Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-19

Solvability of dynamic contact problems for elastic von Kármán plates

I. Bock and J. Jarušek*

Abstract: The existence of solutions is proved for unilateral dynamic contact problems of elastic von Kármán plates. Boundary conditions for a free and clamped plate are considered.

Keywords: elastic von Kármán plates, unilateral dynamic contact, existence of solutions, penalization of contact condition, limit process.

AMS Subject Classification: 35L85, 73C50, 73K10.

1 Introduction and notation

The dynamic contact problems are not frequently solved in the framework of variational inequalities. For the elastic problems there is only a very limited amount of results available (cf. [6] and there cited literature). The aim of the present paper is to extend these results to the nonlinear von Kármán plates in contact with a rigid obstacle. The presented results also extend the research made for the quasistatic contact problems for these plates [3] and [4]. The solvability of dynamic contact problems for von Kármán plates with short and long memory has been proved in [1] and [2], respectively.

The existence of solutions is proved for an approximate penalized problem at first. The limit process to the original problem is enabled by an L_1 estimate of the penalty term and by the use of the compact imbedding theorem and by a proper use of the interpolation technique). The idea of the proof is similar to that introduced by K. Maruo in [8].

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal or $C^{3,1}$ domain with a boundary Γ and $I \equiv (0,T)$ a bounded time interval. The unit outer normal vector is denoted by $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{\tau} = (-n_2, n_1)$ is the unit tangent vector. The displacement is denoted by $\boldsymbol{u} \equiv (u_i)$. Strain tensor is defined as $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - x_3 \partial_{ij} u_3,\ i,j = 1,2$, $\varepsilon_{i3} \equiv 0,\ i = 1,2,3$. Emploing the Einstein summation convention the constitutional law has the form

$$\sigma_{ij}(\boldsymbol{u}) = \frac{E}{1-\nu^2}\big((1-\nu)\varepsilon_{ij}(\boldsymbol{u}) + \nu\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})\big). \tag{1}$$

The constants $E > 0$ and $\nu \in \left\langle 0, \frac{1}{2}\right)$ are the Young modulus of elasticity and the Poisson ratio, respectively. We shall use the abbreviation

$$b = \frac{h^2}{12\varrho(1-\nu^2)},$$

where h is the the plate thickness and ϱ is the density of the material. We denote

$$[u,v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v. \tag{2}$$

*The work presented here was partially supported by the Czech Academy of Sciences under grant IAA 1075402 and under the Institutional research plan AVOZ 10190503 and by the Ministry of Education of Slovak Republic under VEGA grant 1/4214/07.

Here and in the sequel the notation as follows:

$$\frac{\partial}{\partial s} \equiv \partial_s,\ \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr},\ \partial_i = \partial_{x_i},\ i = 1, 2,$$
$$\dot{v} = \frac{\partial v}{\partial t},\ \ddot{v} = \frac{\partial^2 v}{\partial t^2}, \qquad Q = I \times \Omega,\ S = I \times \Gamma.$$

is employed.

By $W_p^k(M)$ with $k \geq 0$ and $p \in [1, \infty]$ the Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces are denoted provided they are defined on a domain or an appropriate manifold M. By $\mathring{W}_p^k(M)$ we denote the spaces with zero traces on ∂M. If $p = 2$ we use the notation $H^k(M)$, $\mathring{H}_p^k(M)$. For the anisotropic spaces $W_p^k(M)$ $k = (k_1, k_2) \in \mathbb{R}_+^2$, k_1 is related with the time while k_2 with the space variables (with the obvious consequences for $p = 2$) provided M is a time-space domain. The duals to $\mathring{H}^k(M)$ are denoted by $H^{-k}(M)$. By C we denote the space of continuous functions with the appropriate sup–norm. By $\mathscr{H}$, $\mathring{\mathscr{H}}$ we denote the space $L_\infty(I; H^2(\Omega))$, $L_\infty(I; \mathring{H}^2(\Omega))$, respectively.

Remark 1.1 *In order to apply Lemma 1 from [7] containing the estimate (11) we need the regularity $v \in H^3(\Omega)$ for a weak solution of the Dirichlet problem*

$$\triangle^2 v = g \ \text{ on } \Omega,\ v = \partial_n v = 0 \ \text{ on } \Gamma,\ g \in H^{-1}(\Omega).$$

The regularity result for $C^{3,1}$ domain Ω is due to Theorem 2.2, Chapter 4 from [9]. In the case of convex polygonal domain we apply Theorem 2.1 from [10]. Via the local rectification of the boundary (cf [6]) it seems to be possible to extend the validity of the result to Ω in C^2.

2 Contact of a free plate

2.1 Problem formulation and penalization

Neglecting the rotary inertia of the plate we obtain the classical formulation for the bending function u and the Airy stress function v composed of the system

$$\left.\begin{aligned} \ddot{u} + bE\triangle^2 u - [u, v] &= f + g, \\ u \geq 0,\ g \geq 0,\ ug &= 0, \\ \triangle^2 v + E[u, u] &= 0 \end{aligned}\right\} \text{ on } Q, \tag{3}$$

the boundary value conditions

$$u \geq 0,\ \Sigma(u) \geq 0,\ u\Sigma(u) = 0,\ \mathscr{M}(u) = 0,\ v = 0 \text{ and } \partial_n v = 0 \text{ on } S, \tag{4}$$

$$\begin{aligned} \mathscr{M}(u) &= bEM(u), \\ M(u) &= \triangle u + (1 - \nu)(2n_1 n_2 \partial_{12} u - n_1^2 \partial_{22} u - n_2^2 \partial_{11} u); \\ \Sigma(u) &= bEV(u), \\ V(u) &= \partial_n \triangle u + (1 - \nu)\partial_\tau[(n_1^2 - n_2^2)\partial_{12} u + n_1 n_2(\partial_{22} u - \partial_{11} u)] \end{aligned}$$

and the initial conditions

$$u(0, \cdot) = u_0 \geq 0,\ \dot{u}(0, \cdot) = u_1 \text{ on } \Omega. \tag{5}$$

For $u, y \in L_2(I; H^2(\Omega))$ we define the following bilinear form

$$A : (u, y) \mapsto b\big(\partial_{kk} u \partial_{kk} y + \nu(\partial_{11} u \partial_{22} y + \partial_{22} u \partial_{11} y) + 2(1 - \nu)\partial_{12} u \partial_{12} y\big) \tag{6}$$

almost everywhere on Q and introduce a cone $\mathscr{C}$ as

$$\mathscr{C} := \{y \in \mathscr{H};\ y \geq 0\}. \tag{7}$$

Then the variational formulation of the problem (3–5) has the following form:

Look for $\{u,v\} \in \mathscr{C} \times L_2(I; \mathring{H}^2(\Omega))$ *such that*

$$\int_Q (EA(u, y_1-u) + \ddot{u}(y_1-u) - [u,v](y_1-u))\, dx\, dt \geq \int_Q f(y_1-u)\, dx\, dt, \tag{8}$$

$$\int_\Omega \big(\triangle v \triangle y_2 + E[u,u]y_2\big)\, dx = 0 \ \forall\ (y_1,y_2) \in \mathscr{C} \times \mathring{H}^2(\Omega). \tag{9}$$

We define the bilinear operator $\Phi : H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$ by means of the variational equation

$$\int_\Omega \triangle\Phi(u,v)\triangle\varphi\, dx = \int_\Omega [u,v]\varphi\, dx,\ \ \varphi \in \mathring{H}^2(\Omega). \tag{10}$$

The equation (10) has a unique solution, because $[u,v] \in L_1(\Omega) \hookrightarrow H^2(\Omega)^*$. The well-defined operator Φ is evidently compact and symmetric. The domain Ω fulfils the assumptions enabling to apply Lemma 1 from [7] due to which $\Phi : H^2(\Omega)^2 \to W_p^2(\Omega),\ 2 < p < \infty$ and

$$\|\Phi(u,v)\|_{W_p^2(\Omega)} \leq c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)}\ \forall u \in H^2(\Omega),\ v \in W_p^1(\Omega). \tag{11}$$

With its help we reformulate the system (8,9) into the following variational inequality:

Problem $\mathscr{P}$. *We look for* $u \in \mathscr{C}$ *such that* $\ddot{u} \in \mathscr{H}^*$, *the initial conditions* (5) *are satisfied in a certain generalized sense, and the inequality*

$$\begin{aligned}\langle \ddot{u}, y-u\rangle_0 + \int_Q E\big(A(u,y-u) + [u, E\Phi(u,u)](y-u)\big)\, dx\, dt \\ \geq \int_Q f(y-u)\, dx\, dt.\end{aligned} \tag{12}$$

holds for any $y \in \mathscr{C}$.

Here $\langle\cdot,\cdot\rangle_0$ denotes the duality pairing between $\mathscr{H}$ and its dual as a natural extension extension of the scalar product in $L_2(Q)$.

In the sequel we shall prove the existence of solutions to problem $\mathscr{P}$.

For any $\eta > 0$ we define the *penalized problem* which includes the system of equations

$$\left.\begin{aligned}\ddot{u} + bE\triangle^2 u - [u,v] &= f + \eta^{-1}u^-,\\ \triangle^2 v + E[u,u] &= 0\end{aligned}\right\}\ \text{on } Q \tag{13}$$

with $u^- = \max\{0, -u\}$, the boundary value conditions

$$\Sigma(u) = 0,\ \mathscr{M}(u) = 0,\ v = 0 \text{ and } \partial_n v = 0 \text{ on } S \tag{14}$$

and the initial conditions (5). It has the variational formulation:

Look for $\{u,v\} \in L_\infty(I; H^2(\Omega)) \times L_2(I; \mathring{H}^2(\Omega))$ *such that* $\ddot{u} \in L_2(I; (H^2(\Omega))^*)$ *and the following system*

$$\int_Q \big(\ddot{u}z_1 + EA(u,z_1) - [u,v]\, z_1 - \eta^{-1}u^- z_1\big)\, dx\, dt = \int_Q f z_1\, dx\, dt, \tag{15}$$

$$\int_\Omega (\triangle v \triangle z_2 + E[u,u])z_2\, dx = 0 \tag{16}$$

is satisfied for any $(z_1,z_2) \in L_2(I; H^2(\Omega)) \times \mathring{H}^2(\Omega)$ *and there hold the conditions* (5).

With the help of the operator Φ we get the following reformulation of (15), (16):

Problem $\mathscr{P}_\eta$.

We look for $u \in L_\infty(I, H^2(\Omega))$ *such that* $\ddot{u} \in L_2(I; (H^2(\Omega))^*)$, *the equation*

$$\int_Q \big(- \ddot{u}z + EA(u,z) + E[u, \Phi(u,u)]z - \eta^{-1}u^- z\big)\, dx\, dt = \int_Q fz\, dx\, dt, \tag{17}$$

holds for any $z \in L_2(I; (H^2(\Omega))$ *and the initial conditions* (5) *remain valid.*

We shall verify the existence of a solution to the penalized problem.

Theorem 2.1 *Let* $f \in L_2(Q)$, $u_0 \in H^2(\Omega)$, *and* $u_1 \in L_2(\Omega)$ *Then there exists a solution* u *of the problem* $\mathscr{P}_\eta$.

If $v = -E_0\Phi(u,u)$, *then a couple* $\{u, v\}$ *is a solution of the problem* (15), (16), (5).

Proof. Let us denote by $\{w_i \in H^2(\Omega);\ i \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ basis of $H^2(\Omega)$. We construct the Galerkin approximation u_m of a solution in a form

$$u_m(t) = \sum_{i=1}^m \alpha_i(t) w_i,\ \ \alpha_i(t) \in \mathbb{R},\ \ i = 1, ..., m,\ \ m \in \mathbb{N},$$

$$\begin{aligned} \int_\Omega \big(\ddot{u}_m(t) w_i + EA(u_m(t), w_i) \\ + E[u_m(t), w_i]\Phi(u_m, u_m)(t)\,) - \eta^{-1} u_m(t)^- w_i\big)\, dx \\ = \int_\Omega f(t) w_i\, dx,\ \ i = 1, ..., m, \end{aligned} \tag{18}$$

$$u_m(0) = u_{0m},\ \dot{u}_m(0) = u_{1m},\ u_{1m} \to u_1 \text{ in } L_2(\Omega),\ \text{and } u_{0m} \to u_0 \text{ in } H^2(\Omega) \tag{19}$$

The system (18) can then be expressed in the form

$$\ddot{\alpha}_i = F_i(t, \dot{\alpha}_1, ..., \dot{\alpha}_m, \alpha_1, ..., \alpha_m),\ i = 1, ..., m.$$

Its right-hand side satisfies the conditions for the local existence of a solution fulfilling the initial conditions corresponding the functions u_{0m}, u_{1m}. Hence there exists a Galerkin approximation $u_m(t)$ defined on some interval $I_m \equiv [0, t_m]$, $0 < t_m < T$. After multiplying the equation (18) by $\dot{\alpha}_i(t)$, summing up with respect to i, taking in mind

$$\int_\Omega [u, v] y\, dx = \int_\Omega [u, y] v\, dx \tag{20}$$

if at least one element of $\{u, v, y\}$ belongs to $\mathring{H}^2(\Omega)$, cf. [5] and integrating we obtain for $Q_m := I_m \times \Omega$

$$\int_{Q_m} \frac{1}{2}\partial_t \left(\dot{u}_m^2 + EA(u_m, u_m) + \frac{E}{2}(\triangle\Phi(u_m, u_m))^2 + \eta^{-1}(u_m^-)^2 \right) dx\, dt = \int_Q f\dot{u}_m\, dx\, dt \tag{21}$$

which leads to the estimates

$$\|\dot{u}_m\|^2_{L_\infty(I; L_2(\Omega))} + \|u_m\|^2_{L_\infty(I; H^2(\Omega))} + \|\Phi(u_m, u_m)\|^2_{L_\infty(I; H^2(\Omega))} + \eta^{-1}\|u_m^-\|^2_{L_\infty(I; L_2(\Omega))} \le c \equiv c(f, u_0, u_1). \tag{22}$$

The validity of this *a priori* estimate on the time interval I_m is obvious. As the right hand side of such an estimate does not depend on m the prolongation of a solution to the whole interval I is possible and (22) holds as written. Moreover the estimate (11) implies

$$\|\Phi(u_m, u_m)\|_{L_\infty(I; W_p^2(\Omega))} \le c_p \equiv c_p(f, u_0, u_1)\ \forall\, p > 2. \tag{23}$$

The estimate (23) further implies

$$\begin{aligned}&[u_m,\Phi(u_m,u_m)]\in L_2(I;L_r(\Omega)),\ r=\frac{2p}{p+2},\\&\big\|[u_m,\Phi(u_m,u_m)]\big\|_{L_2(I;L_r(\Omega))}\le c_r\equiv c_r(f,u_0,u_1).\end{aligned}\tag{24}$$

From the equation (18) we obtain straightforwardly the estimate

$$\|\ddot u_m\|^2_{L_2(I;V_m^*)}\le c_\eta,\ \ m\in\mathbb{N},\tag{25}$$

where $V_m\subset H^2(\Omega)$ is the linear hull of $\{w_i\}_{i=1}^m$. We proceed with the convergence of the Galerkin approximation. Applying the estimates (22-25) and the compact imbedding theorem we obtain for any $p\in[1,\infty)$ a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$), and a function u the convergences

$$\begin{aligned}u_m&\rightharpoonup^* u &&\text{in } \mathscr{H},\\ \dot u_m&\rightharpoonup^* \dot u &&\text{in } L_\infty(I;L_2(\Omega)),\\ \ddot u_m&\rightharpoonup \ddot u &&\text{in } \big(L_2(I;H^2(\Omega))\big)^*,\\ u_m&\to u &&\text{in } C(I;H^{1-\varepsilon}(\Omega))\cap L_\infty(I;H^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon>0,\\ \Phi(u_m,u_m)&\to\Phi(u,u) &&\text{in } L_2(I;H^2(\Omega)),\\ \Phi(u_m,u_m)&\rightharpoonup^*\Phi(u,u) &&\text{in } L_\infty(I;W_p^2(\Omega))\end{aligned}\tag{26}$$

Indeed, the first two convergences are obvious and imply

$$\begin{aligned}u_m\rightharpoonup u \text{ in } H^{1,2}(Q))\hookrightarrow H^{1/2+\varepsilon'}(I;H^{1-\varepsilon''}(\Omega)) \text{ for } \varepsilon'>0\\ \text{and } 0<\varepsilon''(\varepsilon')\searrow 0 \text{ if } \varepsilon'\searrow 0.\end{aligned}\tag{27}$$

The fourth convergence is a consequence of the convergence in the last space in (27) and the compact imbedding $H^{1/2+\varepsilon'}(I;H^{1-\varepsilon''}(\Omega))\hookrightarrow C(I;H^{1-\varepsilon}(\Omega))$ valid for any $\varepsilon>\varepsilon''$. Clearly $0<\varepsilon$ can be arbitrarily small again. The rest is a result of the interpolation of this result with the first convergence.

The fifth convergence is then the consequence of the compactness of the operator $\Phi: H^2(\Omega)\times H^2(\Omega)\mapsto H^2(\Omega)$. The last convergence follows using (11).

Let $\mu\in\mathbb{N}$ and $z_\mu=\sum_{i=1}^{\mu}\phi_i(t)w_i,\ \phi_i\in\mathscr{D}(0,T),\ i=1,...,\mu$. We have

$$\begin{aligned}&\int_\Omega\Big(\ddot u_m(t)z_\mu(t)+EA\big(u_m(t),z_\mu(t)\big)\\&\qquad+E[u_m(t),z_\mu(t)]\Phi(u_m(t),u_m(t)\,)-\eta^{-1}u_m(t)^-z_\mu(t)\Big)\,dx\\&=\int_\Omega f(t)z_\mu(t)\,dx\ \forall\ m\ge\mu,\ \ t\in T.\end{aligned}$$

The convergence process (26) and the property (20) imply that a function u fulfils

$$\int_Q\big(\ddot u z_\mu+EA(u,z_\mu)+E[u,\Phi(u,u)]z_\mu-\eta^{-1}u^-z_\mu\big)\,dx\,dt=\int_Q fz\,dx\,dt.$$

Functions $\{z_\mu\}$ form a dense subset of the set $L_2(I;H^2(\Omega))$, hence a function u fulfils the identity (17). The initial conditions (5) follow due to (19) and the proof of the existence of a solution is complete.

It is obvious that the estimates

$$\begin{aligned}\|\dot u_\eta\|^2_{L_\infty(I;L_2(\Omega))}+\|u_\eta\|^2_{L_\infty(I;H^2(\Omega))}+&\|\Phi(u_\eta,u_\eta)\|^2_{L_\infty(I;W_p^2(\Omega))}+\eta^{-1}\|u_\eta^-\|^2_{L_\infty(I;L_2(\Omega))}\\&\le c\equiv c(f,u_0,u_1).\end{aligned}\tag{28}$$

with u_η a solution of the penalized problem remain valid. In fact, here c depends on $\|f\|_{\left(L_\infty(I;L_2(\Omega))\right)^*}$. Hence, since $L_2(Q)$ is dense in $\big(L_\infty(I;L_2(\Omega))\big)^*$, for $f\in\big(L_\infty(I;L_2(\Omega))\big)^*$, there is a sequence $\{f_k\}\subset L_2(Q)$ such that $f_k\to f$ in $\big(L_\infty(I;L_2(\Omega))\big)^*$. It is easy to see that the solutions u_k of the penalized problems with f_k satisfy the same convergences as in (26). Hence for any $f \in \big(L_\infty(I;L_2(\Omega))\big)^*$ the penalized problem posesses a solution.

2.2 The limit process to the original problem

We rewrite the penalized problem (13) into the operator form

$$\ddot u_\eta + B(u_\eta) - \eta^{-1}u_\eta^- = f \tag{29}$$

with

$$B : H^2(\Omega) \to H^2(\Omega)^*,\ \langle B(v), w\rangle = E\int_\Omega (A(v,w) + [\Phi(v,v),v]w)\,dx,\ w \in H^2(\Omega)$$

and the initial conditions (5).

Let us multiply the equation (13) by $z = 1$. We get

$$0 \le \int_Q \eta^{-1}u_\eta^-\,dx\,dt = \int_\Omega \dot u_\eta^(T,\cdot)\,dx - \int_\Omega u_1\,dx - \int_Q f\,dx\,dt \le C, \tag{30}$$

where C is independent of η (cf. (28)). Since $B(u_\eta)$ takes its estimate in (28) and $L_1(\Omega) \hookrightarrow L_\infty(\Omega)^* \hookrightarrow H^2(\Omega)^*$, we get the dual estimate of the accelaration term

$$\|\ddot u_\eta\|_{\mathscr{H}^*} \le C \tag{31}$$

with C η-independent. Hence there is a sequence $\eta_k \searrow 0$ such that for $u_k \equiv u_{\eta_k}$ the following convergences hold

$$\begin{aligned}
u_k &\rightharpoonup^* u && \text{in } \mathscr{H},\\
\dot u_k &\rightharpoonup^* \dot u && \text{in } L_\infty(I;L_2(\Omega)),\\
\ddot u_k &\rightharpoonup^* \ddot u && \text{in } \mathscr{H}^*,\\
u_k &\to u && \text{in } C(I;H^{1-\varepsilon}(\Omega)) \cap L_\infty(I;H^{2-\varepsilon}(\Omega)),\ \varepsilon > 0,\\
\Phi(u_k,u_k) &\to \Phi(u,u) && \text{in } L_2(I,H^2(\Omega)),\\
\Phi(u_k,u_k) &\rightharpoonup^* \Phi(u,u) && \text{in } L_\infty(I;W_p^2(\Omega)),\\
\eta_k^{-1}u_k^- &\rightharpoonup^* g && \text{in } L_\infty(Q)^* \hookrightarrow \mathscr{H}^*,
\end{aligned} \tag{32}$$

where g is the corresponding contact force. The fourth convergence in (32) and the estimate (30) yield that $u \ge 0$ on Q and, in particular

$$u_k \to u \text{ in } L_\infty(Q). \tag{33}$$

The last convergence implies $g \ge 0$ in the dual sense. Obviously the expression

$$\|\dot u\|_{L_\infty(I;L_2(\Omega))} + \|u\|_{L_\infty(I;H^2(\Omega))} + \|\Phi(u,u\|_{L_\infty(I;W_p^2(\Omega))} + \|\ddot u\|_{\mathscr{H}^*} \tag{34}$$

is finite. Moreover, $t \mapsto u(t)$ is strongly $(I \to L_2(\Omega))$-continuous, hence it is weakly $(I \to H^2(\Omega))$-continuous. This yields it is strongly $(I \to H^{2-\varepsilon}(\Omega))$-continuous, in particular $u \in C(\bar Q)$. The performed convergences have proved that the limit u satisfies the equation

$$\ddot u + B(u) = f + g \tag{35}$$

in the dual sense in $(L_\infty(I;H^2(\Omega)))^*$. To prove $\langle g,u\rangle = 0$ we take in mind that from the just proved facts $\langle g,u\rangle = \lim_{k\to+\infty} 1/\eta_k \langle u_k^-, u_k^-\rangle = 0$, because $u_k^- \to 0$ in $L_\infty(Q)$ and $\eta_k^{-1}u_k^-$ is bounded in $L_1(Q)$. With this fact it is obvious that putting $v - u$ as a test function in (35) with an arbitrary $v \in \mathscr{C}$ we get the variational inequlity (12). The initial condition for u is satisfied in the sense of a weak limit in $H^2(\Omega)$ while that for $\dot u$ is satisfied in the sense of the integration by parts.

Hence we have proved the following

Theorem 2.2 *Let the domain* Ω *be convex polygonal or* $C^{3,1}$ *domain in* $\mathbb{R}^2$*. Let* u_0 *belong to* $H^2(\Omega)$*,* u_1 *belong to* $L_2(\Omega)$ *and let* f *be an element of* $\big(L_\infty(I;L_2(\Omega))\big)^*$*. Then there exists a solution of Problem* $(\mathscr{P})$*.*

Remark 2.3 *The idea of the proof of this theorem was substantially based on the imbedding* $H^2(\Omega)$ *into* $L_\infty(\Omega)$*. It cannot be extended to contact problems of membranes or bodies.*

3 Contact of a clamped plate

In this section we again treat the system (3) with the Dirichlet boundary value condition

$$u = U, \quad \partial_n u = 0 \text{ on } S. \tag{36}$$

and the initial condition (5). We assume that U is defined on $\overline{Q}$ and satisfies

$$\dot U \in L_2(I;H^2(\Omega)),\ \ddot u \in L_2(Q),\ U \geq U_0 \text{ on } Q,\ U(0,\cdot) = u_0,\ \text{and } \partial_n U = 0\ \text{ on } S. \tag{37}$$

for a fixed constant $U_0 > 0$. To state the variational formulation of this problem we shall use the cone

$$\mathscr{K} := \{y \in \mathring{\mathscr{H}} + U;\ y \geq 0\}, \tag{38}$$

where $\mathring{\mathscr{H}} = L_\infty(I;\mathring{H}^2(\Omega))$. The problem to be solved is

Problem $\tilde{\mathscr{P}}$ *We look for* $u \in \mathscr{K}$ *such that* $\ddot u \in (\mathring{\mathscr{H}})^*$*, the initial conditions* (5) *are satisfied in a certain generalized sense, and*

$$\begin{aligned}\langle \ddot u, y-u\rangle_0 + \int_Q E\big(A(u,y-u) + [u, E\Phi(u,u)](y-u)\big)\,dx\,dt \\ \geq \int_Q f(y-u)\,dx\,dt.\end{aligned} \tag{39}$$

holds for any $y \in \mathscr{K}$*.*

As earlier we apply the penalization of the contact condition. The classical formulation of the penalized problem is (13), (36), and (5). This leads to its variational formulation

Problem $\tilde{\mathscr{P}}_\eta$ *We look for* $u \in L_2(I;\mathring{H}^2(\Omega)) + U$ *such that* $\ddot u \in L_2(I;(H^{-2}(\Omega)))$*, the equation*

$$\int_Q \big(\ddot u z + EA(u,z) + [u, E\Phi(u,u)]z - \eta^{-1}u^- z\big)\,dx\,dt = \int_Q fz\,dx\,dt, \tag{40}$$

holds for any $z \in L_2(I;\mathring{H}^2(\Omega))$ *and the conditions* (5) *remain valid.*

To derive the *a priori* estimate for this problem we put $z = \chi_t(\dot u - \dot U)$ for $t \in (0,T]$ in (40), where

$$\chi_t : \mathbb{R} \to \{0,1\};\ \chi_t : s \mapsto \begin{cases} 1, & s \in [0,t], \\ 0, & \text{elsewhere.}\end{cases}$$

With the assumption (37) it is not difficult to prove the *a priori* estimates (28).

The existence of solutions to $\tilde{\mathscr{P}}_\eta$ is again proved via Galerkin approximations. Since all convergences in (26) remain valid, such existence is proved as earlier. Also the uniqueness result is analogously derived.

We proceed with the convergence of the penalization method. We write u_η for the solution of $\tilde{\mathscr{P}}_\eta$. To get the estimate of the penalty term we put $u = u_\eta,\ z = U - u_\eta$ in

(40). We arrive at the estimate

$$U_0 \int_Q \eta^{-1} u_\eta^- \, dx\, dt \le \int_Q \eta^{-1} u_\eta^- (U - u_\eta)\, dx\, dt \; =$$
$$+ \int_Q \big(- \dot u_\eta(\dot U - \dot u_\eta) + EA(u_\eta, U - u_\eta) + +\big(([u_\eta, E\Phi(u_\eta, u_\eta)] - f)(U - u_\eta)\big)\, dx\, dt$$
$$+ \int_\Omega \big(\dot u_\eta (U - u_\eta)\big)(T, \cdot)\, dx.$$

Applying the *a priori* estimates (28) we obtain in a same way as in Section 2 the estimate

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \le c(f, u_0, u_1, U) \tag{41}$$

We proceed similarly as in the case of a free plate. We obtain from (40), (41) the dual estimate

$$\|\ddot u_\eta\|_{(\mathring{\mathscr{H}})^*} \le c \tag{42}$$

with c $\eta-$independent. Combining the last estimate with (28) we obtain the convergences (32) with $(\mathring{\mathscr{H}})^*$ instead of $\mathscr{H}^*$. Applying the same approach as in the case of free plate we are proving that the limit function u is a solution of $\tilde{\mathscr{P}}$:

Theorem 3.1 *Let the domain* Ω *be convex polygonal or* $C^{3,1}$ *domain in* $\mathbb{R}^2$*. Let* u_0 *belong to* $H^2(\Omega)$*,* u_1 *belong to* $L_2(\Omega)$*,* f *be an element of* $\big(L_\infty(I; L_2(\Omega))\big)^*$ *and let* U *satisfy the assumption* (37)*. Then there exists a solution to the problem* $\tilde{\mathscr{P}}$*.*

Remark 3.1. The presented method can also prove the solvability of the unilateral dynamic contact problem for simply supported von Kármán plates. Different combinations of the presented boundary value conditions are admissible, too.

Remark 3.2. The nonuniqueness of solutions of the dynamic contact problem, due to the lack of information about the quality of the response of the system to the contact, is a well-known fact (cf. [6], Chapter 4 and the references cited there). The only hope is to get the uniqueness in the class of elastic reactions (energy conserving solutions), because the penalty method is well assumed to lead to such kind of solution.

References

[1] I. Bock and J. Jarušek: Unilateral dynamic contact of viscoelastic von Kármán plates *Adv. Math. Sci. Appl.* **16** (1) (2006), 175–187.

[2] I. Bock and J. Jarušek: Unilateral dynamic contact of von Kármán plates with singular memory. *Appl. Math.* **52** (6) (2007), 515-527.

[3] I. Bock and J. Lovíšek: On unilaterally supported viscoelastic von Kármán plate with a long memory. *Math. Comput. Simul.* **61** (2003), 399-407.

[4] I. Bock and J. Lovíšek: On a contact problem for a viscoelastic von Kármán plate and its semidiscretization. *Appl. Math.* **50** (2005), 203-217.

[5] P.G. Ciarlet and P. Rabier: *Les équations de von Kármán.* Springer, Berlin 1980.

[6] C. Eck, J. Jarušek and M. Krbec: *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems.* Monographs & Textbooks in Pure & Appl. Math. No. 270 (ISBN 1-57444-629-0). Chapman & Hall/CRC (Taylor & Francis Group), Boca Raton – London – New York – Singapore 2005.

[7] H. Koch and A. Stachel: Global existence of classical solutions to the dynamical von Kármán equations. *Math. Methods in Applied Sciences* 16 (1993), 581–586.

[8] K. Maruo: Existence of solutions of some nonlinear wave equations. *Osaka Math. J.* **22**, 21–30, 1984.

[9] J. Nečas: *Les Méthodés Directes en Théorie des Equations Elliptiques.* Mason, Paris 1967.

[10] A. Oukit and R. Pierre: Mixed finite element for the linear plate problem: the Hermann-Miyoshi model revisited. *Numer. Math.***74** (1996), 453–477.

Author's addresses: Igor Bock: Department of Mathematics, FEI, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava 1, Slovak Republic; e-mail: igor.bock@stuba.sk
Jiří Jarušek: Math. Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic; e-mail: jarusek@math.cas.cz