0,T)$. Showing the same for the other terms on the right hand side of (8.4) and considering an appropriate extension of function $\mathbf{a}$ from Ω^t_c to Ω^t, we verify conditions (a4)–(a6) and (a8). Moreover, we find out that function θ_1 in condition (a5) is in $L^q(0,T)$ for each $1\le q<+\infty$.

9.4 Condition (a7). Of inequalities (2.9) and (2.10) in condition (a7), we focus on (2.10), which, as we shall see, induces a certain restriction on the size of $\dot{\delta}^t$ in the neighbourhood of the critical time instant t^c of the collision. Thus, let $\boldsymbol{\phi}\in W^{1,2}_\sigma(\Omega^t)$.

Using assumption (a4) and the continuous imbedding $L^6(\Omega^t)\hookrightarrow W^{1,2}(\Omega^t)$, we can derive that

$$\bigg|\int_{\Omega^t}\boldsymbol{\phi}\cdot\nabla\mathbf{a}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\bigg|\ \le\ c_{15}\,a(t)\,\|\boldsymbol{\phi}\|^2_{2;\,\Omega^t}+\frac{\nu}{10}\,\|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t} \tag{8.7}$$

where $a(t):=\theta_1(t)+\theta_1^4(t)$. Inequality (8.7) holds at times $t\ne t^c$ when domain Ω^t has the cone property and $W^{1,2}(\Omega^t)^3$ is therefore continuously imbedded into $L^6(\Omega^t)^3$. Constant c_{15} depends on ν and it also generally depends on t through the cone parameters appearing in the definition of the cone property of Ω^t, see e.g. [1, p. 103]. However, if we use (8.7) only at times t such that $|t-t^c| > \tau$ then c_{15}, although depending on τ, can be considered to be independent of t. Consequently, inequality (2.10) is satisfied (with $\theta_5(t) := c_{15}\, a(t)$) for $t \in [0, t^c-\tau) \cup (t^c+\tau, T]$.

On the other hand, since c_{15} blows up for $t \to t^c$, we use another techniques in order to estimate the integral of $\boldsymbol{\phi} \cdot \nabla \mathbf{a} \cdot \boldsymbol{\phi}$ for $t \in (t^c-\tau, t^c) \cup (t^c, t^c+\tau)$. As we have already mentioned, we confine ourselves only to the critical sub–domain Ω^t_c of Ω^t. We have

$$\left| \int_{\Omega^t_c} \boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi} \cdot \mathbf{a}(\,.\,,t)\; \mathrm{d}\mathbf{y} \right| \;\leq\; \|\nabla \boldsymbol{\phi}\|_{2;\,\Omega^t_c} \left(\int_{\Omega^t_c} |\widetilde{\mathbf{a}}(\,.\,,t)|^2\, |\boldsymbol{\phi}|^2\, \mathrm{d}\mathbf{y} \right)^{1/2}. \tag{8.8}$$

Following the restriction that we explain the important steps only with the first term $\eta\widetilde{\mathbf{a}}_1$ on the right hand side of (8.4), instead with the whole right hand side, and taking into account that the decisive contribution to $|\eta\widetilde{\mathbf{a}}_1|$ comes from the the component of $\eta\widetilde{a}_{1\rho}$, we get

$$\begin{aligned}
\int_{\Omega^t_c} |\eta\widetilde{a}_{1\rho}|^2\, |\boldsymbol{\phi}|^2\, \mathrm{d}\mathbf{y} \;&=\; |\dot{\delta}^t| \int_0^r \rho\, \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} |\eta|\, \frac{\rho^2}{4g_1^2(\rho,\delta^t)} \left[\phi_\rho^2 + \phi_\varphi^2 + \phi_3^2\right] \mathrm{d}y_3 \\
&\leq\; |\dot{\delta}^t| \int_0^r \frac{\rho^3\, \mathrm{d}\rho}{4g_1^2(\rho,\delta^t)} \int_0^{2\pi} \mathrm{d}\varphi \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \sum_{j\in\{\rho;\,\varphi;\,3\}} \left[\phi_j\big(\rho,\varphi,g_1(\rho)\big) + \int_{g_1(\rho)}^{y_3} \partial_3\phi_j(\rho,\varphi,\xi)\, \mathrm{d}\xi \right]^2 \mathrm{d}y_3 \\
&\leq\; |\dot{\delta}^t| \sum_{j\in\{\rho;\,\varphi;\,3\}} \int_0^r \frac{\rho^3\, \mathrm{d}\rho}{4g_1(\rho,\delta^t)} \int_0^{2\pi} \mathrm{d}\varphi \Bigg[\big[g_1(\rho,\delta^t) + g_2(\rho,\delta^t)\big]\, \phi_j^2\big(\rho,\varphi,g_1(\rho,\delta^t)\big) \\
&\qquad + \big[g_1(\rho,\delta^t) + g_2(\rho,\delta^t)\big]^2 \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \big[\partial_3\phi_j(\rho,\varphi,\xi)\big]^2\, \mathrm{d}\xi \Bigg] \\
&\leq\; c_9(R_1,R_2)\, |\dot{\delta}^t| \int_0^r \rho\; \mathrm{d}\rho \int_0^{2\pi} \big|\boldsymbol{\phi}\big(\rho,\varphi,g_1(\rho,\delta^t)\big)\big|^2\, \mathrm{d}\varphi \\
&\qquad + c_{10}(R_1,R_2)\, |\dot{\delta}^t|\, r^2 \int_0^r \rho\; \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \big|\partial_3\boldsymbol{\phi}(\rho,\varphi,y_3)\big|^2\, \mathrm{d}y_3 \\
&\leq\; c_9(R_1,R_2)\, |\dot{\delta}^t|\, \|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t} + c_{10}(R_1,R_2)\, |\dot{\delta}^t|\, r^2\, \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t}.
\end{aligned}$$

Considering all the terms in the expansion (8.4) of $\widetilde{\mathbf{a}}$ (and not only the term $\eta\widetilde{\mathbf{a}}_1$), with all their components (i.e. not only with the component $\eta\widetilde{a}_{1\rho}$), we can derive the same inequality, only with different constants c_{11} and c_{12} instead of c_9 and c_{10}:

$$\left| \int_{\Omega^t_c} |\widetilde{\mathbf{a}}(\,.\,,t)|^2\, |\boldsymbol{\phi}|^2\; \mathrm{d}\mathbf{y} \right| \;\leq\; c_{11}(R_1,R_2)\, |\dot{\delta}^t|\, \|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t} + c_{12}(R_1,R_2)\, |\dot{\delta}^t|\, r^2\, \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t}.$$

Substituting these estimates to (8.8), we obtain

$$\left| \int_{\Omega^t_c} \boldsymbol{\phi} \cdot \nabla\boldsymbol{\phi} \cdot \mathbf{a}\; \mathrm{d}\mathbf{y} \right| \;\leq\; |\dot{\delta}^t|\, \|\nabla\boldsymbol{\phi}\|_{2;\,\Omega^t} \left[c_{11}\, \|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t} + c_{12}\, r^2\, \|\nabla\boldsymbol{\phi}\|_{2;\,\Omega^t} \right]^{1/2}$$

$$\leq \; |\dot{\delta}^t| \sqrt{c_{11}} \, \|\nabla\boldsymbol{\phi}\|_{2;\,\Omega^t} \, \|\boldsymbol{\phi}\|_{2;\,\Gamma^t} + |\dot{\delta}^t| \sqrt{c_{12}} \, r \, \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t}$$

$$\leq \; |\dot{\delta}^t| \left(\epsilon^t \, c_{11} + \sqrt{c_{12}} \, r \right) \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t} + \frac{|\dot{\delta}^t|}{4\epsilon^t} \, \|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t}$$

for any $\epsilon^t > 0$. Considering an appropriate smooth extension of function $\mathbf{a}$ from Ω^t_c to Ω^t, we get additional terms on the right hand side which are analogous to (8.6) and we arrive at the estimate

$$\left| \int_{\Omega^t} \boldsymbol{\phi} \cdot \nabla\boldsymbol{\phi} \cdot \mathbf{a} \, \mathrm{d}\mathbf{x} \right| \;\leq\; \left[|\dot{\delta}^t| \left(\epsilon^t \, c_{11} + \sqrt{c_{12}} \, r \right) + \xi \right] \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t} + \frac{|\dot{\delta}^t|}{4\epsilon^t} \, \|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t} + c_{13}(\xi) \, \|\boldsymbol{\phi}\|^2_{2;\,\Omega^t} \tag{8.9}$$

which is valid for $t \in (t^c - \tau, t^c) \cup (t^c, t^c + \tau)$ and $\xi > 0$.

Except for (8.9), there is another possibility how one can estimate the integral of $\boldsymbol{\phi} \cdot \nabla\boldsymbol{\phi} \cdot \mathbf{a}$ in Ω^t: integrating by parts, we show that this integral equals the negative integral of $\boldsymbol{\phi} \cdot \nabla\mathbf{a} \cdot \boldsymbol{\phi}$ in Ω^t. Confining again ourselves to the critical sub–domain Ω^t_c of Ω^t and to the part $\eta \nabla \widetilde{\mathbf{a}}_1$ of $\nabla\widetilde{\mathbf{a}}$, we find out that the decisive contribution to the integral of $\boldsymbol{\phi} \cdot (\eta \nabla \widetilde{\mathbf{a}}_1) \cdot \boldsymbol{\phi}$ comes from the term $\eta \, (\partial_\rho \widetilde{a}_{1\rho}) \, \phi^2_\rho$. Using the explicit form of $\partial_\rho \widetilde{a}_{1\rho}$ and applying Poincaré's inequality (see e.g. [5, R. Dautray and J. L. Lions, p. 127]), we obtain

$$\left| \int_{\Omega^t_c} \eta \, (\partial_\rho \widetilde{a}_{1\rho}) \, \phi^2_\rho \, \mathrm{d}\mathbf{y} \right| \;\leq\; C(R_1) \, |\dot{\delta}^t| \int_0^r \frac{\rho}{2 g_1(\rho, \delta^t)} \, \mathrm{d}\rho \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \mathrm{d}y_3 \left[\int_0^{2\pi} \phi^2_\rho \, \mathrm{d}\varphi \right]$$

$$\leq \; C(R_1) \, |\dot{\delta}^t| \int_0^r \frac{\rho}{2 g_1(\rho, \delta^t)} \, \mathrm{d}\rho \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \mathrm{d}y_3 \left[4\pi \int_0^{2\pi} (\partial_\varphi \phi_\rho)^2 \, \mathrm{d}\varphi + \frac{1}{2\pi} \, |\widetilde{\phi}_\rho|^2 \right]$$

$$\leq \; C(R_1, R_2) \, |\dot{\delta}^t| \, \|\nabla\phi_\rho\|^2_{2;\,\Omega^t_c} + C(R_1) \, |\dot{\delta}^t| \int_0^r \frac{\rho}{2 g_1(\rho, \delta^t)} \, \mathrm{d}\rho \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} |\widetilde{\phi}_\rho|^2 \, \mathrm{d}y_3 \tag{8.10}$$

where $\widetilde{\phi}_\rho(\rho, y_3) := \int_0^{2\pi} \phi_\rho(\rho, \varphi, y_3) \, \mathrm{d}\varphi$. Using the incompressibility of the flow $\boldsymbol{\phi}$ and the condition $\boldsymbol{\phi} \cdot \mathbf{n} = 0$ on $S^t_{1c} \cup S^t_{2c}$, we can deduce that $\int_{-g_2}^{g_1} \widetilde{\phi}_\rho \, \mathrm{d}y_3 = 0$. This implies that to each $\rho \in (0, r)$ there exists y^ρ_3 between $-g_2$ and g_1 such that $\widetilde{\phi}_\rho(\rho, y^\rho_3) = 0$. Thus, the second term on the right hand side of (8.10) can be estimated:

$$\ldots \;\leq\; C(R_1) \, |\dot{\delta}^t| \int_0^r \frac{\rho}{2 g_1(\rho, \delta^t)} \, \mathrm{d}\rho \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} \left(\widetilde{\phi}_\rho(\rho, y^\rho_3) + \int_{y^\rho_3}^{y_3} \partial_3 \widetilde{\phi}_\rho(\rho, \xi) \, \mathrm{d}\xi \right)^2 \mathrm{d}y_3$$

$$\leq \; C(R_1, R_2) \, |\dot{\delta}^t| \int_0^r \rho \, \mathrm{d}\rho \int_{-g_2(\rho,\delta^t)}^{g_1(\rho,\delta^t)} (\partial_3 \widetilde{\phi}_\rho)^2 \, \mathrm{d}\xi \;\leq\; C(R_1, R_2) \, |\dot{\delta}^t| \, \|\nabla\phi_\rho\|^2_{2;\,\Omega^t_c}.$$

This estimate, (8.10) and analogous estimates of all other terms in the expansion of $\boldsymbol{\phi} \cdot \nabla\mathbf{a} \cdot \boldsymbol{\phi}$ successively enable us to arrive at the inequality

$$\left| \int_{\Omega^t} \boldsymbol{\phi} \cdot \nabla\mathbf{a} \cdot \boldsymbol{\phi} \, \mathrm{d}\mathbf{x} \right| \;\leq\; c_{14} \, |\dot{\delta}^t| \, \|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t}. \tag{8.11}$$

Multiplying inequality (8.9) by κ (where $1 \leq \kappa \leq 1$), inequality (8.11) by $1-\kappa$ and summing afterwards both the inequalities, we get the estimate which generalizes (8.9) and (8.11):

$$\left|\int_{\Omega^t} \boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}\cdot\mathbf{a}\,\mathrm{d}\mathbf{x}\right| \;\leq\; \Big[\big(\kappa\epsilon^t c_{11}+\kappa\sqrt{c_{12}}\,r\big)\,|\dot{\delta}^t|+\kappa\xi+(1-\kappa)c_{14}\,|\dot{\delta}^t|\Big]\,\|\nabla\boldsymbol{\phi}\|^2_{2;\,\Omega^t} + \kappa\,\frac{|\dot{\delta}^t|}{4\epsilon^t}\,\|\boldsymbol{\phi}\|^2_{2;\,\Gamma^t}+\kappa c_{13}\,\|\boldsymbol{\phi}\|^2_{2;\,\Omega^t}. \tag{8.12}$$

Comparing (8.12) with (2.10), we observe that we need

$$\begin{aligned}\big(\kappa\epsilon^t c_{11}+\kappa\sqrt{c_{12}}\,r\big)\,|\dot{\delta}^t|+\kappa\xi+(1-\kappa)c_{14}\,|\dot{\delta}^t| \;&\leq\; \frac{\nu}{10},\\ \kappa\,\frac{|\dot{\delta}^t|}{4\epsilon^t}\;&\leq\;\frac{\gamma}{4}\end{aligned}$$

for $t\in(t^c-\tau,t^c)\cup(t^c,t^c+\tau)$. The second inequality is satisfied if we choose $\epsilon^t := \kappa|\dot{\delta}^t|/\gamma$. Substituting this ϵ^t to the first inequality, we obtain the condition

$$c_{11}\,\frac{\kappa^2}{\gamma}\,|\dot{\delta}^t|^2+\kappa\sqrt{c_{12}}\,r\,|\dot{\delta}^t|+\kappa\xi+(1-\kappa)c_{14}\,|\dot{\delta}^t| \;\leq\; \frac{\nu}{10}.$$

Since we can work with r and ξ arbitrarily small, it is sufficient to have

$$c_{11}\,\frac{\kappa^2}{\gamma}\,|\dot{\delta}^t|^2+(1-\kappa)c_{14}\,|\dot{\delta}^t| \;\leq\; \frac{\nu}{20} \tag{8.13}$$

fulfilled for $t\in(t^c-\tau,t^c)\cup(t^c,t^c+\tau)$ and some $\kappa\in[0,1]$. For instance, the choice $\kappa=0$ leads to the requirement that $|\dot{\delta}^t|\leq\nu/20c_{14}$. The choice $\kappa=1$ yields the condition $|\dot{\delta}^t|\leq\sqrt{\gamma\nu/20c_{11}}$. Generally, (8.13) is satisfied for some $\kappa\in[0,1]$ if

$$|\dot{\delta}^t| \;\leq\; \sup_{0<\kappa\leq 1}\;\frac{-(1-\kappa)c_{14}\gamma+\sqrt{(1-\kappa)^2c_{14}^2\gamma^2+\frac{1}{5}\kappa^2c_{11}\gamma\nu}}{2\kappa^2c_{11}} \tag{8.14}$$

for $t\in(t^c-\tau,t^c)\cup(t^c,t^c+\tau)$. Thus, we can conclude that the auxiliary function $\mathbf{a}$ satisfies condition (a4), namely inequality (2.10), if

(a9) there exists $\tau>0$ such that $|\dot{\delta}^t|$ satisfies inequality (8.14) for $t\in(t^c-\tau,t^c+\tau)\smallsetminus\{t^c\}$.

Condition (a9) can be interpreted as the condition on smallness of the speed with which bodies B_1^t and B_2^t collide at time instant t^c. We observe that the larger is the coefficient γ of friction between the fluid and the surface of bodies B_1^t, B_2^t or the coefficient of viscosity ν, the larger can be the speed $|\dot{\delta}^t|$.

Applying Theorem 1, we obtain:

Theorem 2. *Suppose that domain Ω^t satisfies the conditions (a1), (a2) with the specifications described at the beginning of Section 8, i.e. with $D=\mathbb{R}^3$ and $K=2$. Suppose further that conditions (a0) and (a9) are fulfilled. Then the weak solution of the problem (1.1)–(1.5), introduced in Definition 1, exists.*

Remark 8. We recall that the same theorem cannot hold if no–slip Dirichlet's boundary condition is considered instead of Navier's boundary condition (1.3), due to the results of V. N. STAROVOITOV [24]. Thus, the boundary condition (1.3) enables us to consider a larger class of collisions of bodies, moving in the viscous incompressible fluid, that the traditional no–slip boundary condition.

9.5 A note to the stroke of two compact bodies with general C^2 front surfaces. In this case, the shapes of the bodies B_1^t, B_2^t are not necessarily ball–like in the neighbourhoods of points P_1^t and P_2^t with which they collide at the time instant t^c. Taking also into account relative positions and motions of B_1^t and B_2^t in the time interval $(t^c-\tau, t^c+\tau)$, we can deduce that there exist two functions $y_3 = g_1^t(y_1, y_2, \delta^t)$ and $y_3 = g_2^t(y_1, y_2, \delta^t)$ (for $y_1^2+y_2^2 \leq r^2$) of the class C^2, whose forms generally depend on t and which play the same role as the functions g_1 and g_2 in the previous part of this section. By analogy with g_1 and g_2, the functions g_1^t and g_2^t satisfy:

$$\min_{y_1,\, y_2} g_1^t \,=\, g^t(0,0,\delta^t) \,=\, \min_{y_1,\, y_2} g_2^t \,=\, g_2^t(0,0,\delta^t) \,=\, \tfrac{1}{2}\delta^t.$$

The only restriction we have to impose on the shapes of bodies B_1^t and B_2^t in the neighbourhoods of points P_1^t and P_2^t (in addition to their smoothness) is:

(a10) $\exists\, r>0\ \exists\, c_{15}>0\ :\ g_k^t(y_1,y_2,\delta^t) \geq \frac{1}{2}\delta^t + c_{15}\left(y_1^2+y_2^2\right)$ for $k=1,\,2$ and $y_1^2+y_2^2 \leq r^2$.

We define, by analogy with (8.1),

$$\mathbf{w}_k^t(\mathbf{y}^t) \,:=\, \pm\left(\frac{(\partial_2 h_k^t)\, y_3^t}{2g_k^t(y_1^t,y_2^t,\delta^t)},\, -\frac{(\partial_1 h_k^t)\, y_3^t}{2g_k^t(y_1^t,y_2^t,\delta^t)},\, 0\right) \tag{8.15}$$

for $k=1,\,2$ in $\overline{\Omega_c^t}$, where the sign "$+$" holds for $k=2$ and "$-$" holds for $k=1$. The functions h_1^t and h_2^t of the variables y_1, y_2 are chosen so that

$$\frac{\partial^2 h_k^t}{\partial y_1^2} + \frac{\partial^2 h_k^t}{\partial y_2^2} \,=\, -\dot{g}_k^t - \frac{\partial g_k^t}{\partial \delta^t}\,\dot{\delta}^t \qquad \text{for }\ y_1^2+y_2^2 \leq r^2, \tag{8.16}$$

$$\frac{\partial h_k^t}{\partial y_1}(0,0) \,=\, \frac{\partial h_k^t}{\partial y_2}(0,0) \,=\, 0 \tag{8.17}$$

for $k=1,\,2$, where $\dot{g}_k^t$ denotes the partial derivative of g_k^t with respect to t. (Functions h_1^t and h_2^t, satisfying both the conditions (8.16) and (8.17), can be simply expressed as sums of appropriate Newton's potentials and single layer potentials.) We can arrive at the same theorem as Theorem 2, with condition (a9) replaced by condition (a10). To do that, we proceed with the construction of functions $\widetilde{\mathbf{a}}$ and $\mathbf{a}$ in the same way as in the first part of this section. Obviously, function $\widetilde{\mathbf{a}}$ is divergence–free due to the same reason as the function given by (8.3). Equation (8.16) implies the validity of condition (2.7), while the identities in (8.17) play the important role in the verification of conditions (a4)–(a8). However, the verification of validity of (a4)–(a8) requires more space and we prepare a special paper on this theme.

9 Appendices A1–A3

Appendix A1: Proof of Lemma 1. We write for simplicity only $\mathbf{U}$ and d instead of $\mathbf{U}_n$ and d_n. Since $t_n \notin \mathcal{T}^c$, domain Ω_n is Lipschitzian and consequently, $W_\sigma^{1,2}(\Omega_n) \hookrightarrow L^q(\Omega_n)^3$ for $1 \leq q \leq 6$. Moreover, there exists a continuous operator of traces from $W_\sigma^{1,2}(\Omega_n)$ into $L^2(\Gamma_n)^3$. Using assumption (a5), one can verify for given $\mathbf{U} \in W_\sigma^{1,2}(\Omega_n)$ that

$$\int_{\Omega_n} \big\{\mathbf{U}\cdot\boldsymbol{\Phi} - d\,\mathbf{U}\cdot\nabla\boldsymbol{\Phi}\cdot\mathbf{a}(\,.\,,t_n) + d\,\mathbf{U}\cdot\nabla\mathbf{U}\cdot\boldsymbol{\Phi} + 2d\nu\,(\nabla\mathbf{U})_s : \nabla\boldsymbol{\Phi}\big\}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_n} d\gamma\,\mathbf{U}\cdot\boldsymbol{\Phi}\,\mathrm{d}S$$

is a bounded linear functional in dependence on $\mathbf{\Phi} \in W^{1,2}_\sigma(\Omega_n)$. Thus, it can be written in the form $\big\langle \mathcal{A}(\mathbf{U}_n), \mathbf{\Phi} \big\rangle_{\Omega_n}$, where $\mathcal{A}(\mathbf{U}_n)$ belongs to the dual space $W^{-1,2}_\sigma(\Omega_n)$ to $W^{1,2}_\sigma(\Omega_n)$ and $\langle\, .\,,\, .\,\rangle_{\Omega_n}$ denotes the duality between $W^{-1,2}_\sigma(\Omega_n)$ and $W^{1,2}_\sigma(\Omega_n)$. Similarly, the difference

$$\int_{\Omega_n} \big\{ \mathbf{U}_{n-1} \circ \mathbf{X}(t_{n-1};\, t_n,\, .) \cdot \mathbf{\Phi} - 2d\nu\, [\nabla \mathbf{a}]_n : \nabla \mathbf{\Phi} \big\}\, \mathrm{d}\mathbf{x} \\ - \int_{\Gamma_n} d\gamma\, \big[\mathbf{a}_n - \mathbf{V}_n \big] \cdot \mathbf{\Phi}\, \mathrm{d}S + d \int_{\Omega_n} \mathbf{g}_n \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{x}$$

can be expressed as $\langle\, \mathcal{F}, \mathbf{\Phi} \rangle_{\Omega_n}$ where $\mathcal{F} \in W^{-1,2}_\sigma(\Omega_n)$. We can now write equation (5.2) (for the unknown $\mathbf{U}$) in the equivalent form as an operator equation in the space $W^{-1,2}_\sigma(\Omega_n)$:

$$\mathcal{A}(\mathbf{U}) \;=\; \mathcal{F}. \tag{9.1}$$

$\mathcal{A}$ is a bounded and demicontinuous operator from $W^{1,2}_\sigma(\Omega_n)$ to $W^{-1,2}_\sigma(\Omega_n)$. Using condition (a5), we can verify that operator $\mathcal{A}$ is coercive if d is sufficiently small. (The integral of $2d\nu(\nabla \mathbf{U})_s : \nabla \mathbf{U}$ must be treated by analogy with (4.4).)

There exists a bounded mapping $\mathcal{B} : W^{1,2}_\sigma(\Omega_n) \times W^{1,2}_\sigma(\Omega_n) \longrightarrow W^{-1,2}_\sigma(\Omega_n)$ such that

$$\big\langle \mathcal{B}(\mathbf{U}^1, \mathbf{U}^2),\, \mathbf{\Phi} \big\rangle_{\Omega_n} \;=\; \int_{\Omega_n} \big\{ \mathbf{U}^1 \cdot \mathbf{\Phi} - d\, \mathbf{U}^1 \cdot \nabla \mathbf{\Phi} \cdot \mathbf{a}(\, .\,, t_n) + d\, \mathbf{U}^2 \cdot \nabla \mathbf{U}^1 \cdot \mathbf{\Phi} \\ + 2d\nu\, (\nabla \mathbf{U}^1)_s : \nabla \mathbf{\Phi} \big\}\, \mathrm{d}\mathbf{x} + \int_{\Gamma_n} d\gamma\, \mathbf{U}^1 \cdot \mathbf{\Phi}\, \mathrm{d}S$$

for $\mathbf{U}^1$, $\mathbf{U}^2$ and $\mathbf{\Phi} \in W^{1,2}_\sigma(\Omega_n)$. Then, obviously, $\mathcal{A}(\mathbf{U}) = \mathcal{B}(\mathbf{U}, \mathbf{U})$. Mapping $\mathcal{B}$ has the following properties:

a) *Given* $\mathbf{U}^1$, $\mathbf{U}^2$, $\mathbf{\Phi}$, $\mathbf{\Psi} \in W^{1,2}_\sigma(\Omega_n)$, *the real–valued function* $\big\langle \mathcal{B}(\mathbf{U}^1 + s\mathbf{\Psi}, \mathbf{U}^2),\, \mathbf{\Phi} \big\rangle_{\Omega_n}$ *of variable* s *is continuous at the point* $s = 0$. Indeed, due to the linearity of $\mathcal{B}$ in the first variable, this is equivalent to the continuity of the function $s \big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^2),\, \mathbf{\Phi} \big\rangle_{\Omega_n}$ at the point $s = 0$, which follows from the finiteness of $\big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^2),\, \mathbf{\Phi} \big\rangle_{\Omega_n}$.

b) *If* d *is sufficiently small then mapping* $\mathcal{B}$ *is monotone in its main part.* It means that $\big\langle \mathcal{B}(\mathbf{U}^2, \mathbf{U}^2) - \mathcal{B}(\mathbf{U}^1, \mathbf{U}^2),\, \mathbf{U}^2 - \mathbf{U}^1 \big\rangle_{\Omega_n} \geq 0$ for all $\mathbf{U}^1$, $\mathbf{U}^2 \in W^{1,2}_\sigma(\Omega_n)$. This can be verified by means of the linearity of $\mathcal{B}(\, .\,, \mathbf{U}^2)$: denoting $\mathbf{U} = \mathbf{U}^2 - \mathbf{U}^1$, we have $\big\langle \mathcal{B}(\mathbf{U}^2, \mathbf{U}^2) - \mathcal{B}(\mathbf{U}^1, \mathbf{U}^2),\, \mathbf{U}^2 - \mathbf{U}^1 \big\rangle_{\Omega_n} = \big\langle \mathcal{B}(\mathbf{U}, \mathbf{U}^2),\, \mathbf{U} \big\rangle_{\Omega_n}$. The non–negativity of this expression for d small enough can be again proved by means of conditions (a5)–(a7) and by estimating the integral of $2d\nu(\nabla \mathbf{U})_s : \nabla \mathbf{U}$ in the same way as in (4.4).

c) *If* $\mathbf{U}^r \rightharpoonup \mathbf{U}$ *for* $r \to +\infty$ *weakly in* $W^{1,2}_\sigma(\Omega_n)$ *then* $\big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^r) - \mathcal{B}(\mathbf{\Psi}, \mathbf{U}),\, \mathbf{\Phi} \big\rangle_{\Omega_n} \longrightarrow 0$ *as* $r \to +\infty$ *for each* $\mathbf{\Phi}$, $\mathbf{\Psi} \in W^{1,2}_\sigma(\Omega_n)$. This property of mapping $\mathcal{B}$ follows from the identity

$$\big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^r) - \mathcal{B}(\mathbf{\Psi}, \mathbf{U}),\, \mathbf{\Phi} \big\rangle_{\Omega_n} \;=\; d \int_{\Omega_n} (\mathbf{U}^r - \mathbf{U}) \cdot \nabla \mathbf{\Psi} \cdot \mathbf{\Phi}\, \mathrm{d}\mathbf{x},$$

from the weak convergence $\mathbf{U}^r \rightharpoonup \mathbf{U}$ in $L^6(\Omega_n)^3$ and from the inclusion $\nabla \mathbf{\Psi} \cdot \mathbf{\Phi} \in L^3(\Omega_n)^3$. (Both are the consequences of the continuous imbedding $W^{1,2}_\sigma(\Omega_n) \rightharpoonup L^6(\Omega_n)^3$.)

d) *If* $\mathbf{U}^r \rightharpoonup \mathbf{U}$ *for* $r \to +\infty$ *weakly in* $W^{1,2}_\sigma(\Omega_n)$, $\mathbf{\Psi} \in W^{1,2}_\sigma(\Omega_n)$, $\mathbf{z} \in W^{-1,2}_\sigma(\Omega_n)$ *and* $\mathcal{B}(\mathbf{\Psi}, \mathbf{U}^r) \rightharpoonup \mathbf{z}$ *for* $r \to +\infty$ *weakly in* $W^{-1,2}_\sigma(\Omega_n)$ *then* $\big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^r),\, \mathbf{U}^r \big\rangle_{\Omega_n} \longrightarrow \langle \mathbf{z}, \mathbf{U} \rangle_{\Omega_n}$ *for* $r \to +\infty$. In order to verify this statement, we use the estimate

$$\big| \big\langle \mathcal{B}(\mathbf{\Psi}, \mathbf{U}^r),\, \mathbf{U}^r \big\rangle_{\Omega_n} - \langle \mathbf{z}, \mathbf{U} \rangle_{\Omega_n} \big|$$

$$\leq \quad \left|\left\langle \mathcal{B}(\boldsymbol{\Psi},\mathbf{U}^r),\, \mathbf{U}^r-\mathbf{U}\right\rangle_{\Omega_n}\right| + \left|\left\langle \mathcal{B}(\boldsymbol{\Psi},\mathbf{U}^r),\, \mathbf{U}\right\rangle_{\Omega_n} - \langle \mathbf{z},\mathbf{U}\rangle_{\Omega_n}\right| \tag{9.2}$$

and the identity

$$\left|\left\langle \mathcal{B}(\boldsymbol{\Psi},\mathbf{U}^r),\, \mathbf{U}^r-\mathbf{U}\right\rangle_{\Omega_n}\right| \;=\; \biggl| \int_{\Omega_n} \bigl\{ \boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U}) - d\,\boldsymbol{\Psi}\cdot\nabla(\mathbf{U}^r-\mathbf{U})\cdot\mathbf{a}(\,.\,,t_n)$$
$$+\, d\,\mathbf{U}^r\cdot\nabla\boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U}) + 2d\nu\,(\nabla\boldsymbol{\Psi})_s : \nabla(\mathbf{U}^r-\mathbf{U})\bigr\}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_n} d\gamma\,\boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U})\,\mathrm{d}\mathbf{x}\biggr|$$

The second term on the right hand side of (9.2) tends to zero due to the weak convergence of $\mathcal{B}(\boldsymbol{\Psi},\mathbf{U}^r)$ to $\mathbf{z}$ in $W^{-1,2}_{\sigma}(\Omega_n)$. All terms on the right hand side of the identity above, except for the integral of $d\mathbf{U}^r\cdot\nabla\boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U})$, tend to zero due to the weak convergence of $\mathbf{U}^r$ to $\mathbf{U}$ in $W^{1,2}_{\sigma}(\Omega_n)$. The integral of $d\mathbf{U}^r\cdot\nabla\boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U})$ can be estimated as follows:

$$\biggl|\int_{\Omega_n} d\mathbf{U}^r\cdot\nabla\boldsymbol{\Psi}\cdot(\mathbf{U}^r-\mathbf{U})\,\mathrm{d}\mathbf{x}\biggr| \;\leq\; d\,\|\mathbf{U}^r\|_{6;\,\Omega_n}\,\|\nabla\boldsymbol{\Psi}\|_{2;\,\Omega_n^R}\,\|\mathbf{U}^r-\mathbf{U}\|_{3;\,\Omega_n^R}$$
$$+\, d\,\|\mathbf{U}^r\|_{6;\,\Omega_n}\,\|\nabla\boldsymbol{\Psi}\|_{2;\,\Omega_n\smallsetminus\Omega_n^R}\,\|\mathbf{U}^r-\mathbf{U}\|_{3;\,\Omega_n\smallsetminus\Omega_n^R}, \tag{9.3}$$

where $R>0$ and $\Omega_n^R := \Omega_n\cap B_R(\mathbf{0})$. The sequences $\bigl\{\|\mathbf{U}^r\|_{6;\,\Omega_n}\bigr\}$ and $\bigl\{\|\mathbf{U}^r-\mathbf{U}\|_{3;\,\Omega_n}\bigr\}$ are bounded due to the weak convergence of $\{\mathbf{U}^r\}$ in $W^{1,2}_{\sigma}(\Omega_n)$ and the continuous imbedding of $W^{1,2}_{\sigma}(\Omega_n)$ into $L^6(\Omega_n)^3$ and $L^3(\Omega_n)^3$. The norm $\|\nabla\boldsymbol{\Psi}\|_{2;\,\Omega_n\smallsetminus\Omega_n^R}$ can be made arbitrarily small by choosing R large enough. Thus, the second term of the right hand side of inequality (9.3) can also be made arbitrarily small by choosing R large enough. The norms $\|\mathbf{U}^r-\mathbf{U}\|_{3;\,\Omega_n^R}$ tend to zero as $r\to+\infty$ because $W^{1,2}_{\sigma}(\Omega_n^R)$ is compactly imbedded into $L^3(\Omega_n^R)^3$ and consequently, $\mathbf{U}^r\to\mathbf{U}$ strongly in $L^3(\Omega_n^R)^3$. Hence the first term on the right hand side of (9.3) can be made arbitrarily small by choosing r sufficiently large. The proof of statement d) is completed.

We have verified the assumptions of the Leray–Lions theorem, see e.g. J. LERAY, J. L. LIONS [20] or S. FUČÍK, A. KUFNER [7, p. 231], for d small enough. Due to this theorem, equation (9.1) has a solution $\mathbf{U}\in W^{1,2}_{\sigma}(\Omega_n)$.

Appendix A2: Proof of Lemma 2. a) Let $\mathbb{F}\in C_0^{\infty}(Q_{(0,T)})^9$. Let us extend $\mathbb{F}$ by zero to $[\mathbb{R}^3\times(0,T)]\smallsetminus Q_{(0,T)}$. The support of $\mathbb{F}$ belongs to $\cup_{n=1}^N\,\Omega_n\times[t_{n-1},t_n)$ for all sufficiently large N, hence $\mathbb{U}^N=\nabla\mathbf{u}^N$ on $\operatorname{supp}\mathbb{F}$. Statement a) now follows from the identities

$$\int_0^T\int_{\Omega^t}\mathbb{F}:\mathbb{U}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; \lim_{N\to+\infty}\int_0^T\int_{\mathbb{R}^3}\mathbb{F}:\nabla\mathbf{u}^N\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$=\; -\lim_{N\to+\infty}\int_0^T\int_{\mathbb{R}^3}\operatorname{Div}\mathbb{F}\cdot\mathbf{u}^N\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; -\int_0^T\int_{\Omega^t}\operatorname{Div}\mathbb{F}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

b) Since $\mathbf{u}\in L^{\infty}(0,T;\,L^2(\Omega^t)^3)$, $\mathbb{U}\in L^2(0,T;\,L^2(\Omega^t)^9)$ and $\nabla\mathbf{u}=\mathbb{U}$ a.e. in $Q_{(0,T)}$, we deduce that $\mathbf{u}\in L^2(0,T;\,W^{1,2}(\Omega^t)^3)$. It suffices to show that $\mathbf{u}(\,.\,,t)\in L^2_{\sigma}(\Omega^t)$ for a.a. $t\in(0,T)$. Thus, let $t\in(0,T)\smallsetminus\mathcal{T}^c$ be fixed. The sequence $\{\mathbf{u}^N(\,.\,,t)\}$ is bounded in $L^2(\mathbb{R}^3)^3$, hence there exists a subsequence (we denote it again $\{\mathbf{u}^N(\,.\,,t)\}$) and a limit function $\mathbf{u}^t$ such that $\mathbf{u}^N(\,.\,,t)\rightharpoonup\mathbf{u}^t$ weakly in $L^2(\mathbb{R}^3)^3$. Due to (6.6), $\mathbf{u}^t=\mathbf{u}(\,.\,,t)$ for a.a. $t\in(0,T)\smallsetminus\mathcal{T}^c$. Suppose that the fixed t is chosen so that it is one of the instants of time when this equality holds. Let p be an arbitrary function from $W^{1,2}(\Omega^t)$. Function p can be extended to $\mathbb{R}^3$ so that the gradient of the extension belongs to $L^2(\mathbb{R}^3)^3$. (See e.g. E. M. STEIN [26], p. 181.) In order not to complicate the notation, we denote the extended function again by p. Now we have

$$\int_{\Omega^t} \mathbf{u}(\,.\,,t)\cdot\nabla p\,\mathrm{d}\mathbf{x} \;=\; \int_{\mathbb{R}^3} \mathbf{u}^t\cdot\nabla p\,\mathrm{d}\mathbf{x} \;=\; \lim_{N\to+\infty}\int_{\mathbb{R}^3} \mathbf{u}^n(\,.\,,t)\cdot\nabla p\,\mathrm{d}\mathbf{x} \\ =\; \lim_{N\to+\infty}\int_{\Omega_n} \mathbf{U}_N\cdot\nabla p\,\mathrm{d}\mathbf{x} \;=\; 0, \tag{9.4}$$

where the appropriate n is determined by t and N so that $t\in(t_{n-1},t_n]$. (Points t_{n-1} and t_n belong to the partition of $[0,T]$ defined at the beginning of Section 5. The partition depends on N.) The set of gradients of all functions $p\in W^{1,2}(\Omega^t)$ is dense in the orthogonal complement to $L^2_\sigma(\Omega^t)$ in $L^2(\Omega^t)^3$. Thus, we have verified that $\mathbf{u}(\,.\,,t)\in L^2_\sigma(\Omega^t)$.

c) Let $k\in\{0;\,1;\,\dots;\,K\}$, $\mathcal{J}$ be an closed interval in $(0,T)\smallsetminus\mathcal{T}^c$ and $t'\in\mathcal{J}$. All these quantities are considered to be arbitrarily chosen, however fixed. There exists $\xi>0$ so small that the ξ–neighbourhood $U_\xi(B^t_k)$ of the k–the body B^t_k has the empty intersection with $\overline{B^s_j}$ for all time instants $t,s\in\mathcal{J}$ and $j\in\{0;\,1;\,\dots;\,K\}$, $j\neq k$. The mapping $\mathbf{Y}(t;\,s,\,.)$, which is an isometry of B^s_k onto B^t_k, can be extended to the isometry of $U_\xi(B^s_k)$ onto $U_\xi(B^t_k)$.

Denote $\mathbf{v}^N(\mathbf{x}',t) := \mathbf{u}^N\big(\mathbf{Y}(t;\,t',\mathbf{x}'),t\big)$ for $t\in\mathcal{J}$ and $\mathbf{x}'\in U_\xi(B^{t'}_k)\smallsetminus\overline{B^{t'}_k}$. In accordance with the definition of function $\mathbf{u}^N$, we have $\mathbf{v}^N(\mathbf{x}',t)=\mathbf{U}_n\big(\mathbf{Y}\big(t_n;\,t,\mathbf{Y}(t;\,t',\mathbf{x}')\big)\big)=\mathbf{U}_n\big(\mathbf{Y}(t_n;\,t',\mathbf{x}')\big)$ for $t\in\mathcal{J}_n:=\mathcal{J}\cap(t_{n-1},t_n]$. It follows from (6.6), (6.7) and from statement a) of Lemma 2 that $\mathbf{v}^N(\mathbf{x}',t)\rightharpoonup\mathbf{u}\big(\mathbf{Y}(t;\,t',\mathbf{x}'),t\big)$ weakly in $L^2\big(\mathcal{J};\,W^{1,2}\big(U_\xi(B^{t'}_k)\smallsetminus\overline{B^{t'}_k}\big)^3\big)$ for $N\to+\infty$.

Let $\boldsymbol{\psi}$ be an arbitrary vector function from $L^2(\mathcal{J};\,\partial B^t_k)$. Then we have

$$\begin{aligned}
&\int_{\mathcal{J}}\int_{\partial B^t_k}\mathbf{u}_*(\mathbf{x},t)\cdot\boldsymbol{\psi}(\mathbf{x},t)\,\mathrm{d}S(\mathbf{x})\,\mathrm{d}t \;=\; \lim_{N\to+\infty}\int_{\mathcal{J}}\int_{\partial B^t_k}\mathbf{u}^N_*(\mathbf{x},t)\cdot\boldsymbol{\psi}(\mathbf{x},t)\,\mathrm{d}S(\mathbf{x})\,\mathrm{d}t\\
&=\; \lim_{N\to+\infty}\sum_{n=1}^N\int_{\mathcal{J}_n}\int_{\partial B^t_k}\mathrm{tr}\big[\mathbf{U}_n\big(\mathbf{Y}(t_n;\,t,\mathbf{x})\big)\big]\cdot\boldsymbol{\psi}(\mathbf{x},t)\,\mathrm{d}S(\mathbf{x})\,\mathrm{d}t\\
&=\; \lim_{N\to+\infty}\sum_{n=1}^N\int_{\mathcal{J}_n}\int_{\partial B^{t'}_k}\mathrm{tr}\big[\mathbf{U}_n\big(\mathbf{Y}(t_n;\,t',\mathbf{x}')\big)\big]\cdot\boldsymbol{\psi}\big(\mathbf{Y}(t;\,t',\mathbf{x}'),t\big)\,\mathrm{d}S(\mathbf{x}')\,\mathrm{d}t\\
&=\; \int_{\mathcal{J}}\int_{\partial B^{t'}_k}\mathrm{tr}\big[\mathbf{u}\big(\mathbf{Y}(t;\,t',\mathbf{x}'),t\big)\big]\cdot\boldsymbol{\psi}\big(\mathbf{Y}(t;\,t',\mathbf{x}'),t\big)\,\mathrm{d}S(\mathbf{x}')\,\mathrm{d}t\\
&=\; \int_{\mathcal{J}}\int_{\partial B^t_k}\mathrm{tr}\big[\mathbf{u}(\mathbf{x},t)\big]\cdot\boldsymbol{\psi}(\mathbf{x},t)\,\mathrm{d}S(\mathbf{x})\,\mathrm{d}t.
\end{aligned}$$

This completes the proof of Lemma 2. As usually, we mostly omit the denotation "tr" for traces of functions on the boundary.

Appendix A3: Proof of Lemma 3. Recall that $\mathcal{E}^N(\boldsymbol{\phi})$ is defined by (6.9), where $\mathcal{I}(\mathbf{u}^N,\mathbf{u}^N_*,\boldsymbol{\phi}) = \mathcal{I}_1(\mathbf{u}^N,\boldsymbol{\phi})+\dots+\mathcal{I}_4(\mathbf{u}^N,\boldsymbol{\phi})+\mathcal{I}_5(\mathbf{u}^N_*,\boldsymbol{\phi})$.

We shall further denote $\boldsymbol{\phi}_n:=\boldsymbol{\phi}(\,.\,,t_n)$, where $0=t_0<t_1<\dots<t_N=T$ is the partition of $[0,T]$, corresponding to natural number N.

The integral $\mathcal{I}_1(\mathbf{u}^N,\boldsymbol{\phi})$ can be expressed:

$$\mathcal{I}_1(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; -\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\int_{\Omega^t}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}\big(\mathbf{X}(t;\,\xi,\mathbf{x}),t\big)\Big]_{\xi=t}\cdot\mathbf{u}^N(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$
\begin{aligned}
&= -\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\Omega_{n-1}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}\big(\mathbf{X}(t;\,t_{n-1},\mathbf{x}_{n-1}),t\big)\cdot\mathbf{U}_{n-1}(\mathbf{x}_{n-1})\;\mathrm{d}\mathbf{x}_{n-1}\,\mathrm{d}t \\
&= -\sum_{n=1}^{N}\int_{\Omega_{n-1}}\big[\boldsymbol{\phi}_n\big(\mathbf{X}(t_n;\,t_{n-1},\mathbf{x}_{n-1})\big)-\boldsymbol{\phi}_{n-1}(\mathbf{x}_{n-1})\big]\cdot\mathbf{U}_{n-1}(\mathbf{x}_{n-1})\;\mathrm{d}\mathbf{x}_{n-1} \\
&= \int_{\Omega_0}\mathbf{U}_0(\mathbf{x}_0)\cdot\boldsymbol{\phi}_0(\mathbf{x}_0)\;\mathrm{d}\mathbf{x}_0+\sum_{n=1}^{N-1}\int_{\Omega_n}\big[\mathbf{U}_n(\mathbf{x}_n)-\mathbf{U}_{n-1}\big(\mathbf{X}(t_{n-1};\,t_n,\mathbf{x}_n)\big)\big]\cdot\boldsymbol{\phi}_n(\mathbf{x}_n)\;\mathrm{d}\mathbf{x}_n.
\end{aligned}
$$

The integral $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})$ equals

$$
\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; -\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\Omega^t}\mathbf{U}_n(\mathbf{x})\cdot\nabla\boldsymbol{\phi}(\mathbf{x},t)\cdot\mathbf{a}(\mathbf{x},t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t.
$$

The integral on Ω^t can be replaced by the same integral on Ω_n because $\mathbf{U}_n$ equals zero outside Ω_n and $\mathbf{a}(.\,,t)$ equals zero outside Ω^t. Thus, we can further write $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}) = \mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_1 + \mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_2+\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_3$ where

$$
\begin{aligned}
\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_1 \;&:=\; -\sum_{n=1}^{N} d_n\int_{\Omega_n}\mathbf{U}_n(\mathbf{x})\cdot\nabla\boldsymbol{\phi}_n(\mathbf{x})\cdot\mathbf{a}(\mathbf{x},t_n)\;\mathrm{d}\mathbf{x}, \\
\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_2 \;&:=\; -\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\Omega_n}\mathbf{U}_n(\mathbf{x})\cdot\nabla[\boldsymbol{\phi}(\mathbf{x},t)-\boldsymbol{\phi}(\mathbf{x},t_n)]\cdot\mathbf{a}(\mathbf{x},t)\;\mathrm{d}\mathbf{x}\,\mathrm{d}t, \\
\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_3 \;&:=\; -\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\int_{\Omega_n}\mathbf{U}_n(\mathbf{x})\cdot\nabla\boldsymbol{\phi}(\mathbf{x},t_n)\cdot[\mathbf{a}(\mathbf{x},t)-\mathbf{a}(\mathbf{x},t_n)]\;\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
$$

Due to the infinite differentiability of function $\boldsymbol{\phi}$, assumption (a5) and estimate (6.4), the sum $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_2$ satisfies

$$
\big|\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_2\big| \;\leq\; C(\boldsymbol{\phi},c_5)\sum_{n=1}^{N} d_n\,\|\mathbf{a}(.\,,t)\|_{2;\,\Omega^t}\;\mathrm{d}t \;\longrightarrow\; 0 \qquad \text{for } N\to+\infty.
$$

In order to understand the behaviour of $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_3$ as $N\to+\infty$, we choose a "small" number $\kappa>0$ and we use the cut–off function η defined by (7.2). (Recall that $\eta(t)$ equals zero in the $\frac{1}{2}\kappa$–neighbourhood of critical points $t_1^c,\,\dots,\,t_M^c$, which form the set $\mathcal{T}^c$, and $\eta(t)$ equals one at times t whose distance from $\mathcal{T}^c$ is at least κ.) We put $\boldsymbol{\phi}^1:=\eta\cdot\boldsymbol{\phi}$ and $\boldsymbol{\phi}^2:=(1-\eta)\cdot\boldsymbol{\phi}$. Thus, we have $\boldsymbol{\phi}=\boldsymbol{\phi}^1+\boldsymbol{\phi}^2$ where function $\boldsymbol{\phi}^1$ coincides with $\boldsymbol{\phi}$ at times t such that $\mathrm{dist}(t;\mathcal{T}^c)\geq\kappa$ and it equals zero at times t such that $\mathrm{dist}(t;\mathcal{T}^c)\leq\frac{1}{2}\kappa$. Now $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi})_3$ equals the sum $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^1)_3+\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^2)_3$. The first term $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^1)_3$ can be estimated by means of the uniform continuity of function $\mathbf{a}$ on a subset of $Q^*_{[0,T]}$, containing only points $(\mathbf{x},t)$ such that $\mathrm{dist}(t;\mathcal{T}^c)\geq\kappa$:

$$
\big|\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^1)_3\big| \;\leq\; \sum_{n=1}^{N} c_7(d_n,\kappa)\int_{t_{n-1}}^{t_n}\int_{\Omega_n}|\mathbf{U}_n(\mathbf{x})|\,|\nabla\boldsymbol{\phi}^1(\mathbf{x},t_n)|\;\mathrm{d}\mathbf{x}\,\mathrm{d}t,
$$

where $c_7(d_n,\kappa)\to0$ as $d_n\to0$ for each $\kappa>0$. Hence $\big|\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^1)_3\big|\to0$ for $N\to+\infty$. The second term $\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^2)_3$ can be estimated:

$$\left|\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}^2)_3\right| \;\le\; C(c_5)\,\big(\max_{Q^*_{[0,T]}}|\nabla\boldsymbol{\phi}^2|\big)\left(\sum_{n=1}^{N}\right)_{\kappa}\int_{t_{n-1}}^{t_n}\int_{\Omega_n}\left|\mathbf{a}(\mathbf{x},t)-\mathbf{a}(\mathbf{x},t_n)\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

where $\big(\sum_{n=1}^{N}\big)_{\kappa}$ denotes the sum over n running from 1 to N such that $\mathrm{dist}(t_n;\mathcal{T}^c)\le\kappa$. The right hand side can be made arbitrarily small uniformly with respect to N by choosing $\kappa>0$ sufficiently small. Thus, we can conclude that

$$\mathcal{I}_2(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; -\sum_{n=1}^{N} d_n\int_{\Omega_n}\mathbf{U}_n\cdot\nabla\boldsymbol{\phi}_n\cdot\mathbf{a}(.\,,t_n)\,\mathrm{d}\mathbf{x}+\mathcal{R}_2^N(\kappa)+\mathcal{S}_2(\kappa) \tag{9.5}$$

where $\mathcal{R}_2^N(\kappa)\to 0$ as $N\to+\infty$ for each $\kappa>0$ and $\mathcal{S}_2(\kappa)\to 0$ as $\kappa\to 0+$ independently of N.

We can proceed in the same spirit and show that

$$\mathcal{I}_3(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; \sum_{n=1}^{N} d_n\int_{\Omega_n}\mathbf{U}_n\cdot\mathbb{U}^N\cdot\boldsymbol{\phi}_n\,\mathrm{d}\mathbf{x}+\mathcal{R}_3^N(\kappa)+\mathcal{S}_3(\kappa), \tag{9.6}$$

$$\mathcal{I}_4(\mathbf{u}^N,\boldsymbol{\phi}) \;=\; \sum_{n=1}^{N} d_n\int_{\Omega_n}2\nu\big([\nabla\mathbf{a}]_n+\mathbf{U}_n\big)_s:\nabla\boldsymbol{\phi}_n\,\mathrm{d}\mathbf{x}+\mathcal{R}_4^N(\kappa)+\mathcal{S}_4(\kappa), \tag{9.7}$$

$$\mathcal{I}_5(\mathbf{u}_*^N,\boldsymbol{\phi}) \;=\; \sum_{n=1}^{N} d_n\int_{\Gamma_n}\gamma(\mathbf{A}_n+\mathbf{U}_n-\mathbf{V}_n)\cdot\boldsymbol{\phi}_n\,\mathrm{d}S\,\mathrm{d}t+\mathcal{R}_5^N(\kappa)+\mathcal{S}_5(\kappa), \tag{9.8}$$

where $\mathcal{R}_3^N(\kappa)$, $\mathcal{R}_4^N(\kappa)$, $\mathcal{R}_5^N(\kappa)$ behave in the same way as $\mathcal{R}_2^N(\kappa)$ and $\mathcal{S}_3(\kappa)$, $\mathcal{S}_4(\kappa)$, $\mathcal{S}_5(\kappa)$ behave in the same way as $\mathcal{S}_2(\kappa)$. Similarly, using assumption (a4) and the smoothness of function $\boldsymbol{\phi}$, one can express the integral of $\mathbf{g}\cdot\boldsymbol{\phi}$ on the right hand side of (3.2):

$$\int_0^T\int_{\Omega^t}\mathbf{g}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; \sum_{n=1}^{N} d_n\int_{\Omega_n}\mathbf{g}_n\cdot\boldsymbol{\phi}_n\,\mathrm{d}\mathbf{x}+\mathcal{R}_6^N(\kappa)+\mathcal{S}_6(\kappa), \tag{9.9}$$

where $\mathcal{R}_6^N(\kappa)$, respectively $\mathcal{S}_6(\kappa)$, also behaves in the same way as $\mathcal{R}_2^N(\kappa)$, respectively $\mathcal{S}_2(\kappa)$. It means that $\mathcal{R}_6^N(\kappa)\to 0$ as $N\to+\infty$ for each $\kappa>0$ and $\mathcal{S}_6(\kappa)\to 0$ as $\kappa\to 0+$. Summing now $\mathcal{I}_1(\mathbf{u}^N,\boldsymbol{\phi})$, $\dots$, $\mathcal{I}_4(\mathbf{u}^N,\boldsymbol{\phi})$ and $\mathcal{I}_5(\mathbf{u}_*^N,\boldsymbol{\phi})$, expressing them by means of (9.5)–(9.8) and using (9.1) (with $\boldsymbol{\Phi}_n=\boldsymbol{\phi}_n$) and (9.9), we verify that the approximations $\mathbf{u}^N$ (respectively $\mathbf{u}_*^N$ on the boundary of Ω^t) satisfy (6.9) with

$$\mathcal{E}^N \;=\; \mathcal{R}_1^N(\kappa)+\ldots+\mathcal{R}_5^N(\kappa)-\mathcal{R}_6^N(\kappa)+\mathcal{S}_1(\kappa)+\ldots+\mathcal{S}_5(\kappa)-\mathcal{S}_6(\kappa).$$

The statement of the lemma now follows from the asymptotic behaviour of $\mathcal{R}_j^N(\kappa)$ and $\mathcal{S}_j(\kappa)$ $(j=1,\dots,6)$ for $N\to+\infty$ and $\kappa\to 0+$.

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